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INEQUALITIES CONCERNING POLYNOMIALS HAVING ZEROS IN CLOSED INTERIOR OF A CIRCLE

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In this paper we have obtained certain inequalities for polynomials having zeros in closed interior of a circle. Our result improves upon the known result.

Key Words : Polynomials; Derivative; Zeros; Inequalities, Extremal Problems

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z)$ be a polynomial of degree n and let $M(p, R) = \max_{|z|=R} |p(z)|$, then we have, as a simple deduction from maximum modulus principle [2, p. 158 problem III 267 & 269]

$$M(p, r) \geq r^n M(p, 1), r < 1$$

with equality only for $p(z) = \lambda z^n, |\lambda| = 1$.

For polynomials not vanishing in $|z| < 1$, Rivlin[4] obtained stronger inequality

$$M(p, r) \geq \left[\frac{1+r}{2}\right]^n M(p, 1), r < 1. \qquad \ldots (1.1)$$

Here equality holds for $p(z) = (\alpha + \beta z)^n \; |\alpha| = |\beta|$.

Aziz[1] obtained the following generalization of inequality (1.1) for polynomials not vanishing in $|z| < K, K > 0$.

Theorem A — *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < K, K > 0$ then*

$$M(p, r) \geq \left[\frac{r+K}{1+K}\right]^n M(p, 1), \text{ for } K \geq 1 \text{ and } r < 1 \text{ or } K < 1 \text{ and } r \leq K^2.$$

Here equality holds for $p(z) = (z + K)^n$.

On applying Theorem A to the polynomial $z^n p(1/z)$ Aziz[1] obtained

Theorem *A′* — *If $p(z)$ is a polynomial of degree n, having all it zeros in $|z| \leq K, K > 0$ then*

$$M(p, R) \geq \left[\frac{R+K}{1+K}\right]^n M(p, 1), \text{ for } K \leq 1 \;\&\; R > 1 \text{ or } K > 1 \text{ and } R \geq K^2.$$

Here equality holds for the polynomial $p(z) = (z+K)^n$.

Recently Jain[5] proved the following theorem, related to Theorem A'.

Theorem B : *Let p (z) be a polynomial of degree n, having all its zeros in* $|z| \leq K, K > 1$ *then for* $K < R < K^2$,

$$M(p, R) \geq R^s \left[\frac{R+K}{1+K}\right] M(p, 1), \text{ for } S < n.$$

where s is the order of a possible zero p(z) at z = 0.

In this paper, we have obtained the following refinement of Theorem B. More precisely, we prove.

Theorem C — *Let p(z) is a polynomial of degree n, having all its zeros in* $|z| \leq K, K > 1$ *then for* $K < R < K^2$,

$$M(p, R) \geq R^S \left[\frac{R+K}{1+K}\right] M(p, 1) + \frac{R^{n-1}}{K^{n-1}} \left\{\frac{R^{n-s}-1}{KR^{n-s-1}+1}\right\} m$$

where $m = \min_{|z|=K} |p(z)|$ *and s is the order of a possible zero of p(z) at z = 0.*

2. Lemmas

The following lemma is due to Govil[3].

Lemma 1 — If $p(z)$ be a polynomial of degree n, having no zeros in $|z| < K, K \geq 1$ then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+K}\left\{\max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)|\right\}.$$

The result is best possible and equality holds for $p(z) = (z+K)^n$.

Lemma 2 — If $p(z)$ be a polynomial of of degree n, having all its zeros in $|z| \geq K, K > 0$ then for $r \leq K \leq R$,

$$\frac{M(p, r)}{r^n + Kr^{n-1}} \geq \frac{M(p, R)}{R^n + KR^{n-1}} + m\left[\frac{R^n - r^n}{(r^n + Kr^{n-1})(R^n + Kr^{n-1}}\right],$$

where $m = \min_{|z|=K} |p(z)|$.

PROOF OF LEMMA 2 — Let $r \leq K \leq R$, then obviously the polynomial $T(z) = p(rz)$ has no zeros in $|z| < K/r$. As $K/r \geq 1$, we have by Lemma 1,

$$M(T', 1) \leq \frac{n}{1+K/r}\left\{M(T, 1) - \min_{|z|=K/r} |T(z)|\right\}$$

or

$$M(p', r) \le \frac{n}{K+r}\{M(p, r) - m\}. \qquad \therefore (2.1)$$

As $p'(z)$ is a polynomial of degree $(n - 1)$, we have by maximum modulus principle[2],

$$\frac{M(p', t)}{t^{n-1}} \le \frac{M(p', r)}{r^{n-1}}, t \ge r \qquad \ldots (2.2)$$

Combining (2.1) and (2.2), we get

$$M(p', t) \le \frac{nt^{n-1}}{(K+r)\, r^{n-1}}\{M(p, r) - m\},\ t \ge r. \qquad \ldots (2.3)$$

Now we have, for $0 \le \theta < 2\pi$

$$p(Re^{i\theta}) - p(re^{i\theta}) = \int_r^R p'(te^{i\theta})\, e^{i\theta}\, dt$$

or

$$|p(Re^{i\theta}) - p(re^{i\theta})| = \left| \int_r^R p'(te^{i\theta})\, e^{i\theta}\, dt \right|$$

or

$$|p(Re^{i\theta})| - |p(re^{i\theta})| \le \int_r^R |p'(te^{i\theta})|\, dt,$$

or

$$|p(Re^{i\theta}) - p(re^{i\theta})| \le \int_r^R |p'(te^{i\theta})|\, dt,$$

which implies by (2.3)

$$|p(Re^{i\theta})| - |p(re^{i\theta})| \le \int_r^R \frac{nt^{n-1}}{r^n + Kr^{n-1}}\{M(p, r) - m\}\ dt.$$

$$= \frac{(M(p, r) - m}{r^n + Kr^{n-1}}\, n \int_r^R t^{n-1}\, dt$$

$$= \frac{R^n - r^n}{r^n + Kr^{n-1}}\{M(p, r) - m\}.$$

Hence

$$|p(Re^{i\theta})| \leq \frac{R^n - r^n}{r^n + Kr^{n-1}} \{M(p, r) - m\} + |p(re^{i\theta})|$$

which implies

$$M(p, R) \leq \left(\frac{R^n + Kr^{n-1}}{r^n + Kr^{n-1}}\right) M(p, r) - \left(\frac{R^n - r^n}{r^n + Kr^{n-1}}\right) m$$

or
$$M(p, R) \leq \left(\frac{R^n + Kr^{n-1}}{r^n + Kr^{n-1}}\right)\left[M(p, r) - \left(1 - \frac{r^n - Kr^{n-1}}{R^n + Kr^{n-1}}\right) m\right]$$

or
$$\frac{\left[M(p, r) - \left(1 - \dfrac{r^n + Kr^{n-1}}{R^n + Kr^{n-1}}\right) m\right]}{r^n + Kr^{n-1}} \geq \frac{M(p, R)}{R^n + KR^{n-1}}$$

or
$$\frac{M(p, r)}{r^n + Kr^{n-1}} \geq \frac{M(p, R)}{R^n + KR^{n-1}} + \frac{\left(1 - \dfrac{r^n + Kr^{n-1}}{R^n + Kr^{n-1}}\right) m}{r^n + Kr^{n-1}}$$

$$= \frac{M(p, R)}{R^n + KR^{n-1}} + m\left(\frac{1}{r^n + Kr^{n-1}} - \frac{1}{R^n + Kr^{n-1}}\right)$$

and this completes the proof of Lemma 2.

3. Proof of the Theorem

The polynomial $q(z) = z^n \overline{(p(1/\bar{z}))}$ has all its zeros in $|z| \geq \frac{1}{K}, \frac{1}{K} < 1$ and is of degree $n - s$. On applying Lemma 2 to the polynomial $q(z)$ with $R = 1$, we have

$$\frac{M(q, r)}{r^{n-s} + (1/K)\, r^{n-s-1}} \geq \frac{M(q, 1)}{1 + (1/K)}$$

$$+ \min_{|z| = 1/K} |q(z)| \left[\frac{1 - r^{n-s}}{(r^{n-s} + (1/K)\, r^{n-s-1})(1 + (1/K)\, r^{n-s-1})}\right], \quad \frac{1}{k^2} < r < \frac{1}{k}$$

which is equivalent to

$$M(q, r) \geq \frac{r^{n-s-1}(r + 1/K)}{1 + 1/K} M(q, 1) + \frac{1 - r^{n-s}}{(1 + (1/K)\, r^{n-s-1})} \min_{|z| = 1/K} |q(z)|, \quad \frac{1}{k^2} < r < \frac{1}{k}$$

which on simplification gives

$$M\left(p, \frac{1}{r}\right) \geq \frac{1}{r^{s+1}} \left\{\frac{r + (1/K)}{1 + (1/K)}\right\} \max_{|z| = 1} |p(z)|$$

$$+\ \frac{1-r^{n-s}}{1+(1/K)\,r^{n-s-1}}\left\{\frac{1}{(rk)^n}\right\}\min_{|z|=K}|p(z)|$$

Replacing r by $1/R$ in the above inequality, we get

$$M(p,R)\geq\frac{R^{s+1}\left(\frac{1}{R}+\frac{1}{K}\right)}{1+1/K}M(p,1)+(R/K)^n\frac{1-\frac{1}{R^{n-s}}}{1+\frac{1}{KR^{n-s-1}}}\min_{|z|=K}|p(z)|,\ K>1$$

and $\qquad K<R<K^2$

which is equivalent to

$$M(p,R)\geq\frac{R^s\,(R+K)}{1+K}M(p,1)+(R/K)^{n-1}\left\{\frac{R^{n-s}-1}{KR^{n-s-1}+1}\right\}m,\ K>1\text{ and }K<R<K^2$$

which proves the desired result.

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