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LRS BIANCHI I COSMOLOGICAL MODELS IN THE PRESENCE OF ZERO-MASS SCALAR FIELD

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An L R S Bianchi I spacetime filled with a perfect fluid is considered in the presence of a zero-mass scalar field and it is shown that the field equations are solvable for any arbitrary cosmic scale function. Some physical and geometrical features of the models are discussed.

Key Words : Cosmology; Zero-Mass Scalar Fields; LRS Bianchi I Models; Zel'dovich Universe

1. INTRODUCTION

Following the development of inflationary models, the importance of scalar fields (mesons) in cosmology has become well known[1]. The study of interacting fields, one of the fields being a zero-mass scalar field, is basically an attempt to look into the yet unsolved problem of the unification of gravitational and quantum theories[2 & 3]. Considerable interest has been focused on a set of field equations representing zero-mass scalar fields coupled with the gravitational field for the last three decades. Bergmann and Leipnik[4], and Brahmachary[5] have investigated the spherically symmetric fields associated with zero-rest-mass. The static solutions for axially symmetric fields have been investigated by Buchdahl[6]. Janis *et al.*[7 & 8], in an attempt to present an extension of Israel's idea of a singular event horizon[9], have considered the spherically symmetric solutions of the field equations of general relativity containing zero-rest-mass meson fields. Penney[10] and Gautreau[11] have extended the study of the case of axially symmetric fields and have found that the scalar fields obey a flat space Laplace equation and a large class of solutions exist. Singh[12], Patel[13] and Reddy[14] have investigated plane-symmetric solutions of the field equations corresponding to zero-mass scalar fields. Stephenson[15], Rao *et al.*[16], Chatterjee and Roy[17], Reddy and Rao[18], Verma[19], Shanthi and Rao[20], Pradhan *et al.*[21] are some of the authors who have studied various aspects of interacting fields in the framework of general relativity.

At the present state of evolution, the universe is spherically symmetric and the matter distribution in it is isotropic and homogeneous. But in its early stages of evolution, it could have not had a smoothed out picture. Close to the big bang singularity, neither the assumption of spherically symmetry nor of isotropy can be strictly valid. So, we consider plane symmetry, which is less restrictive than spherical symmetry and provides an avenue to study in homogeneities. For simplification and description of the large scale behaviour of the actual universe, locally rotationally symmetric [henceforth referred as L R S] Bianchi I spacetime has been widely studied. Recently, Mazumder[22] has obtained solutions of an L R S Bianchi I spacetime filled with a perfect fluid. Hajj-Boutros and Sfeila[23] and Sri Ram[24] also obtained some solutions for the same field equations by using their solution-generating techniques.

Cosmological models based on scalar fields of various kinds have had enormous success in solving cosmological problems, among which are the causality, entropy, initial singularity and cosmological constant problems. Motivated by it, in the present investigation, we have considered an L R S Bianchi I universe in the presence of zero-mass scalar fields associated with a perfect fluid distribution in it. With the introduction of the zero-mass scalar fields we see that the effective pressure and energy density terms are found to be different from earlier works[22-24]. We can now study separately the behaviour of the pressure and density terms during the early evolution of the universe, i.e., for small times and at late times. One way to generalize the results of Refs. [22-24] is to consider zero-mass scalar fields. We shall also show below that if the scalar field depends only on time and has a particular relation to the time-dependent function in the metric, then our model predicts new interesting properties. The model represents a plane symmetric Zel'dovich universe in the presence of zero-mass scalar fields.

2. Field Equations

The metric for the L R S Bianchi I spacetime is of the form[25]

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 (dy^2 + dz^2), \qquad \text{... (1)}$$

where A and B are functions of the cosmic time t. The energy momentum tensor of a perfect fluid together with a zero-mass scalar field is given by

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(s)}_{\mu\nu}, \qquad \text{... (2)}$$

where

$$T^{(m)}_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu n} \qquad \text{... (3)}$$

is the energy momentum tensor corresponding to perfect fluid distribution with the four vector velocity u^μ satisfying $u_\mu u^\mu = -1$, p the pressure and ρ the mass-energy density. The energy momentum tensor $T^{(S)}_{\mu\nu}$ corresponds to zero-mass scalar fields ϕ and is

$$T^{(s)}_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta}, \qquad \text{... (4)}$$

where $\phi(t)$ (a function of t only) is the zero-mass scalar field which satisfies the wave equation

$$g^{\mu\nu}\phi_{;\mu\nu} = 0. \qquad \text{... (5)}$$

The scalar field ϕ is not directly coupled to matter. It interacts with matter indirectly through gravity. The Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} \qquad \text{... (6)}$$

together with energy momentum tensor defined by eq. (2) give the following equations

$$-kp+\dot{\phi}^2=\frac{2\ddot{B}}{B}+\frac{\dot{B}^2}{B^2} \qquad \text{... (7)}$$

$$-kp+\dot{\phi}^2=\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{\ddot{A}}{A} \qquad \text{... (8)}$$

and

$$k\rho-\dot{\phi}^2=\frac{2\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2} \qquad \text{... (9)}$$

where $k=8\pi G$, G the gravitational constant. The overdot indicates a derivative with respect to time t. The wave eq. (5) yields

$$\left(\frac{\dot{A}}{A}+\frac{2\dot{B}}{B}\right)\dot{\phi}+\ddot{\phi}=0 \qquad \text{... (10)}$$

and the energy conservation equation for the matter $T^{(m)}_{\mu\nu;\mu}=0$, leads to

$$\dot{\rho}+\left(\frac{\dot{A}}{A}+\frac{2\dot{B}}{B}\right)(\rho+p)=0. \qquad \text{... (11)}$$

3. Solutions of the Field Equations

From eqs. (7) and (8), we obtain

$$\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}-\frac{\ddot{A}}{A}-\frac{\dot{A}\dot{B}}{AB}=0 \qquad \text{... (12)}$$

which has first integeral

$$B^2\dot{A}-AB\dot{B}=\lambda, \qquad \text{... (13)}$$

where λ is an integrating constant.

Eq. (13) is a linear differential equation in $A(t)$ and has an exact solution

$$A=k_1B+\lambda B\int\frac{dt}{B^3(t)}. \qquad \text{... (14)}$$

Similarly eq. (13) is also a linear differential equation in $B(t)$, which has an exact solution

$$B^2=k_2A^2-2\lambda A^2\int\frac{dt}{A^3(t)}. \qquad \text{... (15)}$$

On integration, eq. (10) yields

$$\phi = k_4 + \int \frac{k_3\, dt}{A(t)\, B^2(t)}, \qquad \ldots (16)$$

where k_1, k_2, k_3 and k_4 are integration constants.

Thus, for any arbitrary $B(t)$, eq. (14) gives $A(t)$ and then ϕ is known from eq. (16). Similarly for an arbitrary $A(t)$ one can calculate $B(t)$ and ϕ from eqs. (15) and (16). Then from eqs. (7) and (9), p and ρ can be obtained and hence the solution of the field equations is completely known.

To illustrate our analysis, we choose $B = t^{\frac{1}{2}(1-\alpha)}$[22]. From eqs. (14) and (16) we obtain

$$A = k_1\, t^{\frac{1}{2}(1-\alpha)} + \frac{2\lambda}{(3\alpha - 1)}\, t^{\alpha} \qquad \ldots (17)$$

and

$$\dot{\phi}^2 = k_3^2 \left(k_1\, t^{\frac{1}{2}(1-\alpha)^2} + \frac{2\lambda}{(3\alpha-1)}\, t^{\alpha(1-\alpha)} \right)^{-2}, \qquad \ldots (18)$$

where α is a real number satisfying $\alpha \neq \frac{1}{3}$. So, in this case, the geometry of our universe is described by the line-element

$$ds^2 = -dt^2 + \left(k_1\, t^{\frac{1}{2}(1-\alpha)} + \frac{2\lambda}{(3\alpha-1)}\, t^{\alpha} \right)^2 dx^2 + t^{(1-\alpha)}\,(dy^2 + dz^2). \qquad \ldots (19)$$

For the metric (19), from the eqs. (7)-(9), we obtain the expressions for p and ρ

$$kp = k_3^2 \left(k_1\, t^{\frac{1}{2}(1-\alpha)^2} + \frac{2\lambda}{(3\alpha-1)}\, t^{\alpha(1-\alpha)} \right)^{-2} + \frac{(1-\alpha)(1+3\alpha)}{4t^2} \qquad \ldots (20)$$

$$k\rho = k_3^2 \left(k_1\, t^{\frac{1}{2}(1-\alpha)^2} + \frac{2\lambda}{(3\alpha-1)}\, t^{\alpha(1-\alpha)} \right)^{-2}$$

$$+ \frac{2\lambda(1-\alpha)(1+3\alpha)\, t^{\frac{1}{2}(3\alpha-1)} + 3k_1(1-\alpha)^2(3\alpha-1)}{4t^2 \left[2\lambda\, t^{\frac{1}{2}(3\alpha-1)} + k_1(3\alpha-1) \right]}. \qquad \ldots (21)$$

It is readily seen, from eqs. (20) and (21), that the first term dominates during early times whereas the second term dominates at late times. Using the expressions for A, B, p and ρ from the above equations, we observe that eq. (11) is satisfied. The model is greatly affected by inclusion of the scalar field. The behaviour of ρ and p is same at early and late times when $\phi = 0$. It can be seen that for $\alpha = 1$, $\rho = p =$ constant, whereas one gets $\rho = p = 0$ in the absence of a scalar field[22].

The energy conditions[26]

(i) $(\rho + p) > 0$

(ii) $(\rho+3p)>0$

and (iii) $\rho>0$

are satisfied when $k_1>0, \lambda>0$, and $\frac{1}{3}<\alpha<1$ and the dominant energy conditions[27]

(i) $(\rho-p)\geq 0$

and (ii) $(\rho+p)\geq 0$,

when $\lambda>0, k_1>0$ and $\frac{1}{3}<\alpha\leq\frac{2}{3}$.

The expansion scalar θ, the shear tensor σ_{ik}, the rotation ω_{ik} and the acceleration vector f_i for the velocity field u_i are defined by

$$\theta=u^{i}_{;i}, \qquad \text{... (22)}$$

$$\sigma_{ik}=\frac{1}{2}(u_{i;k}+u_{k;i})-\frac{1}{2}(u_i f_k+u_k f_i)-\frac{1}{3}\theta(g_{ik}+u_i u_k), \qquad \text{... (23)}$$

$$\omega_{ik}=u_{i;k}-\sigma_{ik}-\frac{1}{3}\theta(g_{ik}+u_i u_k)-u_{i;\alpha}u^{\alpha}u_k \qquad \text{... (24)}$$

and

$$f_i=u^k u_{i;k}. \qquad \text{... (25)}$$

Here the semicolon indicates covariant differentiation. The spatial volume is given by $V=A\,B^2$.

For the velocity field u_i these kinematical parameters are found to have the following expressions :

$$V=\frac{t\left(k_1(3\alpha-1)+2\lambda t^{\frac{1}{2}(3\alpha-1)}\right)}{(3\alpha-1)\,t^{\frac{1}{2}(3\alpha-1)}}, \qquad \text{... (26)}$$

$$\theta=\frac{4\lambda t^{\frac{1}{2}(3\alpha-1)}+3k_1(1-\alpha)(3\alpha-1)}{2t\left[2\lambda t^{\frac{1}{2}(3\alpha-1)}+k_1(3\alpha-1)\right]}, \qquad \text{... (27)}$$

$$\sigma=\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(\frac{2\lambda\left(\alpha-\frac{1}{3}\right)t^{\frac{1}{2}(3\alpha-1)}}{t\left[2\lambda r^{\frac{1}{2}(3\alpha-1)}+k_1(3\alpha-1)\right]}\right), \qquad \text{... (28)}$$

and

$$\omega=0 \qquad \text{... (29)}$$

$$f_i=[0,0,0,0]. \qquad \text{... (30)}$$

Hence, the model (19) is expanding, shearing and non-rotating. The acceleration vector f_i is zero and consequently the stream lines of the perfect fluid are geodetic. As the shear tensor is not zero, the model is clearly anisotropic.

For $\alpha = 1$, the metric (19) represents a non static cosmological model filled with a stiff fluid, the pressure and density of which are given by

$$kp = k\rho = \frac{k_3^2}{(k_1 + \lambda)^2}. \quad \text{... (31)}$$

The models with $\rho = p$ are important in relativistic cosmology for the description of very early stages of the universe.

For $\alpha \neq 1, \frac{-1}{3}, \frac{1}{3}$ from eqs. (20), (21), (26), (27) and (28) it is seen that at the singularity $t = 0$, $V \to 0$ and p, ρ, θ and σ are infinitely large. As $t \to \infty$, $V \to \infty$ and p, ρ, θ and σ vanish. Therefore, for $\alpha \neq 1, \frac{-1}{3}, \frac{1}{3}$, the solution (19) represents an anisotropic universe exploding from $t = 0$, which expands for $0 < t < \infty$ and after a large time t, would give essentially an isotropic empty universe.

Choosing $A = \varepsilon_1 t^{\frac{1}{2}(1-\alpha)} + \varepsilon_2 t^{\alpha}$ and $\lambda = 0$ in eq. (14), we get

$$B^2 = l_1 t^{(1-\alpha)} + l_2 t^{\frac{1}{2}(1+\alpha)} + l_3 t^{2\alpha}, \quad \text{... (32)}$$

where $\quad l_1 = k_2 \varepsilon_1^2, l_2 = 2k_2 \varepsilon_1 \varepsilon_2, l_3 = k_2 \varepsilon_2^2$.

Hence, in this case, the geometry of our universe is described by the line-element

$$ds^2 = -dt^2 + (\varepsilon_1 t^{\frac{1}{2}(1-\alpha)} + \varepsilon_2 t^{\alpha})^2 \, dx^2 + (l_1 t^{(1-\alpha)} + l_2 t^{\frac{1}{2}(1+\alpha)} + l_3 t^{2\alpha}) (dy^2 + dz^2) \quad \text{... (33)}$$

From eq. (16), we can obtain the value of $\dot{\phi}^2$, and eqs. (8) and (9) give us the values of the physical parameters p and ρ. The expressions for these parameters are quite lengthy and complicated. Therefore, we shall not report them here but it is seen that the properties of the model (33) are the same as that of solution (19).

Choosing $B = t^{\frac{1}{3}}$ in eq. (14), we obtain the geometry of the universe as

$$ds^2 = -dt^2 + t^{\frac{2}{3}} (k_1 + \lambda \ln(t))^2 \, dx^2 + t^{\frac{2}{3}} (dy^2 + dz^2). \quad \text{... (34)}$$

4. Conclusion

In this paper we have generalized the solutions of Refs. [22], [23] and [24]. For $\phi = 0$ and $\alpha = -\frac{1}{3}$, from (19), one can obtain the solutions of Hajj-Boutros and Sfeila[23]. For $\phi = 0$, $\alpha = 0$

and $\alpha = -\frac{1}{3}$, from (33), we obtain the solutions of Sri Ram[24]. For $\lambda = 0$, $k_1 = 1$, the solution (34) represents the Einstein-de Sitter universe.

For $\phi = 0$, from eq. (19) we recover the model of Mazumder[22], and thus our solutions represent a generalization of Ref. [22].

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