Indian J. pure appl. Math., **32**(6) : 779-787, June 2001

A MEIR-KEELER TYPE FIXED POINT THEOREM

R. P. PANT*, P. C. JOSHI**

**Department of Mathematics, Statistics & Computer Science, G. B. Pant University, Pant Nagar 263 145, India*
***Department of Mathematics, Kumaon University, Nainital 263 002, India*

AND

VIJAY GUPTA[†]

[†]Department of Mathematics, Netaji Subhash Institute of Technology, New Delhi, India

(*Received* 25 *January* 2000; *after Revision* 6 *July* 2000; *Accepted* 8 *January* 2001)

The aim of the present paper is to obtain a fixed point theorem for a sequence of mappings satisfying a Meir-Keeler type (ε, δ) contractive condition. As compared to the analogous results, the present theorem has been obtained without assuming additional conditions on the function δ and, threfore, generalizes a number of Meir-Keeler type theorems. As a corollary of the main theorem we proved an answer to a question, which remained an open problem for more than a decade, on the existence of a contractive definition which generates a fixed point but does not force the map to be continuous at the fixed point.

Key Words and Phrases : Common Fixed Points; Compatible Mappings; Contractive Conditions; Meir-Keeler Type Condition

INTRODUCTION

During the last decade, a large body of literature has grown on common fixed points of compatible maps satisfying various contractive conditions. The most general results of this type deal with common fixed points of four mappings, say A, B, S, T of a metric space (X, d), and use either a Meir-Keeler type (ε, δ) contractive condition of the form

(1) given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \le \max\{d(Sx, Ty), d\,(Ax, Sx), (By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\} < \varepsilon + \delta$$

$$\Rightarrow d(Ax, By) < \varepsilon$$

or a ϕ-contractive condition of the form

(2) $d(Ax, By) \le \phi$ (max $\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}$),

where $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$ or some generalized version of these conditions which is applicable to sequences of mappings. The contractive condition (2) does not ensure the existence of a fixed point unless some additional condition is assumed on the function ϕ. The following conditions on the function ϕ, which were introduced by various authors, are known to ensure a common fixed point under the contractive condition (2) :

(I) $\phi(t)$ is nondecreasing and $t/(t-\phi(t))$ is nonincreasing (Carbone *et al.*[2]),

(II) $\phi(t)$ is nondecreasing and $\lim_n \phi^n(t) = 0$ for each $t > 0$ {Jachymski[3] (Theorem 3.8), Matkowski[7] (Theorem 1.2)},

(III) ϕ is upper semicontinuous and $\phi(t) < t$ for each $t > 0$ {Jachymski[3] (Theorems 3.3, 5.1)}, Pant[12] {(Theorem 3), Boyd and Wong[1]}, or equivalently

(IV) ϕ is nondecreasing and continuous from the right (Park and Rhoades[14]).

Jachymski[3] (Lemma 3.1) has shown that if ϕ satisfies any of the conditions (II), (III) or (IV) then the contractive condition (2) implies the contractive condition (1). Similarly, if $\phi(t)$ satisfies condition (I) then given $\varepsilon > 0$ there exists $k > 1$ such that $t/(t - \phi(t)) \le k$ for each $t \ge \varepsilon$, that is, $\phi(t) \le ((k-1)/k)\, t$ for each $t \ge \varepsilon$. In view of this and the fact that $\phi(t)$ is nondecreasing it follows that the contractive condition (2) implies the (ε, δ) contractive condition (1) with $\delta(\varepsilon) = \varepsilon/k$. We thus see that if any of the conditions (I), (II), (III) or (IV) is assumed on ϕ in (2) then the ϕ-contractive condition (2) implies the Meir-Keeler type (ε, δ) condition (1) and both are satisfied simultaneous. Similarly, as shown in Example 3 below (see also Rao and Rao[15]), a Meir-Keeler type (ε, δ) contractive condition (1) does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, common fixed point theorems for four mappings using Meir-Keeler type conditions assume δ to be nondecreasing or lower semicontinuous. For example, Jungck[5] and Jungck *et al.*[6] assume δ to be lower semicontinuous and Pant[9&10] assumed δ to be nondecreasing. Jachymski[3] (Proposition 4.2) has shown that the (ε, δ) condition (1) with a nondecreasing δ implies the ϕ-contractive condition (2). It is also true that the (ε, δ) condition (1) with a lower semicontinuous δ implies the ϕ-contractive condition (2). To see this, suppose that (1) holds and δ is lower semicontinuous. Then given $\varepsilon_0 > 0$, we have

$$\lim_{\varepsilon \to \varepsilon_0} \inf \delta(\varepsilon) = \delta(\varepsilon_0).$$

This means that given $\varepsilon_0 > 0$ there exists an interval $(\varepsilon_1, \varepsilon_0)$ and a number $k > 0$ such that $\delta(\varepsilon) \ge k$ for each ε in $(\varepsilon_1, \varepsilon_0)$. That is, given $\varepsilon_0 > 0$ there exists an ε_2 such that $\varepsilon_2 < \varepsilon_0 < \varepsilon_2 + k \le \varepsilon_2 + \delta(\varepsilon_2)$. This means that we can define $\phi : R_+ \to R_+$ such that $\phi(\varepsilon_0) = \varepsilon_2 < \varepsilon_0$ and, consequently, (2) is satisfied. This has been shown by Jachymski[4] also, but in a slightly different form. Thus, if additional conditions are assumed on δ, then the (ε, δ) condition (1) implies the ϕ-contractive condition (2) and both hold simultaneously.

From the above discussion it follows that new common fixed point theorems can be obtained if, instead of assuming one of the contractive conditions (1) or (2) with additional hypothesis on δ or ϕ, we assume both (1) and (2) but without additional hypothesis on δ and ϕ. We do this in the present paper.

Two self-mappings A and S of a metric space (X, d) are called compatible (see Jungck[5]) if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X.

RESULTS

Let $\{A_i : i = 1, 2, 3, ...\}$, S and T be selfmappings of a complete metric space (X, d). In the sequel we shall denote

$$M_{1i}(x, y) = \max\ \{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), [d(A_1x, Ty) + d(A_iy, Sx)]/2\}.$$

Theorem — *Let $\{A_i,\ i = 1, 2, 3, ...\}$ S and T be self-mappings of a complete metric space (X, d) such that*

(i) $A_1 X \subset TX, A_i X \subset SX, i > 1$

(ii) given $\varepsilon > 0$ *there exits* $\delta > 0$ *such that*

$$\varepsilon \le M_{12}(x, y) < \varepsilon + \delta \Rightarrow d(A_1x, A_2y) < \varepsilon$$

(iii) $d(A_1x, A_iy) \le \phi_i(M_{1i}(x, y)), i > 2,$

where $\phi_i : R_+ \to R_+$ is such that $\phi_i(t) < t$ for each $t > 0$. Let A_1 and S be compatible and T be compatible with A_k for each $k > 1$. If one of the mappings is continuous then all the A_i, S and T have a unique common fixed point.

PROOF : Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = A_1x_{2n} = Tx_{2n+1}, y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of *(i)*. We claim that $\{y_n\}$ is a Cauchy sequence. Two cases arise. Either $y_n = y_{n+1}$ for some n or $y_n \neq y_{n+1}$ for each n. If $y_n = y_{n+1}$ for some n then, as shown by Rhoades *et al.*[16 p. 485], $y_n = y_{n+k}$ for each $k \ge 1$. For instance suppose that $y_{2m} = y_{2m+1}$. Then $y_{2m+1} = y_{2m+2}$. Otherwise, using *(ii)* we get

$$d(y_{2m+1}, y_{2m+2}) < M_{12}(x_{2m+2}, x_{2m+1}) = d(y_{2m+1}, y_{2m+2}),$$

a contradiction. Hence, $y_{2m+1} = y_{2m+2}$. Similarly, $y_{2m+1} = y_{2m+2}$ implies that $y_{2m+2} = y_{2m+3}$. Proceeding in this manner it follows that $y_{2m} = y_{2m+k}$ for each $k \ge 1$ and $\{y_n\}$ is a Cauchy sequence. Let us, therefore, consider the case when $y_n \neq y_{n+1}$ for each n. In this case, using (ii), we obtain

$$d(y_{2n}, y_{2n+1}) < d(y_{2n-1}, y_{2n}),\ d(y_{2n-1}, y_{2n}) < d(y_{2n-2}, y_{2n-1}).$$

Thus $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and, therefore, tends to a limit $r \ge 0$. If possible, suppose $r > 0$. Then, given $\delta > 0$ there exists a positive integer N such that for each $n \ge N$ we have

$$r < d(y_{2n}, y_{2n+1}) = M_{12}(x_{2n+2}, x_{2n+1}) < r + \delta. \qquad ...(3)$$

Selecting δ in (3) in accordance with *(ii)*, for each $n \ge N$ we get $d(y_{2n+2}, y_{2n+1}) = d(A_1x_{2n+2}, A_2x_{2n+1}) < r$. This, in turn, gives $d(y_{2n+3}, y_{2n+2}) < d(y_{2n+1}, y_{2n+2}) < r$, contradicting (3). Hence, $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$.

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists an $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\varepsilon$. Select δ in *(ii)* so that $0 < \delta \le \varepsilon$. Since $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$, there exists an integer N such that $d(y_n, y_{n+1}) < \delta/6$ whenever $n \ge N$.

Let $n_i \ge N$. Then, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \ge \varepsilon + (\delta/3)$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \le d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}})$$

$$< \varepsilon + (\delta/3) + (\delta/6) < 2\varepsilon,$$

a contradiction. Without loss of generality, we can assume n_i to be odd. Let m_i be the smallest even integer such that $d(y_{n_i}, y_{m_i}) \ge \varepsilon + (\delta/3)$. Then $d(y_{n_i}, y_{m_i} - 2) < \varepsilon + (\delta/3)$ and

$$\varepsilon + (\delta/3) \le d(y_{n_i}, y_{m_i}) \le d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) +$$

$$d(y_{m_i-1}, y_{m_i}) < \varepsilon + (\delta/3) + (\delta/6) + (\delta/6) = \varepsilon + (2\,\delta/3). \qquad \ldots (4)$$

Also, $$d(y_{n_i}, y_{m_i}) \le M_{12}(x_{n_i+1}, x_{m_i+1}) < \varepsilon + (2\,\delta/3) + (\delta/6) < \varepsilon + \delta,$$

that is, $\varepsilon + (\delta/3) \le M_{12}(X_{n_i+1}, X_{m_i+1}) < \varepsilon + \delta$. In view of (ii), this yields $d(y_{n_i+1}, y_{m_i+1}) < \varepsilon$. But then

$$d(y_{n_i}, y_{m_i}) \le d(y_{n_i}, y_{n_i+1}) + d(y_{n_i+1}, y_{m_i+1}) + d(y_{m_i+1}, y_{m_i})$$

$$< (\delta/6) + \varepsilon + (\delta/6) = \varepsilon + (\delta/3),$$

which contradicts (4). Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $y_n \to z$. Also

$$y_{2n} = A_1 x_{2n} = Tx_{2n+1} \to z,\ y_{2n+1} = A_2 x_{2n+1} = Sx_{2n+2} \to z.$$

We now show that $A_i x_{2n+1} \to z$ for each $i > 1$. If $\lim_{n \to \infty} A_i x_{2n+1} \ne z$ for some $i > 1$, then either $\lim_n A_i x_{2n+1} = w \ne z$ or $\lim_n A_i x_{2n+1}$ does not exist. In the latter case either the sequence $\{A_i x_{2n+1}\}$ is unbounded or has at least two limit points. However, in each of these cases there exists a subsequence $\{A_i x_{2m+1}\}$ of $\{A_i x_{2n+1}\}$ and a number $r > 0$ such that

$$d(A_1 x_{2m}, A_i x_{2m+1}) \ge r,\ d(A_i x_{2m+1}, z) \ge r$$

while, by virtue of (5), $d(A_1 x_{2m}, Sx_{2m}) < r/6$, $d(A_1 x_{2m+1}, Tx_{2m+1}) < r/6$ and $d(Sx_{2m}, Tx_{2m+1}) < r/6$, for all m sufficiently large. Using (ii) and (iii), for all large m, we get

$$d(A_1 x_{2m}, A_i x_{2m+1}) < \max\ \{d(Sx_{2m}, Tx_{2m+1}), d(A_1 x_{2m}, Sx_{2m}),$$
$$d(A_i x_{2m+1}, Tx_{2m+1}),$$
$$[d(A_1 x_{2m}, Tx_{2m+1}) + d(A_i x_{2m+1}, Sx_{2m})]/2\}$$
$$\leq \max\ \{r/6, r/6, d(A_i x_{2m+1}, Tx_{2m+1}),$$
$$[0 + d(A_i x_{2m+1}, Sx_{2m})]/2\}$$
$$\leq \max\ \{d(A_i x_{2m+1}, A_1 x_{2m}), [d(A_i x_{2m+1}, A_1 x_{2m}) + d(A_1 x_{2m}, Sx_{2m})]/2\}$$
$$\leq \max\ \{d(A_i x_{2m+1}, A_1 x_{2m}), [d(A_i x_{2m+1}, A_1 x_{2m}) + r/6]/2\}$$
$$= d(A_1 x_{2m}, A_i x_{2m+1}),$$

a contradiction. Hence

$$\lim_n A_i x_{2n+1} = z, i > 1. \qquad \text{... (6)}$$

Suppose that T is continuous. Then for any $k > 1$ we get $TTx_{2n+1} \to Tz$, $TA_k X_{2n+1} \to Tz$ and the compatibility of T and A_k implies that $A_k Tx_{2n+1} \to Tz$. Since $A_k X \subset SX$, corresponding to each value of n there exists z_{2n+1} such that $A_k Tx_{2n+1} = Sz_{2n+1}$. Thus $A_k Tx_{2n+1} = Sz_{2n+1} \to Tz$ and $TT_{2n+1}^{x} \to Tz$. Using an argument similar to that used in proving (6), it follows that $A_1 z_{2n+1} \to Tz$. We show that $A_k z = Tz$ for each $k > 2$. If $A_k z \neq Tz$ for some $k > 2$, then by virtue of (iii), for sufficiently large values of n we get

$$d(A_1 z_{2n+1}, A_k z) \leq \phi_k(M_{1k}(z_{2n+1}, z)) = \phi_k(d(A_k z, Tz)).$$

On letting $n \to \infty$ this yields $d(Tz, A_k z) \leq \phi_k(d(A_k z, Tz)) < d(A_k z, Tz)$, a contradiction. Hence, $A_k z = Tz$ for each $k > 2$. For any fixed $k > 2$, since $A_k X \subset SX$, there exists a w in X such that

$$A_k z = Sw$$

If $A_1 w \neq Sw$, using (iii) we get

$$d(A_1 w, A_k z) < M_{1k}(w, z) = d(A_1 w, A_k z),$$

a contradiction. Thus $A_1 w = Sw = A_k z = Tz, k > 2$. Further, if $A_2 z \neq Sw$, using (ii) we get

$$d(A_1 w, A_2 z) < M_{12}(w, z) = d(A_2 z, A_1 w)$$

a contradiction. Hence for each $i > 1$ we get

$$A_i z = Tz = Sw = A_1 w. \qquad \text{... (7)}$$

Now, the compatibility of A_1 and S implies that $A_1 Sw = SA_1w$ and $A_1 A_1 w = A_1 Sw = SA_1w = SSw$. Similarly, the compatibility of A_i and T implies that $A_i Tw = TA_i w$ and $A_i A_i w$ $A_i Tw = TA_i w = TTw$ for each $i > 1$. If $A_1 w \neq A_1 A_1 w$, by (*ii*) or (*iii*) we get

$$d(A_1 A_1 w, A_1 w) = d(A_1 A_1 w, A_i z) < M_{1i}(A_1 w, z) = d(A_1 A_1 w, A_i z)$$

a contradiction. Hence $A_1 w = A_1 A_1 w = SA_1 w$ and $A_1 w$ is thus a common fixed point of A_1 and S. Similarly, $A_i z\,(= A_1 w)$ is a common fixed point of A_i and T for each $i > 1$. Uniqueness of the common fixed point follows easily. The proof is similar when S is assumed continuous.

Next, suppose that A_p is continuous for some $p > 1$. Then

$A_p A_p x_{2n+1} \to A_p z$, $A_p Tx_{2n+1} \to A_p z$ and the compatibility of A_p and T implies that $TA_p x_{2n+1} \to A_p z$. Since $A_p X \subset SX$, $A_p z = Sw$ for some w in X and corresponding to each x_{2n+1} there exists a z_{2n+1} such that $A_p A_p x_{2n+1} = S_{z_{2n+1}}$. Thus $A_p A_p x_{2n+1} = S_{z_{2n+1}} \to Sw$ and $TA_p x_{2n+1} \to Sw$. Using an argument similar to that used in proving (6), it follows that $A_1 z_{2n+1} \to Sw$. Again since $A_1 X \subset TX$, corresponding to each z_{2n+1} there exists u_{2n+1} such that $A_1 z_{2n+1} = Tu_{2n+1}$.

Thus, $A_1 z_{2n+1} = Tu_{2n+1} \to Sw$ and $Sz_{2n+1} \to Sw$. Again using an argument similar to that used in proving (6) it follows that $\lim_n A_i u_{2n+1} = Sw$ for each $i > 1$. We claim that $A_1 w = Sw$. If $A_1 w \neq Sw$, then by virtue of (iii), for large values of n we get

$$d(A_1 w, A_i u_{2n+1}) \leq \phi_i(M_{1i}(w, u_{2n+1})) = \phi_i(d(A_1 w, Sw))$$

for every $i > 2$. Taking $n \to \infty$ yields, $d(A_1 w, Sw) \leq \phi_i(d(A_1 w, Sw)) < d(A_1 w, Sw)$, a contradiction. Hence $A_1 w = Sw$. Since $A_1 X \subset TX$, there exists a u in X such that $A_1 w = Tu$; that is, $Sw = A_1 w = Tu$. We assert that $A_i u = Tu$ for each $i > 1$. If $A_i u \neq Tu$ for some $i > 1$, then by virtue of (*ii*) and (*iii*) we get

$$d(A_1 w, A_i u) < M_{1i}(w, u) = d(A_i u, Tu) = d(A_i u, A_1 w),$$

a contradiction. Hence $A_i u = Tu$ for each $i > 1$ and

$$Sw = A_1 w = Tu = A_i u,\ i > 1.$$

Now, the remaining part of the proof is similar to that in the lines following eq. (7) in the case of continuity of T. A similar proof is again applicable when A_1 is assumed continuous. This establishes the theorem.

We now give an example to illustrate the above theorem.

Example 1 — Let $X = [2, 20]$ and d be the usual metric on X. Define $A_i, S, T : X \to X$, $i = 1, 2, \ldots$, as follows

$$A_1 x = 2 \text{ for each } x$$

$Sx = x$ is $x \leq 8$, $Sx = 8$ if $8 < x \leq 14$, $Sx = (x + 10)/3$ if $x > 14$.

$Tx = 2$ if $x = 2$ or ≥ 5, $Tx = 12 + x$ if $2 < x < 4$, $Tx = 9 + x$ if $4 \leq x < 5$

$A_2x = 2$ if $x < 4$ or ≥ 5, $A_2 x = 3 + x$ if $4 \leq x < 5$

and for each $i > 2$

$A_i x = 2$ if $x = 2$ or ≥ 4, $A_i x = (30 + x)/4$ if $2 < x < 4$.

Then $\{A_i\}$, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$.

It can be seen in this example that A_1, A_2, S and T satisfy the condition

$$\varepsilon \leq M_{12}(x, y) < \varepsilon + \delta \Rightarrow d(A_1x, A_2y) < \varepsilon,$$

where $\delta(\varepsilon) = 14 - \varepsilon$ if $\varepsilon \geq 6$ and $\delta(\varepsilon) = 6 - \varepsilon$ if $\varepsilon < 6$. We see that $\delta(\varepsilon)$ is neither nondecreasing nor lower semicontinuous. However, A_1, A_2, S and T do not satisfy the contractive condition

$$d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$$

since the required function ϕ does not satisfy $\phi(t) < t$ at $t = 6$. To see this take $x > 8$ and $4 \leq y < 5$ then $M_{12}(x, y) = 6$ and $d(A_1x, A_2) = 1 + y \to 6$ as $y \to 5$. Therefore, $\phi(t)$ cannot be defined at $t = 6$. It can similarly be shown that $A_1, A_i, S, T, i > 2$, satisfy the contractive condition $d(A_1x, A_iy) \leq \phi_i(M_{1i}(x, y))$ with $\phi_i(t) = t/2$ for $t \leq 6$ and $\phi_i(t) = (12 + t)/3$ for $t > 6$. The function ϕ_i is not upper semicontinuous at $t = 6$. To see this take $8 < x \leq 14$ and $2 < y < 4$. Then $d(A_1x, A_iy) = (22 + y)/4 \to 6, M_{1i}(x, y) = 4 + y \to 6$ as $y \to 2$, $\phi_i(4 + y) = (16 + y)/3 > (22 + y)/4$ and $\limsup_{t \to 6} \phi_i(t) = 6$. It also becomes clear from these calculations that the mappings $A_1, A_i, S, T, i > 2$, do not satisfy the contractive condition

$$\varepsilon \leq M_{1i}(x, y) < \varepsilon + \delta \Rightarrow d(A_1x, A_iy) < \varepsilon$$

as δ cannot be defined at $\varepsilon = 6$. Moreover, $t/(t - \phi_i(t))$ failts to be nonincreasing in any interval containing $t = 6$. Similarly, it may be observed that $\lim_n \phi^n(t) = \lim_{n \to \infty} [(1 + 3 + 3^2 + \ldots + 3^{n-1}) 12 + t]/3^n = 6$ when $t > 6$. We thus see that the mappings in the above example satisfy the contractive conditions (ii) and (iii) of Theorem 1 above but do not satisfy any of the conditions (I), (II), (III) or (IV).

As a particular case of the above theorem we obtain the following corollary which provides an affirmative answer to the question (see e.g. Rhoades[17 p. 242]) on the existence of a contractive definition which is strong enough to generate a fixed point but does not force the map to be continuous at the fixed point. This problem had remained an open problem for more than a decade. Another solution to this problem was obtained by the first author[13] by adopting a different approach which uses the notion of reciprocal continuity.

Corollary Let f be a self-map of a complete metric space (X, d) such that for any x, y in X

(iv) given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \max \{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\} < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon,$$

(v) $d(fx, fy) \leq \phi(\max \{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\})$

where $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$. Then f has a unique fixed point.

PROOF : The corollary follows from the above theorem by taking $S = T$ = identity mapping and $A_i = f$ for each i in the above theorem. This corollary can be established independently of the above theorem also. It may be noted that f need not be continuous in the above corollary. The next example illustrates the corollary.

Example 2 — Let $X = [0, 2]$ and d be the usual metric on X. Define $f : X \to X$ as follows

$$fx = 1 \text{ if } x \leq 1, \; fx = 0 \text{ if } x > 1.$$

Then f satisfies the conditions of the above corollary and has a unique fixed point $x = 1$. It may be verified in this example that $d(fx, fy) = 0$ when $x, y \leq 1$ or $x, y > 1$ while $d(fx, fy) = 1$ and $1 < m(x, y) \leq 2$ when $x \leq 1, y > 1$. Hence, f satisfies the (ε, δ) condition (*iv*) with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$. Also, f satisfies the ϕ-contractive condition (*v*) $\phi(t) = 1$ for $t > 1$ and $\phi(t) = t/2$ for $t \leq 1$. Moreover, f is discontinuous at the fixed point $x = 1$.

We now give an example to show that a Meir-Keeler type (ε, δ) condition is not sufficient to ensure the existence of a fixed point of a contractive type mapping.

Example 3 — Let $X = [0, 2]$ and d be the usual metric on X. Define $f : X \to X$ by

$$fx = (1 + x)/2 \text{ if } x < 1, \; fx = 0 \text{ if } x \geq 1$$

Then f satisfies the contractive condition

$$\varepsilon \leq \max \{d(x, y), d(x, fx), d(y, fy)\} < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon$$

with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$ but f does not have a fixed point.

In view of above examples and the discussion in the Introductory Section, the following observations may be made.

1. The results of Boyd and Wong[1], Carbone *et al*[2], Matkowski[7], Pant[11 & 12] Theorem 3, Park and Rhoades[14] and Singh and Kasahara[18] employ a ϕ-contractive condition and assume one of the conditions (I), (II), (III) or (IV) on the function ϕ. But, as discussed in the section on Introduction, each of these assumptions on ϕ implies both the contractive conditions (ii) and (iii) of the above theorem. However, as shown in Example 1 above, the contractive conditions (ii) and (iii) do not imply any of the conditions (I), (II), (III) or (IV). Hence all such theorems are either special cases of our theorem or can be generalized in a verbatim manner in the spirit of the above theorem since our theorem does not assume conditions on ϕ.

2. The results of Jungck[5], Jungck *et al*[6], and Pant[9, 10] use a Meir-Keeler type (ε, δ) contractive condition and δ is assumed lower semicontinuous[5, 6] and nondecreasing[9, 10] respectively. We have seen in the section on Introduction that an (ε, δ) condition under these assumptions on δ implies both the contractive conditions (ii) and (iii) of our theorem. Our theorem, therefore, generalizes the results of Jungck[5], Jungck *et al*[6], and Pant[9, 10] since the above theorem does not require δ to be lower semicontinuous or non-decreasing.

It can similarly be shown that all the analogous fixed point theorems can either be obtained as special cases of the above theorem or can be improved in the spirit of the above theorem.

ACKNOWLEDGEMENT

The authors are thankful to the referee for his valuable suggestions for improving the content and style of the paper and for suggesting the correct value of $\delta(\varepsilon)$ in Example 1.

REFERENCES

1. D. W. Boyd and J. S. Wong, *Proc. Amer. math. Soc.* **20** (1969) 458-64.
2. A. Carbone, B. E. Rhoades and S. P. Singh, *Indian J. pure appl. Math* **20** (1989) 543-48.
3. J. Jachymski, *Indian J. pure appl. Math* **25** (1994) 925-37.
4. J. Jachymski, *J. math. Anal. Appl.* **194** (1995) 293-303.
5. G. Jungck, *Int. J. Math math. Sci.* **9** (1986) 771-79.
6. G. Jungck, K. B. Moon, S. Park and B. E. Rhoades, *J. math. Anal. Appl.* **180** (1993) 221-22.
7. J. Matkowski, *Diss. Math.* **127** (1975).
8. A. Meir and E. Keeler, *J. math. Anal. Appl.* **28** (1969) 326-29.
9 R. P. Pant, *Indian J. pure appl. Math.* **17** (1986) 187-92.
10. R. P. Pant, *Math. Student* **62** (1993) 97-102.
11. R. P. Pant, *Ganita* **47** (1996) 43-49.
12. R. P. Pant, *J. math. Anal. Appl.* **226** (1998) 251-58.
13. R. P. Pant, *Bull. Calcutta math. Soc.* **90** (1998) 281-86.
14. S. Park and B. E. Rhoades, *Math. Soc. Notes.* **9** (1981) 113-118.
15. I. H. N. Rao and K. P. R. Rao, *Indian J. pure appl. Math.* **16** (1985) 1249-62.
16. B. E. Rhoades, S. Park and K. B. Moon, *J. math. Anal. Appl.* **146** (1990) 482-94.
17. B. E. Rhoades, *Contemporary Math.* **72** (1988) 233-45.
18. S. L. Singh and S. Kasahara, *Indian J. pure appl. Math.* **13** (1982) 757-61.