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INTEGRAL EQUATIONS OF FREDHOLM TYPE WITH SPECIAL FUNCTIONS

V. B. L. CHAURASIA AND RINKU PATNI

Department of Mathematics, University of Rajasthan, Jaipur 302 004, *Rajasthan, India*

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The paper is devoted to the study of the various useful methods of solving the one-dimensional integral equation of Fredholm type. In particular, by applying the reduction techniques with a view to inverting a class of integral equation with Lauricella function in the kernel, Riemann-Liouville fractional integral operators as well as Weyl operators are employed to reduce this class to generalized Stieltjes transform which is then inverted to obtain the solution of the integral equation. The Fredholm integral equation involving the product of the H-function and a general class of polynomials in the kernel is also considered by the Mellin transform technique.

Key Words and Phrases : Fredholm Type Integral Equations; Reduction Techniques; Lauricella Function; Riemann-Liouville Fractional Integral; Weyl Fractional Integral Steiltjes Transform; *H*-function, General Class of Polynomials; Mellin Transform Technique; Multivariable H-function; Mellin-Barnes Contour Integrals; Mellin Inversion Theorem

1. INTRODUCTION

In the last three decades several authors have made significant contribution to the study of integral equations involving certain special functions of one and more variables (see, for example, Buschman. Koul and Gupta[1], Higgins[4], Love[6&7], Prabhakar and Kashyap[8], Srivastava and Buschman[11], Srivastava and Raina[15]).

Srivastava and Raina[15], have also given a systematic and detailed discussion of the various methods of solvability of certain interesting cases of the Fredholm type integral equation :

$$\int_0^\infty t^{-\lambda} H_{P,Q}^{M,N}\left[A\left(\frac{x}{t}\right)^m \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right] f(t)dt = g(x).\ (0<x<\infty). \qquad \text{... (1.1)}$$

The object of the present paper is an extension of these results concerning for the Fredholm integral equations

$$\int_0^\infty y^{-\alpha}\, H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:(B',d');\ldots;(B^{(r)},D^{(r)})}$$

$$\left[\begin{matrix}[(a_j):\theta_j';\dots;\theta_j^{(r)}]_{1,A}: \\ [(c_j):\psi_j';\dots;\psi_j^{(r)}]_{1,C}:\end{matrix}\ \begin{matrix}[(b_j'):\phi_j']_{1,B'};\dots;[(b_j^{(r)}):\phi_j^{(r)}]_{1,B^{(r)}}; \\ \\ [(d_j'):\delta_j']_{1,D'};\dots;[(d_j^{(r)});\delta_j^{(r)}]_{1,D^{(r)}}:\end{matrix}\ z_1\left(\frac{x}{y}\right)^q,\dots,z_r\left(\frac{x}{y}\right)^q\right] f(y)\,dy$$

$$= g(x)\ \ (0<x<\infty) \qquad \dots (1.2)$$

and

$$\int_0^\infty y^{-\alpha} S_n^m\left[E\left(\frac{x}{y}\right)^p\right] H_{B,D}^{u,v}\left[z\left(\frac{x}{y}\right)^q \,\middle|\, \begin{matrix}(b_j,\beta_j)_{1,B} \\ (d_j,\delta_j)_{1,D}\end{matrix}\right]_{0<x<\infty)} f(y)\,dy = g(x) \qquad \dots (1.3)$$

The H-function of r-complex variables $z_1, \dots, z_r$ occurring in (1.2) was introduced by Srivastava and Panda[14] which is defined as a contour integrals of Mellin-Barnes type. For full definition and other details of this function, see [Srivastava *et al.*[13]].

The multivariable H-function in (1.2) converges absolutely and defines an analytic function for

$$|\arg(z_i)| < \frac{1}{2}\pi T_i, \qquad \dots (1.4)$$

where
$$T_i = -\sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{u^{(i)}} \delta_j^{(i)}$$

$$-\sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,\ \forall\ i \in (1,\dots,r). \qquad \dots (1.5)$$

$S_n^m[E]$ in (1.3) denotes the general class of polynomials introduced by Srivastava[10].

$$S_n^m[E] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} E^k,\ n = 0, 1, 2, \dots \qquad \dots (1.6)$$

where m is an arbitrary positive integer and coefficients $A_{n,k}\,(n, k \geq 0)$ are arbitrary constants, real or complex.

Let $\int$ denote the space of all functions f which are defined on $R^+ = [0, \infty)$ and satisfy

(i) $f \in \mathcal{C}^\infty(R^+)$, (ii) $\lim_{x\to\infty} [x^\gamma f^r(x)] = 0$ for all non negative integers γ and r

(iii) $f(x) = O(1)$ as $x \to 0$.

For correspondence to the space of good functions defined on the whole real line $(-\infty, \infty)$ see (Lighthill)[5].

The Riemann-Liouville fractional integral (of order μ) is defined by

$$\mathcal{D}^{-\mu}\{f(x)\} = {}_0\mathcal{D}_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)}\int_0^x (x-w)^{\mu-1} f(w)\,dw, \ (\text{Re}\ (\mu)>0: f\in \textstyle\int), \quad \ldots (1.7)$$

where $\mathcal{D}^{\mu}\{f(x)\} = \phi(x)$ is understood to mean that ϕ is a locally integrable solution of $f(x) = \mathcal{D}^{-\mu}\{\phi(x)\}$, implying that $\mathcal{D}^{\mu}$ is the inverse of the fractionalintegral operator $\mathcal{D}^{-\mu}$ (whenever necessary, we shall simply write $\mathcal{D}_x^{-\mu}$ for ${}_0\mathcal{D}_x^{-\mu}$ for the Riemann-Liouville fractional integral operator defined by eq. (1.7) above).

The Weyl fractional integral (of order h) is defined by

$$\mathcal{W}^{-h}\{f(x)\} = {}_x\mathcal{D}_{\infty}^{-h}\{f(x)\}$$

$$= \frac{1}{\Gamma(h)}\int_x^{\infty} (\zeta-x)^{h-1} f(\zeta)\,d\zeta, \ (Re\ (h) > 0: f\in \textstyle\int). \quad \ldots (1.8)$$

2. Preliminary Results

Lemma 1: Let

(*i*) $\lambda, u^{(i)}, v^{(i)}, A, B^{(i)}, C, D^{(i)}$ be positive integers such that $0\le\lambda\le A, 0\le u^{(i)}\le D^{(i)}, C\ge 0$ and $0\le v^{(i)}\le B^{(i)}$, $i = 1, \ldots, r$;

(*ii*) $Re\ (\alpha) > Re\ (\beta)$; $Re\left[\beta+q\sum_1^r (d_j^{(i)}/\delta_j^{(i)})\right]>0$ $(j = 1, \ldots, u^{(i)}$; $q > 0$;

(*iii*) $|\arg(z_i)| < \frac{1}{2}\pi T_i$, where T_i is given by (1.5).

Then

$$\mathcal{W}^{\beta-\alpha}\left[y^{-\alpha} H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:(B',D');\ldots;(B^{(r)},D^{(r)})}\left[\begin{matrix}[(a_j):\theta_j',\ldots,\theta_j^{(r)}]_{1,A};\\ [(c_j):\psi_j',\ldots,\psi_j^{(r)}]_{1,C};\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}[(b_j'):\phi_j']_{1,B'};\ldots;[(b_j^{(r)}):\phi_j^{(r)}]_{1,B^{(r)}};\\ [(d_j'):\delta_j']_{1,D'};\ldots;[(d_j^{(r)}):\delta_j^{(r)}]_{1,D^{(r)}};\end{matrix}\ z_1\left(\frac{x}{y}\right)^q,\ldots,z_r\left(\frac{x}{y}\right)^q\right]\right]$$

$$= y^{-\beta} H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1;(B',D');\ldots;(B^{(r)},D^{(r)})}\left[\begin{matrix}[1-\beta:q,\ldots,q],\\ [(c_j):\psi_j',\ldots,\psi_j^{(r)}]_{1,C},\end{matrix}\right.$$

$$\left.\begin{matrix} [(a_j') : \theta_j', \ldots, \theta_j^{(r)}]_{1,A} : [(b_j') : \phi_j']_{1,B'} ; \ldots ; [(b_j^{(r)}) : \phi_j^{(r)}]_{1,B^{(r)}} ; z_1 \left(\frac{x}{y}\right)^q \\ [1-\alpha : q, \ldots, q] \qquad : [(d_j') : \delta_j']_{1,D'} ; \ldots ; [(d_j^{(r)}) : \delta_j^{(r)}]_{1,D^{(r)}} ; z_r \left(\frac{x}{y}\right)^q \end{matrix}\right] \quad \ldots (2.1)$$

PROOF : Let $\eta(y)$ denote the first member of the assertion (2.1). Then, making use of (1.8) and the definition of the multivariable H-function[13] we find that

$$\eta(y) = \frac{1}{\Gamma(\alpha-\beta)} \int_y^\infty (t-y)^{\alpha-\beta-1} \left(\frac{1}{(2\pi i)^r} \int_{L_1} \ldots \int_{L_r} \phi_1(s_1) \ldots \phi_r(s_r) \right.$$

$$\left. \times \psi(s_1, \ldots, s_r) z_1^{s_1} \ldots z_r^{s_r} (x)^{q \sum_1^r s_i} (t)^{-\alpha - q \sum_1^r s_i} ds_1 \ldots ds_r \right) dt. \quad \ldots (2.2)$$

The change in the order of integration in (2.2) is assumed to be permissible by absolute (and uniform) convergence of the integrals; we have

$$\eta(y) = \frac{1}{\Gamma(\alpha-\beta)(2\pi i)^r} \int_{L_1} \ldots \int_{L_r} \phi_1(s_1) \ldots \phi_r(s_r) \psi(s_1, \ldots, s_r)$$

$$\times z_1^{s_1} \ldots z_r^{s_r} (x)^{q \sum_1^r s_i} \left(\int_y^\beta (t-y)^{\alpha-\beta-1} t^{-\alpha - q \sum_1^r s_i} dt \right) ds_1 \ldots ds_r. \quad \ldots (2.3)$$

The inner integral in (2.3) can be solved under hypothesis (ii) of Lemma 1, and we obtain

$$\eta(y) = \frac{y^{-\beta}}{(2\pi i)^r} \int_{L_1} \ldots \int_{L_r} \phi_1(s_1) \ldots \phi_r(s_r) \psi(s_1, \ldots, s_r)$$

$$\times \frac{\Gamma\left(\beta + q \sum_l^r s_i\right)}{\Gamma\left(\alpha + q \sum_1^r s_i\right)} z_1^{s_1} \ldots z_r^{s_r} \left(\frac{x}{y}\right)^{q \sum_1^r s_i} ds_1 \ldots ds_r, \quad \ldots (2.4)$$

which yields the second member of (2.1) by reinterpreted the *H*-function of *r*-variables.

The multivariable *H*-function appearing in (2.1) exist (and are analytic) under the conditions (i) and (ii) of Lemma 1, and the Weyl fractional integral converges absolutely under the condition (ii).

Theorem 1 — *With the set of sufficient conditions (i), (ii) and (iii) of Lemma 1,*

$$\int_0^\infty y^{-\beta} H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1\,C+1:(B',D');\ldots:(B^{(r)},D^{(r)})}\left[\begin{matrix}[1-\beta:q,\ldots,q],\\ [(c_j):\psi_j';\ldots;\psi_j^{(r)}]_{1,C},\end{matrix}\right.$$

$$[(a_j):\theta_j',\ldots,\theta_j^{(r)}]_{1,A}:[(b_j'):\phi_j']_{1,B'};\ldots;[(b_j^{(r)});\phi_j^{(r)}]_{1,B^{(r)}};$$

$$\left.[1-\alpha:q,\ldots,q]:[(d_j'):\delta_j']_{1,D'};\ldots;[(d_j^{(r)}):\delta_j^{(r)}]_{1,D^{(r)}}\;;\;z_1\left(\frac{x}{y}\right)^q,\ldots,z_r\left(\frac{x}{y}\right)^q\right]$$

$$\times f(y)\,dy=\int_0^\infty y^{-\alpha} H^{0,\lambda}_{A,C}\left[z_1\left(\frac{x}{y}\right)^q,\ldots,z_r\left(\frac{x}{y}\right)^q\right]\mathcal{D}^{\beta-\alpha}\,\{f(y)\}\,dy, \quad \ldots (2.5)$$

provided further $f\in \int$ and $x > 0$.

PROOF : Let Δ denote the first member of the assertion (2.5). Then using Lemma 1 and applying (1.8), we have

$$\Delta=\int_0^\infty \frac{f(y)}{\Gamma(\alpha-\beta)}\left\{\int_y^\infty (\zeta-y)^{\alpha-\beta-1}\zeta^{-\alpha}H^{0,\lambda}_{A,C}\left[z_1\left(\frac{x}{y}\right)^q,\ldots,z_r\left(\frac{x}{y}\right)^q\right]d\zeta\right\}dy \quad \ldots (2.6)$$

$$=\int_0^\infty \zeta^{-\alpha}H^{0,\lambda}_{A,C}\left[z_1\left(\frac{x}{y}\right)^q,\ldots,z_r\left(\frac{x}{y}\right)^q\right]\left(\int_0^\zeta \frac{(\zeta-y)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(y)\,dy\right)d\zeta. \quad \ldots (2.7)$$

Assuming the inversion of the order of integration to be permissible just as in the proof of Lemma 1.

Now, by definition (1.7), (2.7) gives

$$\Delta=\int_0^\infty \zeta^{-\alpha}H^{0,\lambda}_{A,C}\left[z_1\left(\frac{x}{y}\right)^q,\ldots,z_r\left(\frac{x}{y}\right)^q\right]\mathcal{D}^{\beta-\alpha}\,\{f(\zeta)\}\,d\zeta, \quad \ldots (2.8)$$

which is the second member of (2.5).

3. SOLUTION OF A LAURICELLA FORM OF THE INTEGRAL EQUATION (1.2)

To obtain the solution of a certain Lauricella's hypergeometric form of the integral equation (1.2), we use the reduction technique by which a given integral equation may be reduced to some simpler integral transform with the aid of the result derived in the preceding section.

Indeed in Theorem 1, we first use the relationship[2]

$$H^{0,0;(u',v');\ldots;(u^{(r)},v^{(r)})}_{0,0:(B',D');\ldots;(B^{(r)},D^{(r)})}\left[\begin{matrix}[(a_j):\theta_j';\ldots;\theta_j^{(r)}]_{1,A}:\\ [(c_j):\psi_j';\ldots;\psi_j^{(r)}]_{1,C}:\end{matrix}\right.$$

$$\left.\begin{matrix} [(b_j') : \phi_j']_{1,B'} ; \dots ; [(b_j^{(r)}) : \phi_j^{(r)}]_{1,B^{(r)}} : \\ [(d_j') : \delta_j']_{1,D'} ; \dots ; [(d_j^{(r)}) : \delta_j^{(r)}]_{1,D^{(r)}} : \end{matrix} \; z_1, \dots, z_r \right]$$

$$= \prod_{i=1}^{r} \left\{ H_{B^{(i)}, D^{(i)}}^{\mu^{(i)}, \nu^{(i)}} \left[z_i \left| \begin{matrix} [(b_j^{(i)}) : \phi_j^{(i)}]_{1,B^{(i)}} \\ [(d_j^{(i)}) : \delta_j^{(i)}]_{1,D^{(i)}} \end{matrix} \right. \right] \right\} \qquad \dots (3.1)$$

Also, we set $\phi_j^{(i)} = 1$ $(j = 1, \dots, B^{(i)})$ and $\delta_j^{(i)} = 1$ $(j = 1, \dots, D^{(i)})$, and modify the parameters of the multivariable H-function in such a way that use can be made of the relationship[12] on the left hand side of (2.5) and of the relationship[13]:

$$H_{1,1}^{1,1} \left[z \left| \begin{matrix} (1-\alpha, 1) \\ (0, 1) \end{matrix} \right. \right] = \frac{\Gamma(\alpha)}{(1+z)^{\alpha}} = \Gamma(\alpha)\, {}_1F_0 [\alpha : - : -z] \qquad \dots (3.2)$$

On the right hand side of (2.5). Thus the solution of the Corollary 1 emerges from Theorem 1.

Corollary 1 — If $b_1, \dots, b_r, \alpha$ and β be complex parameters,

$Re\ (\alpha) > Re\ (\beta) > 0, f \in \int$ and $q \in IN$, then for all $x > 0$

$$\int_0^{\infty} \frac{\prod_{i=1}^{r} \Gamma(b_i)\, \Gamma(\beta)}{\Gamma(\alpha)\, y^{\beta}} F\left[b_1, \dots, b_r, \Delta'(q; \beta) ; \Delta'(q, \alpha) ; -\left(\frac{x}{y}\right)^{q}, \dots, -\left(\frac{x}{y}\right)^{q} \right] f(y)\, dy$$

$$= \prod_{i=1}^{r} \Gamma(b_i) \int_0^{\infty} \frac{y^{\sum_1^r q b_i - \alpha}\, \mathcal{D}^{\beta - \alpha} (f(y))}{(x^q + y^q)^{\Sigma b_i}}\, dy$$

OR

$$\int_0^{\infty} \frac{\prod_{i=1}^{r} \Gamma(b_i)\, \Gamma(\beta)}{\Gamma(\alpha)\, y^{\beta}} F_{1:0,\dots,0}^{1:1,\dots,1} \left[\begin{matrix} [\beta : q, \dots, q] : \\ [\alpha : q, \dots, q] : \end{matrix} \right.$$

$$\left. \begin{matrix} [b_1 : 1] : \dots : [b_r : 1] : \\ [0 : 0] : \dots : [0, 0] : \end{matrix} \; -\left(\frac{x}{y}\right)^{q}, \dots, -\left(\frac{x}{y}\right)^{q} \right] f(y)\, dy$$

$$= \prod_{i=1}^{r} \Gamma(b_i) \int_0^\infty \frac{y^{\sum_1^r q b_i - \alpha} \mathcal{D}^{\beta-\alpha}(f(y))}{(x^q + y^q)^{\Sigma b_i}} dy,$$

where $$\Delta'(q : \alpha) = \left\{ \frac{\alpha}{q}, \frac{\alpha+1}{q}, \ldots, \frac{\alpha+q-1}{q} \right\} (q \in I\!N). \qquad \ldots (3.3)$$

Theorem* 2 — *Let *$b_1, \ldots, b_r$, α and β be complex parameters such that* $Re \left(\sum_1^r b_i \right) > 1$ *$(i = 1, \ldots, r)$ and $Re(\alpha) > Re(\beta) > 0$, and $q \in I\!N$, $f \in \int$. Then for all $x > 0$ and $g \in \int$, the integral equation*

$$\int_0^\infty \frac{\prod_{i=1}^{r} \Gamma(b_i) \Gamma(\beta)}{\Gamma(\alpha)} y^{-\beta}$$

$$F \left[b_1, \ldots, b_r, \Delta'(q : \beta) : \Delta'(q; \alpha); -\left(\frac{x}{y}\right)^q, \ldots, -\left(\frac{x}{y}\right)^q \right] f(y)\, dy = g(x) \qquad \ldots (3.4)$$

has a solution given by

$$f(x) = \frac{q\, \Gamma(b_1 + \ldots + b_r)}{\prod_{i=1}^{r} \Gamma(b_i)} \mathcal{D}_x^{\alpha-\beta} \left[x^{q\left(1 - \sum_{i=1}^{r} b_i\right) + \alpha - 1} \mathcal{D}_{x^m}^{1 - \sum_{i=1}^{r} b_i} \right.$$

$$\left. \left\{ (1+x) \lim_{n' \to \infty} L_{n', x^m} [g(x)] \right\} \right], \qquad \ldots (3.5)$$

where $$L_{n', x} [g(x)] = \frac{(-x)^{n'-1}}{n'!\,(n'-2)!} \frac{d^{2n'-1}}{dx^{2n'-1}} \{x^{n'} g(x)\} \;\; (n' = 2, 3, 4, \ldots). \qquad \ldots (3.6)$$

PROOF : With the help of Corollary 1, we may write (3.4) in the form :

$$\prod_{i=1}^{r} \Gamma(b_i) \int_0^\infty \frac{(y)^{\sum_1^r b_i q - \alpha}}{(x^q + y^q)^{\Sigma b_i}} \mathcal{D}^{\beta-\alpha} \{f(y)\}\, dy = g(x). \qquad \ldots (3.7)$$

Using the linear functional relationship of Srivastava[9] gives

$$\frac{\prod_{i=1}^{r} \Gamma(b_i)}{q} \int_0^\infty \frac{y^{\sum_1^r b_i - \frac{\alpha}{q}}}{(x^q+y)^{\sum_1^r b_i}} \mathcal{D}_{y^{1/q}}^{\beta-\alpha} \{f(y^{1/q})\} \frac{dy}{(y)^{\frac{q-1}{q}}} = g(x). \quad \ldots (3.8)$$

Now applying the result of Love[7] in (3.8), we get

$$\frac{\prod_{i=1}^{r} \Gamma(b_i)}{q\,\Gamma(b_1+\ldots+b_r)} \int_0^\infty (x+y)^{-1} \mathcal{D}_y^{\sum_1^r b_i - 1}$$

$$\left\{ y^{\sum_1^r b_i - 1 + \frac{(1-\alpha)}{q}} \mathcal{D}_{y^{1/q}}^{\beta-\alpha} (f(y^{1/q})) \right\} dy = g(x^{1/q}). \quad \ldots (3.9)$$

By appealing appropriately to Theorem 9 of Widder[17] concerning the inversion of the Stieltjes transform, we get

$$\mathcal{D}_y^{\sum_1^r b_i - 1} \left\{ y^{\sum_1^r b_i - 1 + \frac{1+\alpha}{q}} \mathcal{D}_{y^{1/q}}^{\beta-\alpha} (f(y^{1/q})) \right\} = \frac{q\,\Gamma(b_1+\ldots+b_r)}{\prod_{i=1}^{r} \Gamma(b_i)}$$

$$\times (1+y^{1/q}) \lim_{n' \to \infty} L_{n',x}[g(y^{1/q})], \quad \ldots (3.10)$$

where $g(x^{1/q})$ as a function of x is operated upon by $L_{n,x}$ and then x is replaced by y. Thus we get the desired result from (3.10). If we take $r = 1$, Theorem 2 is reduces to a result given by H. M. Srivastava and Raina[15] underless stringent conditions. Also for $r = 1$ and $m = 1$, Theorem 2 is seem to correspond to a result given by Love[7].

4. Use of Other Methods

One-dimensional Fredholm integral equation (1.3) involving the Fox's H-function and a general class of polynomials $S_n^m[E]$ in the kernel can also be solved by resorting to the application of Mellin transforms.

We first prove the following result which will be required in proving Theorem 3 below.

Lemma 2 — Let

(*i*) u, v, B, D be positive integers such that $1 \le u \le D$ and $0 \le v \le B$.

(*ii*) m be an arbitrary positive integer and coefficients $A_{n,k}$ $(n, k \ge 0)$ be arbitrary constants, real or complex.

(*iii*) $Re(\alpha) > Re(\beta)$; $Re[\beta + q(d_j/\delta_j)] > 0$, $(j = 1, \ldots, u^{(i)})$; $q > 0$,

(iv) $|\arg(z)| < \frac{1}{2}\pi T$,

where $$T = \sum_{j=1}^{u} \delta_j - \sum_{j=u+1}^{D} \delta_j + \sum_{j=1}^{v} \beta_j - \sum_{j=v+1}^{B} \beta_j > 0 .$$

Then

$$\mathcal{W}^{\beta-\alpha}\left\{ y^{-\alpha} S_n^m\left[E\left(\frac{x}{y}\right)^p \right] H_{B,D}^{u,v}\left[z\left(\frac{x}{y}\right)^q \middle| \begin{matrix} (b_j, \phi_j)_{1,B} \\ (d_j, \delta_j)_{1,D} \end{matrix} \right] \right\}$$

$$= y^{-\beta} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} E^k \left(\frac{x}{y}\right)^{pk} H_{B+1,D+1}^{u,v+1}$$

$$\left[z\left(\frac{x}{y}\right)^q \middle| \begin{matrix} [1-\beta-pk; q], [(b_j):\phi_j]_{1,B} \\ [(d_j):\delta_j]_{1,D}, [1-\alpha-pk; q] \end{matrix} \right]. \qquad \text{... (4.1)}$$

PROOF : To prove Lemma 2, we first use the definition of Weyl fractional integral given in (1.8) express the Fox's H-function in a contour integral of Mellin-Barnes type and a general class of polynomials $S_n^m[E]$, then we change the order of summation and integrations (which is justified under the stated conditions), evaluate the t-integral and reinterpreting the resulting Mellin-Barnes contour integral in terms of the H-function, we easily arrive at the desired result.

Theorem 3 — *Under the sufficient conditions (i), (ii), (iii) and (iv) of Lemma 2,*

$$\int_0^\infty y^{-\beta} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} E^k \left(\frac{x}{y}\right)^{pk} H_{B+1,D+1}^{u,v+1}$$

$$\left[z\left(\frac{x}{y}\right)^q \middle| \begin{matrix} [1-\beta-pk; q], & [(b_j):\phi_j]_{1,B} \\ [(d_j):\delta_j]_{1,D}, & [1-\alpha-pk:q] \end{matrix} \right] f(y)\, dy$$

$$= \int_0^\infty y^{-\alpha} S_n^m\left[E\left(\frac{x}{y}\right)^p \right] H_{B,D}^{u,v}\left[z\left(\frac{x}{y}\right)^q \right] \mathcal{D}^{\beta-\alpha}\{f(y)\}\, dy, \qquad \text{... (4.2)}$$

provided further $f \in \int$ *and* $x > 0$.

Theorem 3 is established with the help of Lemma 2 and the equation (1.8), on proceeding on similar lines as indicated in the proof of Theorem 1.

Theorem 4 — *If* $f \in \int$, $\mathcal{D}^{\alpha-\beta}\{f(y)$ *exists* $q > 0$, $x > 0$, $|\arg(z)| < \frac{1}{2}\pi T$, $T > 0$ *(T given in Lemma 2), Re* $(\alpha) >$ *Re* $(\beta) > 0$, *and m is an arbitrary positive integer and coefficients* $A_{n,k}$ $(n, k \geq 0)$ *are arbitrary constants, real or complex, then the solution of the integral equation*

$$\int_0^\infty y^{-\alpha} S_n^m \left[F\left(\frac{x}{y}\right)^p \right] H_{B,D}^{\mu,\nu} \left[z\left(\frac{x}{y}\right)^q \,\middle|\, \begin{matrix} (b_j, \phi_j)_{1,B} \\ (d_j, \delta_j)_{1,D} \end{matrix} \right] f(y)\, dy = g(x) \quad (0 < x < \infty). \quad (4.3)$$

is given by

$$f(x) = \sum_{k=0}^{[n/m]} \frac{q}{2\pi i} x^{\alpha-1} \lim_{\rho \to \infty} \int_{\sigma - i\rho}^{\sigma + i\rho} \frac{(-n)_{mk}}{k!} A_{n,k} E^k \left[\theta\left(\frac{-pk-s}{q}\right) \right]^{-1}$$

$$\cdot x^{-s} z^{\left(\frac{pk+s}{q}\right)} \phi(s)\, ds, \quad \ldots (4.4)$$

provided further that

$$\max \{Re [(b_l - 1)/\phi]\} < \text{Re} \left(\frac{pk+s}{q}\right) < \min \{Re\, (d_j/\delta_j)\}.$$

$$(j = 1, \ldots, u) : (l = 1, \ldots, v) \quad \ldots (4.5)$$

PROOF : On replacing f by $\mathcal{D}^{\alpha-\beta}\{f\}$ in (4.2) and applying (4.1), we have

$$g(x) = \int_0^\infty y^{-\beta} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} E^k \left(\frac{x}{y}\right)^{pk} H_{B+1,D+1}^{\mu,\nu+1}$$

$$\left[z\left(\frac{x}{y}\right)^q \,\middle|\, \begin{matrix} [1-\beta-pk; q], [(b_j) : \phi_j]_{1,B} \\ [(d_j) : \delta_j]_{1,D}, [1-\alpha-pk; q] \end{matrix} \right] \mathcal{D}^{\alpha-\beta}\{f(y)\}\ dy. \quad \ldots (4.6)$$

Multiplying both the sides of (4.6) by x^{s-1} and integrating with respect to x from 0 to ∞, we have

$$\phi(s) = \int_0^\infty x^{s-1} g(x)\, dx = \int_0^\infty y^{-\beta} \mathcal{D}^{\alpha-\beta}\ \{f\ (y)\}$$

$$\left(\int_0^\infty x^{s-1} S_n^m \left[E\left(\frac{x}{y}\right)^p \right] \times H_{B+1,D+1}^{\mu,\nu+1} \left[z\left(\frac{x}{y}\right)^q \,\middle|\, \begin{matrix} [1-\beta-pk, q], [(b_j) : \phi_j]_{1,B} \\ [(d_j) : \delta_j]_{1,D}, [1-\alpha-pk, q] \end{matrix} \right] dx \right) dy, \quad \ldots (4.7)$$

where we have assumed the absolute (and uniform) convergence of the integrals involved, with a view to justifying the inversion of the order of integration.

Now evaluate the inner integral in (4.7) by a simple change of variables in the familiar results (c.f., for example, [3] and [13]), eq. (4.7) reduces to

$$\phi(s) = \int_0^\infty y^{-\beta} \mathcal{D}^{\alpha-\beta}\{f\ (y)\} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} E^k (z)^{-\left(\frac{pk+s}{q}\right)}$$

$$\times \frac{y^s \Gamma(\beta-s)}{q\,\Gamma(\alpha-s)}\,\theta\left(\frac{-pk-s}{q}\right)dy, \quad \ldots (4.8)$$

where $\phi(s)$ is given by (4.7).

$$\phi(s) = \frac{1}{q}\sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{\Gamma(\beta-s)}{\Gamma(\alpha-s)} (z)^{-\left(\frac{pk+s}{q}\right)}$$

$$\times\,\theta\left(\frac{-pk-s}{q}\right) \int_0^\infty y^{s-\beta}\, \mathcal{D}^{\alpha-\beta}\,\{f(y)\}\, dy. \quad \ldots (4.9)$$

Inverting (4.9) by applying the Mellin inversion theorem[16], we get

$$\mathcal{D}^{\alpha-\beta}\{f(y)\} = \frac{q}{2\pi i}\sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} E^k \lim_{\rho\to\infty} \int_{\sigma-i\rho}^{\sigma+i\rho} \frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)}$$

$$\times\left[\theta\left(\frac{-pk-s}{q}\right)\right]^{-1} y^{\beta-s-1}\, z^{\left(\frac{pk+s}{q}\right)} \phi(s)\, ds. \quad \ldots (4.10)$$

Operating upon both sides by $\mathcal{D}^{\beta-\alpha}$, (4.10) gives us

$$f(y) = \frac{q}{2\pi i}\sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} E^k\, \mathcal{D}^{\beta-\alpha}$$

$$\left\{ \lim_{\rho\to\infty} \int_{\sigma-i\rho}^{\sigma+i\rho} \frac{\Gamma(\alpha-s)}{\Gamma(\beta-s)} \left[\theta\left(\frac{-pk-s}{q}\right)\right]^{-1} y^{\beta-s-l}\, z^{\left(\frac{pk+s}{q}\right)} \phi(s)\, ds \right\}, \quad \ldots (4.11)$$

which finally yields

$$f(x) = \sum_{k=0}^{[n/m]} \frac{q}{2\pi i} x^{\alpha-1} \lim_{\rho\to\infty} \int_{\sigma-i\rho}^{\sigma+i\rho} \frac{(-n)_{mk}}{k!} A_{n,k} E^k$$

$$\left[\theta\left(\frac{-pk-s}{q}\right)\right]^{-1} \cdot x^{-s}\, (z)^{\left(\frac{pk+s}{q}\right)} \phi(s)\, ds \quad \ldots (4.12)$$

as the solution of the integral equation (1.3).

If we take $n \to 0$, Theorem 4 is seen to correspond to a result given by Srivastava and Raina[15] under less stringent conditions.

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