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# PRE-UNIQUE SEQUENTIAL SPACES

T. M. NOUR

*Department of Mathematics, University of Jordan, Amman-Jordan*

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In this paper, preopen sets are used to define and investigate a new separation axiom in topological spaces. Relationships between this new separation axiom and other separation axioms are obtained. Sequentially preclosed sets and sequentially strongly compact sets are also defined. We also introduce *p*-convergence of a sequence. It is shown that if a sequence $\langle x_n \rangle$ *p*-converges to *x*, then it converges to *x*. A proper example is given to show that the converse is not true.

**Key Words : Pre-unique Sets; Sequential Spaces; Topological Spaces; Separation Axiom; *p*-convergence**

In 1947, Sierpinki[1] studied the spaces with unique sequential limits and referred to those spaces as semi-Hausdorff. In 1965, Cullen[2] studied this class of spaces and named it as the class of unique sequential spaces (US-spaces). This class was also studied by Slepian[3]. Murdeshwar and Naimpally[4] and Wilansky[5]. The object of this paper is to generalize this class of spaces and thus introduce the class of pre-unique sequential spaces (pre-US spaces), using preopen sets.

A subset $A$ of a topological space $(X, \tau)$ is said to be preopen[6] (resp. semi-open[7]) if $A \subseteq \text{cl}\,(\text{int}\, A))$, where int$A$, cl$A$ denote interior of $A$ and closure of $A$ in $X$, respectively. Every open set is preopen but a preopen set is not necessarily open. The union of any family of preopen sets is preopen. The intersection of two preopen sets need not be preopen. However, the intersection of a preopen set with an open set is preopen. The complement of a preopen set is called preclosed[8]. The preclosure of a set $B \subseteq X$, denoted by pcl $(B)$, is the intersection of all preclosed sets that contain $B$. It is evident that a set $A$ is preclosed iff $A = \text{pcl}\,(A)$ and for any set $A \subseteq X$, $\text{pcl}(A) \subseteq \text{cl}\,(A)$.

*Definition* 1[2] — A space $X$ is said to be a US-space if every convergent sequence has exactly one limit to which it converges.

*Definition* 2 — A sequence $\langle x_n \rangle$ is said to *p*-converges to a point $x$ in a space $X$ if $\langle x_n \rangle$ is eventually in every preopen set containing $x$.

It can be easily verified that if a sequence $\langle x_n \rangle$ *p*-converges to $x$, then $\langle x_n \rangle$ converges to $x$. But if $\langle x_n \rangle$ converges to $x$, then it need not *p*-converge to $x$. Following is an example :

*Example* 1 — Let $X = \{a, b, c\}$ and $\tau = \{x, \phi, \{a\}, \{a, b\}\}$ be a topology on $X$. Let $\langle x_n \rangle = (a, c, b, c, ...)$ be a sequence in $X$. Then $\langle x_n \rangle$ converges to the point $c$. But it does not *p*-converge to $c$.

*Definition* 3 — A space $X$ is said to be a pre-US space if every sequence in $X$ *p*-converges to at most one point.

It is evident that every US-space is pre-US. But the converse is not necessarily true. Following is an example:

*Example* 2 — Let $X = \{a, b, c\}$ and let $\tau = \{x, \phi, \{a, b\}\}$ be a topology on $X$. Then the space $(X, \tau)$ is pre-US but not US.

*Definition* 4 — A space $X$ is said to be pre-$T_1$ if for any two distinct points $x$ and $y$ of $X$, there exist preopen sets $U$ and $V$ such that $x \in U$ and $y \notin U$ and also $x \notin V$ and $y \in V$.

***Theorem*** 1 — *Every pre-US space is pre-* $T_1$.

PROOF : Let $X$ be pre-Us and $x$ and $y$ be two distinct points of $X$. Consider the sequence $\langle x_n \rangle$ where $x_n = x$ for all $n$ clearly $\langle x_n \rangle$ $p$-converges to $x$. Also, since $x \neq y$ and $X$ is pre-US, $\langle x_n \rangle$ cannot $p$-converge to $y$. That is, there exists a preopen set $V$ containing $y$ but not $x$. Similarly, if we consider the sequence $\langle y_n \rangle$ where $y_n = y$ for every $n$ and proceeding as before, we get a preopen set $U$ containing $x$ but not $y$. Thus $X$ is pre-$T_1$.

The following example shows that the converse of the above Theorem need not be true.

*Example* 3 — Consider the cofinite topology $\tau$ on an infinite set $X$. Then $(X, \tau)$ is pre-$T_1$. But it is not pre-US since every sequence of distinct points of $X$ $p$-converges to more than one point.

*Definition* 5 — A space $X$ is said to be pre-$T_2$ if for any pair of distinct points $x$, $y$ of $X$, there exist disjoint preopen sets $U$ and $V$ such that $x \in U, y \in V$.

***Theorem*** 2 — *Every pre-$T_2$ space is pre-US.*

PROOF : Let $X$ be pre-$T_2$ and $\langle x_n \rangle$ be a sequence in $X$. Suppose $\langle x_n \rangle$ $p$-converges to points $x$ and $y$. This means that $\langle x_n \rangle$ is eventually in every preopen set containing $x$ and also in every preopen set containing $y$: This is not possible since $X$ is pre-$T_2$. Thus $x = y$.

Converse of Theorem 2 is not true in general as can be seen from the following example:

*Example* 4 — Let $X$ be an uncountable set and Let $\tau$ be the co-countable topology on $X$. Then $(x, \tau)$ is pre-US but not pre-$T_2$.

*Definition* 6 — A set $B$ is said to be sequentially preclosed if every sequence in $B$ $p$-converges to a point in $B$.

***Theorem*** 3 — *A space $X$ is pre-US if and only if the diagonal $\Delta$ is a sequentially preclosed subset of XxX.*

PROOF : Let $X$ be pre-US. Let $\langle (x_n, x_n) \rangle$ be a sequence in $\Delta$. Suppose that $\langle (x_n, x_n) \rangle$ $p$-converges to $(x, y)$. Hence, $\langle x_n \rangle$ $p$-converges to $x$ and $y$. Therefore, $x = y$. Thus $\Delta$ is sequentially preclosed.

Conversely, if $\Delta$ is sequentially preclosed and $X$ is not pre-US, then there exists a sequence $\langle x_n \rangle$ that $p$-converges to $x$ and $y$, $x \neq y$. Hence $\langle (x_n, x_n) \rangle$ $p$-converges to $(x, y) \notin \Delta$, which is a contradiction. Hence, $X$ is pre-US.

***Theorem*** 4 — *Every semi-open subset of a pre-US space is pre-US.*

PROOF : Let $Y$ be a semi-open subset of $X$ and let $\langle x_n \rangle$ be a sequence in $Y$. Suppose that $\langle x_n \rangle$ $p$-converges to $x$ and $y$ in $X$. Let $U$ be a preopen subset of $X$ containing $x$ and $V$ be a preopen subset of $X$ containing $y$. Then $U \cap Y$ and $V \cap Y$ are preopen subsets of $Y$.[9] Therefore, $\langle x_n \rangle$ is eventually in $U \cap Y$ and $V \cap Y$ and so in $U$ and $V$. Since $X$ is pre-US, this implies that $x = y$. Hence $Y$ is pre-US.

*Definition* 7 — [10] A space $X$ is said to be strongly compact if every preopen cover of $X$ has a finite subcover.

*Definition* 8 — A subset $Y$ of $X$ is said to be sequentially strongly compact if every sequence in $Y$ has a subsequence which p-converges to a point in $Y$.

***Theorem*** 5 — *In a pre-US space, every sequentially strongly compact set is sequentially preclosed.*

PROOF : Let $X$ be a pre-US space. Let $Y$ be a sequentially strongly compact subset of $X$. Let $\langle x_n \rangle$ be a sequence in $Y$ that $p$-converges to $x \in X$. Since $Y$ is sequentially strongly compact, there exists a subsequence $\langle x_{n_\mu} \rangle$ of $\langle x_n \rangle$ such that $\langle x_{n_\mu} \rangle$ $p$-converges to a point $y$ in $Y$. Also since $\langle x_{n_\mu} \rangle$ is a subsequence of $\langle x_n \rangle$, therefore $\langle x_{n_\mu} \rangle$ $p$-converges to $x \in X \mid Y$. Since $\langle x_{n_\mu} \rangle$ is a sequence in the pre-US space $X$, $x = y$. Thus $Y$ is sequentially preclosed.

***Theorem*** 6 — *The product of an arbitrary family of pre-US spaces is pre-US.*

PROOF : Let $X = \prod_{\lambda \in I} X_\lambda$, where each $X_\lambda$ is pre-US. Suppose a sequence $\langle x_n \rangle$ in $X$ $p$-converges to $x = (x_\lambda)$ and $y = (y_\lambda)$. Suppose that there exists $\mu \in I$ such that $\langle x_{n_\mu} \rangle$ does not $p$-converge to $x_\mu$. Then there exists a preopen set $U_\mu$ in $X_\mu$ containing $X_\mu$ such that $\langle x_{n_\mu} \rangle$ is not eventually in $U_\mu$. Consider the set $U = \prod_{\lambda \neq \mu} X_\lambda \times U_\mu$. Then $U$ is a preopen subset of $X$ and $x \in U$.[9]

Also, $\langle x_n \rangle$ is not eventually in $U$, which contradicts the fact that $\langle x_n \rangle$ $p$-converges to $x$. Thus $\langle x_{n_\lambda} \rangle$ $p$-converges to $x_\lambda$ and $y_\lambda$ for each $\lambda \in I$. Since $X_\lambda$ is pre-US, $x_\lambda = y_\lambda$ for each $\lambda$. Thus $x = y$. Hence, $X$ is pre-US.

*Definition* 9 — [10] A function $f: X \to Y$ is said to be $M$-preopen if the image of every preopen set in $X$ is preopen in $Y$.

***Theorem*** 7 — *The image of a pre-US space under an M-preopen bijective function is pre-US.*

PROOF : Let $f: X \to Y$ be an $M$-preopen bijective function and $X$ be pre-US. Let $\langle y_n \rangle$ be a sequence in $Y$. Suppose that $\langle y_n \rangle$ $p$-converges to two points $y$ and $y^*$. In this case we shall prove that the sequence $\langle f^{-1}(y_n) \rangle$ $p$-converges to $f^{-1}(y)$ and $f^{-1}(y^*)$. Let $U$ be any preopen set containing $f^{-1}(y)$. Then $f(U)$ is a preopen set containing $y$ and hence $\langle y_n \rangle$ is eventually in $f(U)$.

Therefore, $\langle f^{-1}(y_n) \rangle$ is eventually in $U$. Hence, $\langle f^{-1}(y_n) \rangle$ $p$-converges to $f^{-1}(y)$. Similarly, we can prove that $\langle f^{-1}(y_n) \rangle$ $p$-converges to $f^{-1}(y^*)$. This is not possible, since $X$ is pre-US. Hence, $Y$ is pre-US.

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