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EXACT EXPLICIT FORMULAE FOR THE COEFFICIENTS OF DIFFERENTIATED EXPANSIONS OF DOUBLE AND TRIPLE ULTRASPHERICAL POLYNOMIALS

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The tensor product of orthogonal ultraspherical (Gegenbauer) polynomials is used to approximate a function of more than one variable. Formulae expressing the coefficients of differentiated expansions of double and triple ultraspherical polynomials in terms of the coefficients of the original expansion are stated and proved. The special cases, of double and triple Chebyshev polynomials of the first and second kinds and of Legendre polynomials are also considered.

Key Words and Phrases : Spectral Methods; Gegenbauer Orthogonal Polynomials; Expansion Coefficients

1. INTRODUCTION

Classical orthogonal polynomials are used extensively for the numerical solution of differential equations in spectral and pseudospectral methods, see for instance, Canuto *et al.*[1], Coutsias *et al.*[2], Doha[3-6], Doha & Helal[7], Gottlieb & Orszag[8], Guo Ben-yu[9], Haidvogel & Zang[10] and Horner[11]. For such methods explicit expressions for the expansion coefficients of the derivatives in terms of the expansion coefficients of the solution are required. Two explicit formulae expressing the Chebyshev (Legendre) coefficients of general order derivative of an infinitely differentiable function in terms of its Chebyshev (Legendre) coefficients are given in Karageorghis[12] and Phillips[15] respectively. A more general formula-with its special cases-for ultraspherical coefficients is given in Karageorghis & Phillips[13]. Such a formula has been stated in a more compact form and proved in a simple way in Doha[4]. Formulae expressing the coefficients of expansion of double and triple Chebyshev and Legendre polynomials in terms of the coefficients of the original expansions are given in Doha[5&6]. In Section 2 of the present paper, we state and prove the corresponding formulae expressing the coefficients of expansions of double and triple ultraspherical polynomials which have been partially differentiated any number of times with respect to their variables in terms of the coefficients of the original expansions; results for the Chebyshev polynomials of the first and second kinds and for Legendre polynomials are also deduced. An extension to the expansion in triple ultraspherical polynomials is given in Section 3.

2. RELATIONS BETWEEN THE EXPANSION COEFFICIENTS OF DOUBLE ULTRASPHERICAL SERIES AND ITS DERIVATIVES EXPANSIONS

We define the double ultraspherical polynomials as

$$C_{mn}^{(\alpha)}(x, y) = C_m^{(\alpha)}(x)\, C_m^{(\alpha)}(y), \qquad \dots (1)$$

where $C_m^{(\alpha)}(x)$, $C_n^{(\alpha)}(y)$ are ultraspherical polynomials of degrees m and n associated with the real parameter $\alpha > -\frac{1}{2}$ in the variables x and y respectively. It is worthy to note here that the double Chebyshev polynomials of the first kind $T_{mn}(x, y)$ and of the second kind $U_{mn}(x, y)$ and the double Legendre polynomials $P_{mn}(x, y)$, are particular forms of the double ultraspherical polynomials. Namely, we have

$$T_{mn}(x, y) = C_{mn}^{(0)}(x, y) = T_m(x)\, T_n(y),$$

$$U_{mn}(x, y) = C_{mn}^{(1)}(x, y) = \frac{1}{(m+1)(n+1)} U_m(x)\, U_n(y)$$

and

$$P_{mn}(x, y) = C_{mn}^{\left(\frac{1}{2}\right)}(x, y) = P_m(x)\, P_n(y).$$

Let u, (x, y) be a continuous function defined on the square $S[-1 \leq x, y \leq 1]$, and let it have continuous and bounded partial derivatives of any order with respect to its variables x and y. Then it is possible to express

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y), \qquad \dots (2)$$

and

$$u^{(p,q)}(x, y) = \frac{\partial^{\,p+q} u(x, y)}{\partial x^p\, \partial y^q} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p,q)}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y), \qquad \dots (3)$$

where $a_{mn}^{(p,q)}$ denote the ultraspherical expansion coefficients of $u^{(p,q)}(x, y)$ and $a_{mn}^{(0.0)} = a_{mn}$.

Using the expressions[4]

$$2(m+\alpha)\, C_m^{(\alpha)}(x) = \frac{m+2\alpha}{m+1} D_x\, C_{m+1}^{(\alpha)}(x) - \frac{m}{m+2\alpha-1} D_x\, C_{m-1}^{(\alpha)}(x), \qquad \dots (4)$$

and

$$2(n+\alpha)\, C_n^{(\alpha)}(y) = \frac{n+2\alpha}{n+1} D_y\, C_{n+1}^{(\alpha)}(y) - \frac{n}{n+2\alpha-1} D_y\, C_{n-1}^{(\alpha)}(y), \qquad \dots (5)$$

with the assumptions that

$$D_x \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p-1,q)}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p,q)}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y),$$

and

$$D_y \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p,q-1)}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p,q)}\, C_m^{(\alpha)}(x)\, C_n^{(\alpha)}(y),$$

it is not difficult to derive the expressions

$$\frac{(m+2\alpha-1)}{2m(m+\alpha-1)}a^{(p,q)}_{m-1,n}-\frac{(m+1)}{2(m+\alpha+1)(m+2\alpha)}a^{(p,q)}_{m+1,n}=a^{(p-1,q)}_{mn}, m,p\geq 1, \quad \ldots (6)$$

and

$$\frac{(n+2\alpha-1)}{2n(n+\alpha-1)}a^{(p,q)}_{m,n-1}-\frac{(n+1)}{2(n+\alpha+1)(n+2\alpha)}a^{(p,q)}_{m,n+1}=a^{(p,q-1)}_{mn}, n,q\geq 1. \quad \ldots (7)$$

Now, we define a related set of coefficients $b^{(p,q)}_{mn}$ by writing

$$a^{(p,q)}_{mn}=\frac{(m+\alpha)(n+\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{m!\,n!}b^{(p,q)}_{mn}, m,n\geq 0, p,q=0,1,2,.., \quad \ldots (8)$$

then eqs. (6) and (7) take the simpler forms

$$b^{(p,q)}_{m-1,n}-b^{(p,q)}_{m+1,n}=2(m+\alpha)b^{(p-1,q)}_{m,n}, m,p\geq 1, \quad \ldots (9)$$

and

$$b^{(p,q)}_{m,n-1}-b^{(p,q)}_{m,n+1}=2(n+\alpha)b^{(p,q-1)}_{m,n}, n,q\geq 1. \quad \ldots (10)$$

Repeated application of (9) keeping n and q fixed [see, Orszag (1971)][14] yields

$$b^{(p,q)}_{mn}=2\sum_{i=1}^{\infty}(m+2i+\alpha-1)b^{(p-1,q)}_{m+2i-1,n}, p\geq 1, \quad \ldots (11)$$

and the same with (10) keeping m and p fixed yields

$$b^{(p,q)}_{mn}=2\sum_{j=1}^{\infty}(n+2j+\alpha-1)b^{(p,q)}_{m,n+2j-1}, q\geq 1. \quad \ldots (12)$$

The main result of this section is the following theorem.

***Theorem* 1** — *The coefficients* $b^{(p,q)}_{mn}$ *are related to the coefficients* $b^{(0,q)}_{mn}, b^{(p,0)}_{mn}$ *and the original coefficients* b_{mn} *by :*

$$b^{(p,q)}_{mn}=\frac{2^p}{(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!\,\Gamma(m+i+p+\alpha-1)}{(i-1)!\,\Gamma(m+i+\alpha)}(m+2i+p+\alpha-2)$$

$$b^{(0,q)}_{m+2i+p-2,n}, p\geq 1, \quad \ldots (13)$$

$$b^{(p,q)}_{m,n}=\frac{2^q}{(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\,\Gamma(n+j+q+\alpha-1)}{(j-1)!\,\Gamma(n+j+\alpha)}$$

$$(n+2j+q+\alpha-2)b^{(p,0)}_{m,n+2j+q-2}, q\geq 1, \quad \ldots (14)$$

and
$$b_{mn}^{(p,q)} = \frac{2^{p+q}}{(P-1)\,!\,(q-1)\,!} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(i+p-2)\,!\,(j+q-2)\,!\,\Gamma(m+i+p+\alpha-1)}{(i-1)\,!\,(j-1)\,!\,\Gamma(m+i+\alpha)}$$

$$\times \frac{\Gamma(n+j+q+\alpha-1)}{\Gamma(n+j+\alpha)} (m+2i+p+\alpha-2)\,(n+2j+q+\alpha-2)$$

$$b_{m+2i+p-2,\,n+2j+q-2},\ p, q \geq 1, \qquad \ldots (15)$$

for all $m, n \geq 0$.

In order to prove the theorem, the following two lemmas are required :

Lemma — $$\sum_{i=1}^{M} (m+2i+\alpha-1) \frac{(M-i+p-1)\,!\,\Gamma(m+i+M+p+\alpha-1)}{(M-i)\,!\,\Gamma(m+i+M+\alpha)}$$

$$= \frac{1}{p} \frac{(M+p-1)\,!\,\Gamma(m+M+p+\alpha)}{(M-1)\,!\,\Gamma(m+M+\alpha)},\ m, p \geq 1, \qquad \ldots (16)$$

Lemma — $$\sum_{j=1}^{N} (n+2j+\alpha-1) \frac{(N-j+q-1)\,!\,\Gamma(n+j+N+q+\alpha-1)}{(N-j)\,!\,\Gamma(n+j+N+\alpha)}$$

$$= \frac{1}{q} \frac{(N+q-1)\,!\,\Gamma(n+N+q+\alpha)}{(N-1)\,!\,\Gamma(n+N+\alpha)},\ n, q \geq 1. \qquad \ldots (17)$$

The interested reader is referred to Doha[4] for the proof of the two lemmas (16) and (17).

PROOF OF THEOREM 1 — Firstly, we prove formula (13). For $p = 1$, application of (11) with $p = 1$ yields the required formula. Proceeding by induction, assuming that the relation is valid for p (keeping n and q fixed), we want to show that

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{p\,!} \sum_{i=1}^{\infty} \frac{(i+p-1)\,!\,\Gamma(m+i+p+\alpha)}{(i-1)\,!\,\Gamma(m+i+\alpha)}$$

$$(m+2i+p+\alpha-1)\, b_{m+2i+p-1,\,n}^{(0,q)}. \qquad \ldots (18)$$

From (11), replacing p by $p+1$, and assuming the validity of (13) for p,

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)\,!} \sum_{i=1}^{\infty} (m+2i+\alpha-1)$$

$$\left\{ \sum_{k=1}^{\infty} \frac{(k+p-2)\,!\,\Gamma(m+2i+k+p+\alpha-2)}{(k-1)\,!\,\Gamma(m+2i+k+\alpha-1)} \times (m+2i+2k+p+\alpha-3)\, b_{m+2i+2k+p-3,\,n}^{(p,q)} \right\}, \qquad \ldots (19)$$

let $i + k - 1 = M$, then (19) takes the form

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!} \sum_{M=1}^{\infty}$$

$$\left[\sum_{\substack{i,k=1 \\ i+k=M+1}}^{M} (m+2i+\alpha-1) \frac{(k+p-2)!\,\Gamma(m+2i+k+p+\alpha-2)}{(k-1)!\,\Gamma(m+2i+k+\alpha-1)} \times (m+2M+p+\alpha-1)\, b_{m+2M+p-1,n}^{(0,q)} \right],$$

which may also be written as

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!} \sum_{M=1}^{\infty} \left[\sum_{i=1}^{M} (m+2i+\alpha-1) \frac{(M-i+p-1)!\,\Gamma(m+M+i+p+\alpha-1)}{(M-i)!\,\Gamma(m+M+i+\alpha)} \times (m+2M+p+\alpha-1) \right] b_{m+2M+p-1,n}^{(0,q)}.$$

Application of Lemma (16) to the second series yields equation (18) and the proof of formula (13) is complete.

It can be also shown that formula (14) is true by following the same procedure with (12), keeping m and p fixed. Formula (15) is obtained immediately by substituting (13) into (14). This completes the proof of Theorem 1.

Now, substitution of (13), (14) and (15) into (8) gives the relations between the coefficients $a_{mn}^{(p,q)}$, $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ and a_{mn} as :

$$a_{mn}^{(p,q)} = \frac{2^p (m+\alpha)\,\Gamma(m+2\alpha)}{(p-1)!\,m!} \sum_{i=1}^{\infty} \frac{(i+p-2)!\,\Gamma(m+i+p+\alpha-1)\,(m+2i+p-2)!}{(i-1)!\,\Gamma(m+i+\alpha)\,\Gamma(m+2i+p+2\alpha-2)} \times a_{m+2i+p-2,n}^{(0,q)}, \quad p \ge 1, \qquad \text{... (20)}$$

$$a_{mn}^{(p,q)} = \frac{2^q (n+\alpha)\,\Gamma(n+2\alpha)}{(q-1)!\,n!} \sum_{j=1}^{\infty} \frac{(j+q-2)!\,\Gamma(n+j+q+\alpha-1)\,(n+2j+q-2)!}{(j-1)!\,\Gamma(n+j+\alpha)\,\Gamma(n+2j+q+2\alpha-2)} \times a_{m,n+2j+q-2,n}^{(p,0)}, \quad q \ge 1, \qquad \text{... (21)}$$

$$a_{mn}^{(p,q)} = \frac{2^{p+q} (m+\alpha)(n+\alpha)\,\Gamma(m+2\alpha)\,\Gamma(n+2\alpha)}{(p-1)!\,(q-1)!\,m!\,n!} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{(i+p-2)!\,(j+q-2)!}{(i-1)!\,(j-1)!}$$

$$\times \frac{\Gamma(m+i+p+\alpha-1)\,\Gamma(n+j+q+\alpha-1)\,(m+2i+p-2)!\,(n+2j+q-2)!}{\Gamma(m+i+\alpha)\,\Gamma(n+j+\alpha)\,\Gamma(m+2i+p+2\alpha-2)\,\Gamma(n+2j+q+2\alpha-2)}$$

$$\times\, a_{m+2i+p-2,\, n+2j+q-2},\; p, q \geq 1. \qquad \ldots (22)$$

In particular, the special cases for the "bivariate" Chebyshev polynomials of the first and second kinds may be obtained directly by taking $\alpha = 0, 1$ respectively, and for the "bivariate" Legendre polynomials by taking $\alpha = \frac{1}{2}$. These are given as corollaries of the previous theorem.

Corollary 1 — If $\alpha = 0$ and

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty}{}'' a_{mn}\, T_m(x)\, T_n(y), \qquad \ldots (23)$$

and

$$u^{(p,q)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty}{}'' a_{mn}^{(p,q)}\, T_m(x)\, T_n(y), \qquad \ldots (24)$$

then the coefficients $a_{mn}^{(p,q)}$ which are related to $a_{mn}^{(0,q)}, a_{mn}^{(p,0)}$ and a_{mn} are given in [Doha[5] and pp. 86-87, formulae (16)-(18)]. Note here that the double primes in (23) and (24) indicate that the first term as $\frac{1}{4} a_{00}$, a_{0m} and a_{n0} are to be taken as $\frac{1}{2} a_{m0}$ and $\frac{1}{2} a_{0n}$ for $m, n > 0$ respectively.

Corollary 2 — If $\alpha = 1$ and

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn}\, U_m(x)\, U_n(y), \qquad \ldots (25)$$

and

$$u^{(p,q)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn}^{(p,q)}\, U_m(x)\, U_n(y), \qquad \ldots (26)$$

then the coefficients $A_{mn}^{(p,q)}$ are related to the coefficients $A_{mn}^{(0,q)}, A_{mn}^{(p,0)}$ and A_{mn} by :

$$A_{mn}^{(p,q)} = \frac{2^p (m+1)}{(p-1)\,!} \sum_{i=1}^{\infty} \frac{(i+p-2)\,!\,(m+i+p-1)\,!}{(i-1)\,!\,(m+i)\,!} A_{m+2i+p-2,\, n}^{(p,q)},\; p \geq 1, \qquad \ldots (27)$$

$$A_{mn}^{(p,q)} = \frac{2^p (n+1)}{(q-1)\,!} \sum_{j=1}^{\infty} \frac{(j+q-2)\,!\,(n+j+q-1)\,!}{(j-1)\,!\,(n+j)\,!} A_{m,\, n+2j+q-2,\, n}^{(p,0)},\; q \geq 1, \qquad \ldots (28)$$

and

$$A_{mn}^{(p,q)} = \frac{2^{p+q} (m+1)(n+1)}{(p-1)\,!\,(q-1)\,!} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty}$$

$$\frac{(i+p-2)\,!\,(j+q-2)\,!\,(m+i+p-1)\,!\,(n+j+q-1)\,!}{(i-1)\,!\,(j-1)\,!\,(m+i)\,!\,(n+j)\,!}$$

$$\times A_{m+2i+p-2,\, n+2j+q-2},\; p, q \geq 1, \qquad \ldots (29)$$

for all $m, n \geq 0$.

Corollary 3 — If $\alpha = \frac{1}{2}$ and

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} P_m(x) P_n(y) \qquad \dots (30)$$

and
$$u^{(p,q)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(P,q)} P_m(x) P_n(y), \qquad \dots (31)$$

then the coefficients $a_{mn}^{(p,q)}$ which are related to the coefficients $a_{mn}^{(0,q)}, a_{mn}^{(p,0)}$ and a_{mn} are given in [Doha[6], pp. 31-32, formulae (32)-(34)].

3. Extension to Triple Ultraspherical Series Expansion

Let $u(x, y, z)$ be a continuous function defined on the cube $C[-1, \leq x, y, z \leq 1]$, and let it have continuous and bounded partial derivatives of any order with respect to its variable x, y and z. Then it is possible to express

$$u(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} a_{lmn} C_l^{(\alpha)}(x) C_m^{(\alpha)}(y) C_n^{(\alpha)}(z)$$

and
$$u^{(p,q,r)}(x, y, z) = \frac{\partial^{p+q+r} u(x, y, z)}{\partial x^p \partial y^q \partial z^r}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} a_{lmn}^{(p,q,r)} C_l^{(\alpha)}(x) C_m^{(\alpha)}(y) C_n^{(\alpha)}(z).$$

Further, let

$$a_{lmn}^{(p,q,r)} = \frac{(l+\alpha)(m+\alpha)(n+\alpha)\Gamma(l+2\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{l!\,m!\,n!} b_{lmn}^{(p,q,r)},$$

$$l, m, n \geq 0, p, q, r = 0, 1, 2, \dots, \qquad \dots (32)$$

then it is not difficult to show that

$$b_{l-1,m,n}^{(p,q,r)} - b_{l+1,m,n}^{(p,q,r)} = 2(l+\alpha) b_{lmn}^{(p-1,q,r)}, p \geq 1,$$

$$b_{l,m-1,n}^{(p,q,r)} - b_{l,m+1,n}^{(p,q,r)} = 2(m+\alpha) b_{lmn}^{(p,q-1,r)}, q \geq 1$$

and
$$b_{l,m,n-1}^{(p,q,r)} - b_{l,m,n+1}^{(p,q,r)} = 2(n+\alpha) b_{lmn}^{(p,q,r-1)}, r \geq 1,$$

which, in turn, yield

$$b_{lmn}^{(p,q,r)} = 2 \sum_{i=1}^{\infty} (l + 2i + \alpha - 1)\, b_{l+2i-1,m,n}^{(p-1,q,r)},\ p \geq 1, \quad \text{... (33)}$$

$$b_{lmn}^{(p,q,r)} = 2 \sum_{j=1}^{\infty} (m + 2j + \alpha - 1)\, b_{l,m+2j-1,n}^{(p,aq-1,r)},\ q \geq 1 \quad \text{... (34)}$$

and

$$b_{lmn}^{(p,q,r)} = 2 \sum_{k=1}^{\infty} (n + 2k + \alpha - 1)\, b_{l,m,n+2k-1}^{(p,q,r-1)},\ r \geq 1. \quad \text{... (35)}$$

Now, we state without proof the following theorem, which is to be considered as an extension of Theorem 1 of Section 2.

***Theorem* 2** — *The coefficients* $b_{lmn}^{(p,q,r)}$ *are related to the coefficients with superscripts* $(0, q, r)$, $(p, 0, r)$, $(p, q, 0)$, $(0, 0, r)$, $(0, q, 0)$, $(0, 0, p)$ *and* b_{lmn} *by*

$$b_{lmn}^{(p,q,r)} = \frac{2^p}{(p-1)!} \sum_{i=1}^{\infty} \frac{(i+p-2)!\, \Gamma(l+i+p+\alpha-1)}{(i-1)!\, \Gamma(l+i+\alpha)} (l + 2i + p + \alpha - 2)$$

$$b_{l+2i+p-2,m,n}^{(0,q,r)},\ p \geq 1, \quad \text{... (36)}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^q}{(q-1)!} \sum_{j=1}^{\infty} \frac{(j+q-2)!\, \Gamma(m+j+q+\alpha-1)}{(j-1!\, \Gamma(m+j+\alpha)}$$

$$(m + 2j + q + \alpha - 2)\, b_{l,m+2j+q-2,n}^{(p,0,r)},\ q \geq 1, \quad \text{... (37)}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^r}{(r-1)!} \sum_{k=1}^{\infty} \frac{(k+r-2)!\, \Gamma(n+k+r+\alpha-1)}{(k-1)!\, \Gamma(n+k+\alpha)}$$

$$(n + 2k + r + \alpha - 2)\, b_{l,m,n+2k+r-2}^{(p,q,0)},\ r \geq 1, \quad \text{... (38)}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{p+q}}{(P-1)!\,(q-1)!}$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(i+p-2)!\,(j+q-2)!\, \Gamma(l+i+p+\alpha-1)}{(i-1)!\,(j-1)!\, \Gamma(l+i+\alpha)}$$

$$\times \frac{\Gamma(m+j+q+\alpha-1)}{\Gamma(m+j+\alpha)} (l + 2i + p + \alpha - 2)(m + 2j + q + \alpha - 2)$$

$$b_{l+2i+p-2,m+2j+q-2,n}^{(0,0,r)},\ p, q \geq 1, \quad \text{... (39)}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{p+r}}{(p-1)!\,(r-1)!} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{(i+p-2)!\,(k+r-2)!\,\Gamma(l+i+p+\alpha-1)}{(i-1)!\,(k-1)!\,\Gamma(l+i+\alpha)}$$

$$\times \frac{\Gamma(n+k+r+\alpha-1)}{\Gamma(n+k+\alpha)} (l+2i+p+\alpha-2)\,(n+2k+r+\alpha-2)\, b^{(0,q,0)}_{l+2i+p-2,\,m,\,n+2k+r-2},$$

$$p, r \geq 1 \qquad \text{... (40)}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{q+r}}{(q-1)!\,(r-1)!} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(j+q-2)!\,(k+r-2)!\,\Gamma(m+j+q+\alpha-1)}{(j-1)!\,(k-1)!\,\Gamma(m+j+\alpha)}$$

$$\times \frac{\Gamma(n+k+r+\alpha-1)}{\Gamma(n+k+\alpha)} (m+2j+q+\alpha-2)\,(n+2k+r+\alpha-2)$$

$$b^{(p,0,0)}_{l,\,m+2j+q-2,\,n+2k+r-2}, \; q, r \geq 1, \qquad \text{... (41)}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{p+q+r}}{(p-1)!\,(q-1)!\,(r-1)!} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(i+p-2)!\,(j+q-2)!\,(k+r-2)!}{(i-1)!\,(j-1)!\,(k-1)!}$$

$$\times \frac{\Gamma(l+i+p+\alpha-1)\,\Gamma(m+j+q+\alpha-1)\,\Gamma(n+k+r+\alpha-1)}{\Gamma(l+i+\alpha)\,\Gamma(m+j+\alpha)\,\Gamma(n+k+\alpha)} (l+2i+p+\alpha-2)$$

$$\times (m+2j+q+\alpha-2)\,(n+2k+r+\alpha-2)$$

$$b_{l+2i+p-2,\,m+2j+q-2,\,n+2k+r-2}, \; p, q, r \geq 1. \qquad \text{... (42)}$$

The formulae corresponding to expansions in triple Chebyshev polynomials of the first and second kinds and of triple Legendre polynomials may be obtained as special cases by taking $\alpha = 0, 1, \frac{1}{2}$ respectively in formula (32) with (36)-(42).

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