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PSEUDO-DIFFERENTIAL OPERATOR ON GEL'FAND AND SHILOV SPACES

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Pseudo-differential operator on Gel'fand and Shilov space of type W is defined and its properties are studied. L^p-boundedness of pseudo-differential operator is investigated and some results associated with some certain type of Sobolev spaces are studied.

Key Words : L^p-spaces; Convex Function; Bessel Potential; Pseudo-differential Operator; Fourier Transformation

1. INTRODUCTION

The spaces of type $W_M, W_{,M,a,} W^{\Omega}, W^{\Omega,b}$ were investigated by Gel'fand and Shilov[1]. It has shown that the Fourier transformation

$F : W_{M,a} \to W^{\Omega, 1/a}$, $F : W^{\Omega, b} \to Q_{M, 1/b}$ are linear and continuous. Therefore, the generalized Fourier transformation $F : (W^{\Omega, 1/a}) \to (W_{M,a})$ is linear and continuous, where M, Ω are convex functions and a, b are positive constants.

Zaidman[5] and Wong[4] defined the pseudo-differential operator by using Schwartz theory of Fourier transformation and studied some properties related to pseudo-differential operators.

Now, we recall the definitions of Gel'fand and Shilov spaces, i.e., $W_M, W_{M,a}, W^{\Omega}, W^{\Omega,b}$. Let $\mu(\xi)$ and $\omega(\eta)$ be continuous increasing functions on $(0, \infty]$ such that $\mu(0) = 0, \mu(\infty) = \infty$ and $\omega(0)$, $\omega(\infty) = \infty$. For $x > 0$, $y > 0$ define convex function $M(x)$ and $\Omega(y)$ by

$$M(x) = \int_0^x \mu(\xi)\, d\xi, M(-x) = M(x)$$

and

$$\Omega(y) = \int_0^y \omega(\eta)\, d\eta\, \Omega(-y) = \Omega(y).$$

If the functions $\mu(\xi)$ and $\omega(\eta)$ are mutually inverse, i.e. $\mu[\omega(\eta)] = \eta$ and $\omega[\mu(\xi)] = \xi$, then the corresponding functions $M(x)$ and Ω (y) will be said to be dual in sense of Young.

Therefore, the Young inequality is

$$xy \le M(x) + \Omega(y) \text{ for } x \ge 0, y \ge 0. \qquad \ldots (1.1)$$

Then,

$$M\ (0) = 0,\ M(\infty) = \infty \text{ and } \Omega(0) = 0, \Omega(\infty) = \infty,$$

$$M(x_1) + M(x_2) \le M(x_1 + x_2) \qquad \ldots (1.2)$$

and

$$\Omega(y_1) + \Omega(y_2) \le \Omega(y_1 + y_2). \qquad \ldots (1.3)$$

Now, the space W_M is defined to the set of all infinitely differentiable functions ϕ, which satisfy the inequality

$$|\phi^{(k)}(x)| \le C_k \exp[-M(ax)] \qquad \ldots (1.4)$$

where the constants C_k and depend upon the function ϕ. The space W_M can be regarded as a union of normed linear space $W_{M,a}$ of all those functions belonging to the space W_M which satisfy the inequalities

$$|\phi^{(k)}(x)| \le C_{k,\delta} \exp[-M\ \{(a-\delta)x\}] \text{ for } \delta > 0. \qquad \ldots (1.5)$$

Also, the space W^{Ω} is defined to the set of all entire analytic functions

$\phi(z)$ ($\therefore z = x + iy$) which satisfy the inequalities

$$|z^k \phi(z)| \le C_k \exp[\Omega\ \{(by)\}] \qquad \ldots (1.6)$$

where the constants b and C_k depend upon the function. Clearly W^{Ω} can be regarded as the union of countably normed space $W^{\Omega, b}$ which satisfy the inequality

$$|z^k \phi(z)| \le C_{k,\rho} \exp[\{(b + \rho)y\}] \text{ for } \rho > 0. \qquad \ldots (1.7)$$

In this paper the pseudo-differential operator A_σ associated with symbol $\sigma(x, \xi)$ is defined and under certain condition A_σ acts as a continuous linear mapping from $W_{M, 1/a}$ into $W_{M, 1/(a+a_0)}$. The L^p-boundedness and properties related to Sobolev spaces of p.d.o are studied for $1 \le p < \infty$, $s \in \mathbb{R}$.

2. Pseudo-Differential Operators

In this section we study the p.d.o A_σ associated with symbol $\sigma(x, \xi)$ defined as follows :

The function $\sigma(x, \xi) \in (\mathbb{R}, x\mathbb{C})$ which is assumed to be extendible as an entire analytic function with respect to $\xi = (u + it)$ is said to be in the class of U^m iff for any non-negative integers

α and $b\eta$, there exists a positive constant $C_{\alpha,\beta}$ depending upon α and β only, such that

$$|D_x^{\alpha} D_{\xi}^{\beta} \sigma(x, \xi)| \leq C_{\alpha,\beta}(1+|u|)^{m-|\beta|} \exp[(\Omega(a_0 t)] \quad (\because \ \xi = u + it). \qquad \dots (2.1)$$

An example of the function in U^m is

$$\sigma(x, \xi) = e^{-x}(1+\xi)^m \text{ as } \xi = (u + it).$$

Now for $\phi \in W_{M, 1/a}$, p.d.o. A_{σ} by

$$(A_{\sigma}\phi)(x) = \int_{-\infty}^{+\infty} e^{ix\xi} \sigma(x, \xi) \hat{\phi}(\xi)\, du, \phi \,\varepsilon\, W_{M, 1/a}. \qquad \dots (2.2)$$

***Theorem* 2.1** — *Let $\sigma(x, \xi)$ be a symbol belonging to U^m. Then A_{σ} maps from $W_{M, 1/a}$ into $W_{M, 1/(a+a_0)}$.*

PROOF : Let $\phi \in W_{M, 1/a}$. Then, for non-negative integers α and β, we need only to prove that

$$\sup_{x \in R} |\exp[M[(1/(a+a_0)) - \delta) x]] D_x^{\beta}(A_{\sigma}\phi)(x)| < \infty.$$

So that, the pseudo differential operator is defined by

$$(A_{\sigma}\phi)(x) = \int_{-\infty}^{+\infty} e^{ixu} \sigma(x, u) \hat{\phi}(u)\, du, \ \phi \in W_{M, 1/a}.$$

Now, as in [1, p. 21] we can write

$$(A_{\sigma}\phi)(x) = \int_{-\infty}^{+\infty} e^{ix\xi} \ \sigma(x, \xi) \hat{\phi}(\xi)\, du, \xi = u + it.$$

Therefore,

$$D_x^{\beta}(A_{\sigma}\phi)(x) = \int_{-\infty}^{+\infty} D_x^{\beta}(e^{ix\xi}\sigma(x, \xi)) \hat{\phi}(\xi)\, du$$

$$= \int_{-\infty}^{+\infty} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \xi^{\gamma} e^{ix\xi} (D_x^{\beta-\gamma}\sigma)(x, \xi) \hat{\phi}(\xi)\, du$$

$$= \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \int_{-\infty}^{+\infty} e^{ix\xi} (D_x^{\beta-\gamma}\sigma)(x, \xi) \xi^{\gamma} \hat{\phi}(\xi)\, du.$$

So that,

$$(ix)^{\alpha} D_x^{\beta} (A_{\sigma}\phi)(x) = \sum_{\gamma \le \beta} \binom{\beta}{\gamma} \int_{-\infty}^{+\infty} D_{\xi}^{\alpha} (e^{ix\xi} (D_x^{\beta-\gamma} \sigma)(x, \xi)\, \xi^{\gamma} \hat{\phi}(\xi)\, du$$

$$= \sum_{\gamma \le \beta} \sum_{\delta \le \alpha} (-1)^{|\alpha|} \binom{\beta}{\gamma} \binom{\alpha}{\delta} \int_{-\infty}^{+\infty} e^{ix\xi} (D_{\xi}^{\alpha-\delta} D_x^{\beta-\gamma} \sigma)(x, \xi)\, D_{\xi}^{\delta} (\xi^{\gamma} \hat{\phi}(\xi))\, du$$

$$= \sum_{\gamma \le \beta} \sum_{\delta \le \alpha} (-1)^{|\alpha|} \binom{\beta}{\gamma} \binom{\alpha}{\delta} e^{-xt} \int_{-\infty}^{+\infty} e^{ixu} (D_{\xi}^{\alpha-\delta} D_x^{\beta-\gamma} \sigma)(x, \xi)\, D_{\xi}^{\delta} (\xi^{\gamma} \hat{\phi}(\xi))\, du.$$

Therefore,

$$|x^{\alpha} D_x^{\beta} (A_{\sigma}\phi)(x)| \le \sum_{\gamma \le \beta} \sum_{\delta \le \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} e^{-|x||t|} \int_{-\infty}^{+\infty}$$

$$|(D_{\xi}^{\alpha-\delta} D_x^{\beta-\gamma} \sigma)(x, \xi)|\, |D_{\xi}^{\delta})\xi^{\gamma} \hat{\phi}(\xi))|\, du$$

$$\le \sum_{\gamma \le \beta} \sum_{\delta \le \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} e^{-|x||t|} \int_{-\infty}^{+\infty} |(D_{\xi}^{\alpha-\delta} D_x^{\beta-\gamma} \sigma)(x, \xi)|$$

$$(1 + |u|^k)\, |D_{\xi}^{\delta} (\xi^{\gamma} \hat{\phi}(\xi))|\, (1 + |u|^k)^{-1}\, du.$$

Now, using inequalities $(1 + |u|^k) \le (1 + |\xi|^k)$ for $\xi = u + it$ and

$$(1 + |u|^k)^{-1} \le 2^{k-1} (1 + |u|)^{-k},$$

we obtain

$$|x^{\alpha} D_x^{\beta} (A_{\sigma}\phi)(x)| \le \sum_{\gamma \le \beta} \sum_{\delta \le \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} e^{-|x||t|} \int_{-\infty}^{+\infty}$$

$$C_{\alpha-\delta, \beta-\gamma} (1 + |u|)^{m-|\beta|} \exp [\Omega [(a_0 t)]]$$

$$(1 + |\xi|^k) \left(\left| D_{\xi}^{\delta} (\xi^{\gamma} \hat{\phi}(\xi)) \right| \right) 2^{k-1} (1 + |u|)^{k-1}\, du$$

$$\le \sum_{\gamma \le \beta} \sum_{\delta \le \alpha} \binom{\beta}{\gamma} \binom{\alpha}{\delta} e^{-|x||t|} \int_{-\infty}^{+\infty} C_{\alpha-\delta, \beta-\gamma} 1 + |u|)^{m-|\beta|} \exp [\Omega [(a_0 t)]]$$

$$\left(\left| D_{\xi}^{\delta} (\xi^{\gamma} \hat{\phi}(\xi)) \right| + \left| \xi^{\gamma} D_{\xi}^{\delta} (\xi^{r} \hat{\phi}(\xi) \right| \right) 2^{k-1} (1 + |u|)^{-k}\, du.$$

From (1.7) and [1, pp. 13-14] we get

$$|x^{\alpha} D_x^{\beta}(A_{\sigma}\phi)(x)| \leq \sum_{\gamma\leq\beta}\sum_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}2^{k-1}C_{\alpha-\delta,\beta-\gamma}(C_{\gamma,\rho,0}+C_{\gamma,\rho,k})$$

$$\exp[-|x||t|+\Omega[(a_0 t)]+\Omega[(a+\rho)t]]$$

$$\int_{-\infty}^{+\infty}(1+|u|^{m-k-|\beta|}du$$

$$\leq \sum_{\gamma\leq\beta}\sum_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}2^{k-1}C_{\alpha-\delta,\beta-\gamma}(C_{\gamma,\rho,0}+C_{\gamma,\rho,k})$$

$$\exp[-|x||t|+\Omega[(a+a_0+\rho)t]]$$

$$\int_{-\infty}^{+\infty}(1||u|^{m-k-|\beta|}du,$$

the last integral can be made convergent by assuming k sufficiently large positive integer. Now, using the Young inequality (1.1), and the arguments (1.1), and the arguments [1, pp. 21-22] we find that

$$|x^{\alpha} D_x^{\beta}(A_{\sigma}\phi)(x)| \leq E_{\alpha,\beta,\delta,\rho,k}\exp[-M[(1/(a+a_0)-\delta)x]].$$

Hence,

$$|\exp[M[1/(a+a_0)-\delta)x]]D_x^{\beta}(A_{\sigma}\phi)(x)| \leq E_{\alpha,\beta,\delta,\rho,k}(1+|x|^{\alpha})^{-1}.$$

Thus,

$$\sup_{x\in R}|\exp[M[1/(a+a_0)-\delta)x]]D_x^{\beta}(A_{\sigma}\phi)(x)|<\infty.$$

If $a_0=0$, then A_{σ} maps from $W_{M,1/a}$ into itself.

***Theorem* 2.2** — *Let $\sigma(x,\xi)$ be a symbol in U^0, then for $1<p<\infty$, $A_{\sigma}: L^p(I\!R)\to L^p(I\!R)$ is a bounded linear operator.*

PROOF : Let $l(\xi)\in C^k(\mathbb{C}-\{0\})$, $k\geq 1$ be such that there is a positive constant B for which

$$|D^{\alpha}l(\xi)| \leq B|u|^{-\alpha}\exp[\Omega[a_0 t)]] \quad \text{... (2.3)}$$

for all $|\alpha|\leq k$ and $\xi=u+it$. If we put $t=0$ then we obtain

$$|D^{\alpha}l(u)|\leq B|u|^{-\alpha}, u\neq 0. \quad \text{... (2.4)}$$

Since $\phi\in W_{M,a}$ and $W_{M,a}$ is a subspace of Schwartz space S therefore from [4, Theorem 9.10] the following inequality will be satisfied

$$\| (A_l \phi)(x) \| \leq CB \| \phi \|_p \ \forall \ \phi \in W_{M,a} \subset S$$

where $1 < p < \infty$ and

$$(A_l \phi)(x) = \int_{-\infty}^{+\infty} e^{ixu} \, l(u) \, \hat{\phi}(u) \, du.$$

The proofs of Lemma 9.11 and Lemma 9.12 of Wong[4] will be same for our case. So, with the help of [4, pp. 78-88] we can say that the pseudo-differential operator A_σ which is defined by (2.2) is a bounded linear operator from $L^p(I\!R)$ to $L^p(I\!R)$ when the symbol $\sigma(x, \xi) \in U^0$.

Remark : The condition that $\sigma(x, \xi)$ is extendible as entire function is not necessary for the conclusion of the theorem.

3.1 Bessel Potential

Let $\sigma(\xi)$ the entire analytic function, be a multiplier in W^Ω. Then for $f \in (W^\Omega)$, we can define the product $f\sigma \in W^\Omega)'$ in the following way

$$\langle f\sigma, \phi \rangle = \langle f, \sigma\phi \rangle, \ \forall \ \phi \in W^\Omega.$$

Assume that $\sigma(\xi) = (1 + |\xi|^2)^{-1/2}$ as $\xi = u + it$. Then $|\sigma(\xi)| \leq C(1 + |u|)^s \exp[\Omega[a_0 t]]$.

Now, using the technique of [1, pp. 16-17] we can easily show that $\sigma(\xi)$ is a multiplier in the space W^Ω. So that, from [1, pp. 21-22] the Fourier transformation $F(f\sigma) \in (W_M)'$.

For $s \in I\!R$, we denote by V^s the pseudo-differential operator of which the symbol $\sigma(\xi)$ is given by

$$\sigma(\xi) = (1 + |\xi|^2)^{-s/2} \text{ as } \xi = u + it. \qquad \text{... (3.1)}$$

From (2.1) it follows that the symbol $\sigma(\xi) = (1 + |\xi|^2)^{-s/2} \in V^{-s}$ if s is zero or an even negative integer. Then the operator V^s will be called Bessel potential of order s. Thus the Bessel potential V^s is defined by

$$(V^s f)(x) = F^{-1}[\sigma(\xi) F(f)](x). \qquad \text{... (3.2)}$$

Lemma 3.1 — The Bessel potential V^s is a continuous linear mapping of $W_{M,1/a}$ into $W_{M,(1/a+a_0)}$ for $s \leq 0$ is an even integer.

PROOF : The proof of the theorem will be obtained from [1, *p*. 16] and Theorem 2.1.

Theorem 3.2 — *Let* $f \in (W_{M,1/a})'$. *Then*

$$\text{(i)} \ V^s V^t f = V^{s+t} f \qquad \text{... (3.3)}$$

and

$$\text{(ii)} \ V^0 f = f. \qquad \text{... (3.4)}$$

PROOF : See, [4, p. 90].

Definition 3.1 — For $s \in \mathbb{R}$ and $1 \le p < \infty$, the space $G^{s,p}(\mathbb{R})$ is defined to be the set of all $\phi \in W_{M,1/a}$ for which $V^s \phi$ is a function of $L^p(\mathbb{R})$ defined below. The norm in $G^{s,p}$ is defined by

$$\| \phi \|_{s,p} = \| \phi \|_{G^{s,p}} = \| V^{-s} \phi \|_p = \left(\int_{-\infty}^{+\infty} | V^{-s} \phi |^p \, dy \right)^{1/p}. \quad \ldots (3.5)$$

***Theorem* 3.3** — *$G^{s,p}$ is a Banach space with respect to the norm $\| \, \|_{s,p}$.*

PROOF : The proof is obvious from [4, p. 91].

***Theorem* 3.4** — *V^t is an isometry of $G^{s,p}$ onto $G^{s+t,p}$. More precisely,*

$$\| V^t f \|_{s+t,p} = \| f \|_{s+t,p} = \| f \|_{s,p}, f \in G^{s,p}.$$

PROOF : The proof of the above Theorem can be found in [4, p. 91].

***Theorem* 3.5** — *Let $1 < p < \infty$ and $s \le t$. Then $G^{t,p} \subseteq G^{s,p}$, and $\| f \|_{s,p} \le \| f \|_{t,p}, f \in G^{s,p}$.*

PROOF : For a proof see [4, p. 91].

***Theorem* 3.6** — *Let $s \ge 0$ and $1 < p < \infty$. Then*

$$\| V^s f \|_p \le C_s \| f \|_p, f \in L^p(\mathbb{R}).$$

PROOF : See [4, pp. 94-95].

***Theorem* 3.7** — *Let $\sigma(x, \xi)$ be a symbol in U^m. Then $A_\sigma : G^{m,p} \to G^{0,p}$ is a bounded linear operator for $s \in \mathbb{R}$ and $1 < p < \infty$.*

PROOF : Consider the following operators :

$$V^{-s} : G^{s,p} \to G^{0,p}$$

$$A_\sigma V^m : G^{0,p} \to G^{0,p}$$

$$V^{s-m} : G^{0,p} \to G^{s-m,p}$$

The first and the third operators are bounded by Theorem 3.4 and the second operator is bounded by Theorem 2.2. Hence the product $V^{s-m} A_\sigma V^{m-s}$ is a bounded linear operator from $G^{s,p}$ into $G^{s-m,p}$. Moreover, by Theorem 3.4, the operators V^{m-s} and V^{s-m} are isometric and onto. Hence $A_\sigma : G^{s,p} \to G^{0,p}$ is a bounded linear operator.

A generalization of Theorem 3.7 is given by

***Theorem* 3.8** — *Let σ be a symbol in U^m. Then $A_\sigma : G^{s,p} \to G^{s-m,p}$ is a bounded linear operator for $1 < p < \infty$ and $s \in \mathbb{R}$.*

PROOF : Since $V^{m-s} A_\sigma$ is a pseudo-differential operator by Theorem 3.7, therefore, we have

$$\| A_\sigma f \|_{s-m,p} = \| V^{m-s} A_\sigma f \|_p \leq B \| f \|_{s,p}.$$

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REFERENCES

1. I. M. Gel'fand and G. E. Shilov, *Generalized Functions, Theory of Differential Equations*, Vol. 3, Academic Press, New York, 196.
2. R. S. Pathak, *Integral Transforms of Generalized Functions and their Application*", Gordon and Breach Science Publications, 1997.
3. R. S. Pathak and S. K. Upadhyay, *Proc. Amer. math. Soc.*, **121** (3), pp. 733-39, 1994.
4. M. W. Wong, *Introduction to Pseudo-Differential Operators*, World Scientific, Singapore, 1991.
5. S. Zaidman, *Distributions and Pseudo-Differential Operators*, Longman, Essex, England, 1991.