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MULTI INDEX FIXED CHARGE BICRITERION TRANSPORTATION PROBLEM

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In the present paper a Multi-Index Fixed Charge Bi-criterion Transportation Problem is defined. An algorithm to find efficient cost-time trade-off pairs in the multi-index fixed charge bi-criterion transportation problem is presented. A related fixed charge multi-index transportation problem is formulated and the efficient cost-time trade-off pairs are shown to be derivable from the related problem. An example is worked out in support of the theory.

Key Words : Multi-index Transportation Problem; Optimal Cost-time Trade-off

INTRODUCTION

The classical transportation problem is usually conceived of as a two-dimensional problem,

Haley[3] considered the multi-index transportation problem and presented an algorithm to solve the multi-index transportation problem. The multi-index transportation problem in which there are ***m*** origins, ***n*** destinations and ***p*** type of commodities to be transported is

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk}$$

subject to

$$\sum_{i=1}^{m} x_{ijk} = A_{jk}$$

$$\sum_{j=1}^{n} x_{ijk} = B_{ki}$$

$$\sum_{k=1}^{p} x_{ijk} = E_{ij}$$

and $x_{ijk} \geq 0,\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, p$ where

$$\sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki}, \sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij}, \sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p} \sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij} .$$

The fixed-charge problem was originally formulated by G.B. Dantzig and W. Hirsch in 1954[6]. In a classical transportation problem, the cost of transportation is directly proportional to the number of units transported. But the transportation cost may not be linear on account of price-breaks, quantity discounts etc. Thus in real world situations, when a commodity is transported, a fixed cost is incurred in the objective function.

From a practical point of view, the cost-minimizing transportation problem and the time-minimizing transportation problem cannot be viewed as two independent problems, if one is interested in obtaining a solution which minimizes cost and time simultaneously. Most of the practical transportation problems appear with two objectives, minimizing cost and time.

In this paper an algorithm for identifying the efficient cost-time trade-off pairs in a multi-index fixed-charge bi-criterion transportation is developed.

PROBLEM FORMULATION AND THEORETICAL DEVELOPMENT

The general model of the problem considered is as follows :

$$\text{(P) Minimize} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk} + \sum_{i=1}^{m} \sum_{k=1}^{p} F_{ik}, \max_{\substack{1 \le i \le m \\ 1 \le j \le n \\ 1 \le k \le p}} [t_{ijk} \mid x_{ijk} > 0] \right\}$$

subject to

$$\left.\begin{aligned}
&\sum_{i=1}^{m} x_{ijk} = A_{jk} \\
&\sum_{j=1}^{n} x_{ijk} = B_{ki} \\
&\sum_{k=1}^{p} x_{ijk} = E_{ij} \\
&\text{and } x_{ijk} \ge 0, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, p \\
&\text{where } \sum_{j=1}^{n} A_{jk} = \sum_{i=1}^{m} B_{ki}, \sum_{k=1}^{p} B_{ki} = \sum_{j=1}^{n} E_{ij}, \sum_{i=1}^{m} E_{ij} = \sum_{k=1}^{p} A_{jk} \\
&\sum_{j=1}^{n} \sum_{k=1}^{p} A_{jk} = \sum_{k=1}^{p} \sum_{i=1}^{m} B_{ki} = \sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij}
\end{aligned}\right] \quad \ldots (1)$$

Here

$i = 1, 2, ..., m$ are the origins

$j = 1, 2, ..., n$ are the destinations

$k = 1, 2, ..., p$, are the various types of commodities

x_{ijk} = the amount of k^{th} type of commodity transported from the i^{th} origin to the j^{th} destination

c_{ijk} = the variable cost per unit amount of k^{th} type of commodity from the i^{th} origin to the j^{th} destination which is independent of the amount of commodity transported, so long as $x_{ijk} > 0$

F_{ik} = the fixed cost associated with origin i and commodity k

$$F_{ik} = \sum_{j=1}^{n} F_{ijk}\, \delta_{ijk},\ i = 1, 2, ..., m;\ k = 1, 2, ..., p$$

where $\delta_{ijk} = 1$, if $x_{ijk} > 0 = 0$, if $x_{ijk} = 0$

A_{jk} = the total quantity of k^{th} type of commodity to be sent to the j^{th} destination

B_{ki} = the total quantity of k^{th} type of commodity available at the i^{th} origin

E_{ij} = the total quantity to be sent from i^{th} origin to the j^{th} destination

To solve (P) we separate (P) into two problems (P') and (p'').

Problem (P') is :

$$(P') : \text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk} + \sum_{i=1}^{m} \sum_{k=1}^{p} F_{ik}$$

subject to (1).

Problem (p'') is

$$(p'') : \text{Minimize } T = \max_{\substack{1 \le i \le m \\ 1 \le j \le n \\ 1 \le k \le p}} [t_{ijk} \mid x_{ijk} > 0]$$

subject to (1).

Definition — Efficient Cost-Time Trade-Off Pair

A solution $\bar{x}$ of (P) gives an efficient cost-time trade-off pair if there exists no solution $\bar{y}$ of (P) satisfying the conditions (i) $Z(\bar{y}) \le Z(\bar{x})$ and (ii) $T(\bar{y}) \le T(\bar{x})$ with strict inequality holding in atleast one of the conditions out of (i) and (ii).

Our aim is to obtain the set of efficient cost-time trade-off pairs for problem (P).

For this purpose first solve (P′) and then find the duration of time T with respect to the minimum cost, where T is given by

$$T = \max_{\substack{1 \le i \le m \\ 1 \le j \le n \\ 1 \le k \le p}} [t_{ijk} \mid x_{ijk} > 0]$$

After modifying the costs with respect to the time obtained, problem (P') is solved again. Find the time with respect to the new minimum cost obtained. This is repeated until an infeasible solution is obtained. The above procedure is known as Re-optimization procedure.

To solve (P'), a Related Multi-Index Transportation Problem (RP') is formed. It is given by

$$(RP') : \text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk}$$

subject to (1).

At the first iteration let z_1^* be the optimal cost of problem (P') and T_1^* be the time with respect to the cost Z_1^*, then (Z_1^*, T_1^*) is called the cost-time trade-off pair at the first iteration. Using re-optimization procedure, let after q^{th}-iteration the solution be infeasible. Thus, we get the following set of efficient cost-time trade-off pairs :

$$(Z_1^*, T_1^*), (Z_2^*, T_2^*), \ldots, (Z_q^*, T_q^*)$$

where

$$Z_1^* < Z_2^* < \ldots < Z_q^*$$

$$T_1^* > T_2^* > \ldots > T_q^*$$

ALGORITHM

Step 1 — Set $r = 0$, where r is the number of iterations in the algorithm.

Step 2 — $r = r + 1$, $r = 0, 1, 2, \ldots$

Step 3 — At the r-th iteration find the set of optimal solutions of $(RP')_r$.

Step 4 — Corresponding to each optimal solution of $(RP')_r$, calculate the corresponding fixed cost of (P').

Step 5 — Let X_r^* be the optimal solution of $(RP')_r$ which gives the minimum fixed cost of (P'). Let Z_r^* be the objective function value of (P') corresponding to X_r^*.

Step 6 — Find $T_r^* = \max\limits_{\substack{1 \le j \le m \\ 1 \le j \le n \\ 1 \le k \le p}} [t_{ijk} \mid x_{ijk} > 0 \text{ corresponding to } X_r^*]$

Then the corresponding pair (Z_r^*, T_r^*) is the r^{th} cost-time trade-off pair.

Step 7 (*a*) — Define

$$c^r_{ijk} = \begin{cases} M, & \text{if } t_{ijk} \geq T^*_r \\ c_{ijk}, & \text{if } t_{ijk} < T^*_r \end{cases}$$

where M is a sufficiently large positive number and form the corresponding multi-index transportation problem $(RP')_{r+1}$.

(*b*) :Solve this Multi-Index Transportation Problem. If there exists a basic feasible solution not involving the cells with cost M, go to step 3.

If there does not exist such a basic feasible solution go to step 8.

Step 8 — Let after q^{th}-iteration the solution be infeasible i.e. $Z^*_{q+1} \geq M$. Then, the complete set of efficient trade-off pairs is $(Z^*_1, T^*_1), Z^*_2, T^*_2), \ldots, (Z^*_q, T^*_q)$ where $Z^*_1 < Z^*_2 < \ldots < Z^*_q$ and $T^*_1 > T^*_2 > \ldots > T^*_q$.

NUMERICAL EXAMPLE

Consider a $3 \times 3 \times 3$ Fixed Charge Bi-criterion Multi-Index Transportation Problem.

$$\text{(P) : Minimize} \left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} c_{ijk} x_{ijk} + \sum_{i=1}^{3} \sum_{k=1}^{3} F_{ik}, \max_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3 \\ 1 \leq k \leq 3}} [t_{ijk} \mid x_{ijk} > 0] \right\}$$

subject to $\quad \displaystyle\sum_{i=1}^{3} x_{ijk} = A_{jk},$

$$\sum_{j=1}^{3} x_{ijk} = B_{ki},$$

$$\sum_{k=1}^{3} x_{ijk} = E_{ij}$$

and $\quad x_{ijk} \geq 0, \ i = 1, 2, 3; \ j = 1, 2, 3; \ k = 1, 2, 3,$

where $\quad \displaystyle\sum_{j=1}^{3} \sum_{k=1}^{3} A_{jk} = \sum_{k=1}^{3} \sum_{i=1}^{3} B_{ki} = \sum_{i=1}^{3} \sum_{j=1}^{3} E_{ij}.$

The data of variable cost c_{ijk} and t_{ijk} is given in Table I.

c_{ijk}, t_{ijk} $(i = 1, 2, 3; j = 1, 2, 3; k = 1, 2, 3)$ are at the top left corner and at the bottom right corner respectively.

TABLE I

	j = 1	j = 2	j = 3	B_{ki}
i = 1	8 3 7 5 6 4 E_{11}=10	5 8 6 6 10 8 E_{12}=6	7 7 3 4 11 1 E_{13}=9	6 9 10
i = 2	11 2 8 1 13 6 E_{21}=21	9 4 15 2 12 2 E_{22}=9	13 6 7 1 8 1 E_{23}=14	13 14 17
i = 3	5 4 6 8 7 1 E_{31}=21	8 3 9 2 7 1 E_{32}=13	10 4 6 2 12 8 E_{33}=12	15 13 18
A_{jk}	15 17 20	6 11 9	11 8 16	

The fixed costs are

$F_{111} = 10$	$F_{121} = 30$	$F_{131} = 20$
$F_{112} = 20$	$F_{122} = 20$	$F_{132} = 20$
$F_{113} = 30$	$F_{123} = 20$	$F_{133} = 10$
$F_{211} = 10$	$F_{221} = 20$	$F_{231} = 20$
$F_{212} = 10$	$F_{222} = 10$	$F_{232} = 30$
$F_{213} = 40$	$F_{223} = 10$	$F_{233} = 10$
$F_{311} = 10$	$F_{321} = 40$	$F_{331} = 20$
$F_{312} = 20$	$F_{322} = 10$	$F_{332} = 30$
$F_{313} = 20$	$F_{323} = 10$	$F_{333} = 10$

Form the related transportation problem $(RP')_1$.

Using the North-West Corner rule, we get the initial basic feasible solution given in Table II. The values of corresponding $u_{jk}, v_{ki}, w_{ij} = (i = 1, 2, 3;\ j = 1, 2, 3;\ k = 1, 2, 3)$ are also given in Table II.

TABLE II

8 6 3 7 4 5 $w_{11}=0$ - 4	5 - 8 6 5 6 $w_{12}=-8$ 10 1 8	7 - 7 3 - 4 $w_{13}=-3$ 11 9 1	$v_{11}=-3$ $v_{21}=-1$ $v_{31}=6$
11 9 2 8 10 1 $w_{21}=0$ 13 2 6	9 - 4 15 1 2 $w_{22}=0$ 12 8 2	13 4 6 7 3 1 $w_{23}=0$ 8 7 1	$v_{12}=0$ $v_{22}=0$ $v_{32}=0$
5 - 4 6 3 8 $w_{31}=-1$ 7 18 1	8 8 3 9 5 2 $w_{32}=-5$ 7 - 1	10 7 4 6 5 2 $w_{33}=0$ 12 - 8	$v_{13}=0$ $v_{23}=-1$ $v_{33}=-5$
$u_{11}=11$ $u_{12}=8$ $u_{13}=13$	$u_{21}=16$ $u_{22}=15$ $u_{23}=12$	$u_{31}=13$ $u_{32}=7$ $u_{33}=8$	

The optimal solution of $(RP')_1$ is given in Table III.

TABLE III

8 3 7 5 $w_{11}=0$ 6 10 4	5 - 8 6 6 6 $w_{12}=0$ 16 - 8	7 6 7 3 3 4 $w_{13}=0$ 11 0 1	$v_{11}=-3$ $v_{21}=-3$ $v_{31}=-1$
11 5 2 8 14 1 $w_{21}=0$ 13 2 6	9 8 4 15 2 $w_{22}=-1$ 12 1 2	13 6 7 1 $w_{23}=-10$ 8 14 1	$v_{12}=6$ $v_{22}=2$ $v_{32}=6$
5 10 4 6 3 8 $w_{31}=0$ 7 8 1	8 3 9 5 2 $w_{32}=0$ 7 8 1	10 5 4 6 5 2 $w_{33}=0$ 12 2 8	$v_{13}=0$ $v_{23}=0$ $v_{33}=0$
$u_{11}=5$ $u_{12}=6$ $u_{13}=7$	$u_{21}=4$ $u_{22}=9$ $u_{23}=7$	$u_{31}=10$ $u_{32}=6$ $u_{33}=12$	

The optimal value of $Z=Z_1^* = 865 + 320 = 1185$ and the corresponding time $T_1^* = 8$. The first cost-time trade-off pair is (1185, 8).

Define $$c_{ijk} = M \text{ for } t_{ijk} \geq T_1^* = 8$$

and $$c_{ijk} = c_{ijk} \text{ for } t_{ijk} < T_1^* = 8$$

and form the corresponding related multi-index transportation problem. On solving this problem, the next trade-off pair is (Z_2^*, T_2^*), $Z_2^* = 1204$ and $T_2^* = 7$. Proceeding like this we get (Z_3^*, T_3^*) as the third trade-off pair, $Z_3^* = 1310$ and $T_3^* = 6$.

After that the related problem defined at time T_3^* becomes infeasible. Hence the trade-off pairs are (1185, 8), (1204, 7) and (1310, 6).

Remark 1 : At any stage of the solution, if problem (p') has alternate optimal solutions, then the optimal time T_r^* $(r = 1, 2, ..., q)$ would be calculated as

$$T_r^* = \min_{1 \le w \le k_r} \left\{ \max_{\substack{1 \le i \le m \\ 1 \le j \le n \\ 1 \le k \le p}} [t_{ijk}, x_{ijk} > 0 \text{ according to } X_r^{(w)} \right\}$$

where $X_r^{(w)}$ $(w = 1, 2, ..., k_r)$ are alternative optimal solutions of problem (P').

The solution $X_r^{(w)}$ $(w = 1, 2, ..., k_r)$ giving the optimal time T_r^* will then be taken as the optimal solution.

Remark 2 : An alternative approach to solve problem (P) is to first minimize the time function and then read the corresponding cost (variable + fixed) from the solution which gives the minimum time. Thereafter, keep on increasing the time steadily, at each step reading simultaneously the contribution of the cost function. We continue till further raise is not possible.

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