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GROUND TRACE OF A SATELLITE MOVING UNDER THE GRAVITATIONAL FORCES OF THE SUN, THE MOON, THE EARTH AND THE SOLAR RADIATION PRESSURE

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A theory for the ground trace of a satellite moving under the gravitational forces of the sun, the moon, the earth (including ellipticity of the earth's equator) and the solar radiation pressure has been developed. The ground trace has been drawn in the cases: (*i*) orbit in the reference plane and (*ii*) orbit in the equatorial plane. Both the above cases have been studied for (*i*) $J_2^{(2)} = 0, q = 1$ (*ii*) $J_2^{(2)} \neq 0, q = 1$, (*iii*) $J_2^{(2)} = 0, q \neq 1$ (*iv*) $J_2^{(2)} \neq 0, q \neq 1$ ($J_2^{(2)}$ = equatorial ellipticity of the earth, q = solar radiation parameter). It is shown that the graphs take the shape of Fig. 8 crossing the equator twice, due to earth's equatorial ellipticity ($|\Gamma_0| = 90^{\circ}$, q = 1) alone or the solar radiation pressure (q = 0.9) alone or taking into account both the effects, the ground trace shrinks itself in the direct of the latitude of the satellite and this change is more due to solar radiation pressure than due to earth's equatorial ellipticity.

Key Words : Satellite; Gravitational Forces; The Sun; The Moon; The Earth; Ground Three; Equatorial Ellipticity

Notations

A_i, B_l, C_l, D_l : amplitudes of the *i*th oscillatory components in the equations of motion.

q : solar radiation pressure parameter.

r : vector from the centre of the earth to the satellite

t : time

T : duration of regression

α : inclination of the satellite orbital plane to the reference plane.

α_1 : inclination of the reference plane to the ecliptic.

α_m : inclination of the moon's orbital plane to the ecliptic.

β : Co-efficient due to the solar radiation pressure.

Γ : angle measured from the minor axis of the earth's equatorial ellipse to the satellite.

λ : longitude of the satellite

γ : latitude of the satellite

ϕ : orbital angle of the earth around the sun

θ : orbital angle of the satellite round the earth.

ψ : satellite orbital regression angle

ω_i : oscillatory frequency of the ith component in the equations of the motion.

$\gamma_0, \alpha_0, \theta_0, \psi_0$: steady state values of the corresponding variables

T_R : Regression rate of the orbital plane

G : Universal gravitational constant = 6.668×10^{-8} dyne/cm^2.

J_2 : Coefficient due to the oblateness of the earth = 1.08219×10^{-3}

$J_2^{(2)}$: Coefficient due to the earth's equatorial ellipticity = 2.32×10^{-6}

M_s : mass of the sun = (332, 946 $\times M_E$) gm.

M_E : mass of the earth = 600.06780×10^{25} gm.

M_m : mass of the moon = 7.380×10^{25} gm.

R_0 : mean earth radius = 3963 miles.

R : distance between the centre of the sun to the centre of mass of the earth-moon system = 1.4959965×10^8 gm.

ε : obliquity = 23°27'

$\dot{\theta}_m$: moon's orbital rate = 0.22998 rad/solar day

$\dot{\psi}_m$: regression rate of the moon's = $-\ 9.249 \times 10^{-4}$ rad/solar day orbital plane.

$\dot{\phi}$: earth's angular rate = 0.0172 rad/solar day

$\dot{\theta}_E$: earth's rotation rate = 6.3004 rad/solar day

INTRODUCTION

Bhatnagar and Mehra[1 & 2] have studied the motion of a satellite under the gravitational forces of the sun, the moon, the earth (including ellipticity of the earth's equator) and solar radiation pressure and have further studied the orientation of the orbital plane of a geosynchronous satellite.

In paper (I), they have determined the equations of motion in a synodic coordinate system. Further, they have deduced the equations of motion when (i) orbit is in the equatorial plane and (*ii*) the satellite is a geosynchronous. In paper (II), the orientation of the orbital plane of a geosynchronous satellite is studied. It is found that the orbital inclination α contains a secular term and the effect of the oscillatory terms in α and ψ is such that for values of T which are significantly greater than the oscillatory periods included, the values for the drift ΔD are independent of T and less than 0.5° and there is further change in ΔD of the order of 10^{-1} due to earth's equatorial ellipticity and solar radiation pressure. Bhatnagar and Manjit[3] have shown that the in-plane perturbation of the satellite caused by the attraction of the sun and moon and the oblate earth are in the nature of large amplitude oscillations because of the presence of low-frequency terms in the denominator where as in Frick *et al.*[5] these perturbations are in the nature of small amplitude oscillation which could result in a maximum deviation from the desired synchronous position of about 72.66 km as he has studied only one case, i.e., when $\alpha_0 = 0^o$ and in that case all the low-frequency terms vanish. In another paper, Bhatnagar and Mehra[4] have determined the regression period T_R of the orbital plane. It is noted that the regression period always decreases due to the solar radiation pressure but due to the ellipticity as $|\Gamma_0|$ varies from 45° to 90°, the regression period decreases and as $|\Gamma_0|$ varies from 45° to 0°, it increases.

The equations of motion are[1]

$$\dot{\alpha} = \dot{\alpha}_0 + \sum_{i=1}^{182} A_i \sin \omega_i t, \quad \text{... (1)}$$

$$\dot{\psi} = \dot{\psi}_0 + \frac{1}{\sin \alpha_0} \sum_{i=1}^{182} B_i \cos \omega_i t, \quad \text{... (2)}$$

$$(\ddot{r} - r\dot{\theta}^2) + \frac{GM_E}{r^2} = \xi + \sum_{i=1}^{179} C_i \cos \omega_i t \quad \text{... (3)}$$

and

$$\frac{d}{dt}(r^2 \dot{\theta}) = \eta + \sum_{i=1}^{179} D_i \sin \omega_i t, \quad \text{... (4)}$$

where α, ψ determine the orbital plane and (r, θ) the position of the satellite in the orbital plane

$$\dot{\alpha}_0 = \frac{-3\,\dot{\theta}_0 J_2^{(2)} R_0^2}{r_0^2} \cos(\varepsilon - \alpha_1) \sin \alpha_0 \sin 2\Gamma_0,$$

$$\dot{\psi}_0 = -\frac{3\,\dot{\phi}^2}{8\,\dot{\theta}_0}\left[\left\{1 + \frac{\dot{\theta}^2_m}{2\mu\dot{\phi}^2}(2\text{–}3\sin^2\alpha_m)\right\}(2\text{–}3\sin^2\alpha_1) + \frac{3\beta}{\dot{\phi}^2 R^3}\sin^2\alpha_1\right.$$

$$\left. + \frac{2J_2\,\dot{\theta}_0^2 R_0^2}{\dot{\phi}^2 r_0^2}(2\text{–}3\sin^2(\varepsilon - \alpha_1))\right]\cos\alpha_0 + \frac{3\,\dot{\theta}_0 J_2^{(2)} R_0^2}{2\,r_0^2}\{2\text{-}3\ \sin^2(\varepsilon - \alpha_1)\}$$

$$x \cos \alpha_0 \cos 2\,\Gamma_0, \quad \text{... (5)}$$

$$\xi = \frac{-3\,G M_E J_2^{(2)} R_0^2}{8\,r_0^4}\{2\text{-}3\ \sin^2(\varepsilon - \alpha_1)\}\ (2\text{-}3\ \sin^2\alpha_0) + \frac{r_0\,\dot{\phi}^2}{8}(2\text{–}3\sin^2\alpha_0) \times$$

$$(2\text{-}3\ \sin^2\alpha_1) + \frac{r_0\,\dot{\theta}_m^2}{16\,\mu}(2\text{–}3\sin^2\alpha_0)\,(2\text{–}3\sin^2\alpha_1)\,(2\text{–}3\sin^2\alpha_m) + \frac{r_0\,\beta}{R^3}$$

$$+ \frac{9\, G\, M_E\, J_2^{(2)}\, R_0^2}{4\, r_0^4} [(1 + \cos^2 \alpha_0)\ \{1\ +\ \cos^2 (\varepsilon - \alpha_1)\}$$

$$+\ 2 \sin^2 (\varepsilon - \alpha_1) \sin^2 \alpha_0] \cos 2\, \Gamma_0$$

and
$$\eta = 6\, \dot{\theta}_0^2\, J_2^{(2)}\, R_0^2 \cos (\varepsilon - \alpha_1) \cos \alpha_0 \sin 2\, \Gamma_0.$$

α_1 is defined by

$$\tan 2\alpha_1 = \frac{2\left(K - \frac{2\, \dot{\theta}_0^2\, J_2^{(2)}\, R_0^2}{\dot{\phi}^2\, r_0^2} \cos 2\, \Gamma_0 \right) \sin 2\, \varepsilon}{\left[\frac{1 + \dot{\theta}_m^2}{2\, \mu\, \dot{\phi}^2} (2\text{–}3 \sin^2 \alpha_m) \frac{-\beta}{\dot{\phi}^2\, R^3} \right] + 2\left(k - \frac{2\, \dot{\theta}_0^2\, J_2^{(2)}\, R_0^2}{\dot{\phi}^2\, r_0^2} \cos 2\, \Gamma_0 \right) \cos 2\, \varepsilon} \quad \text{... (6)}$$

We may note that Γ_0 during the course of study has been used in two ways (*i*) Γ_0, the value of Γ at a particular instant (*ii*) Γ_0, the value of Γ at synchronous altitude, that is, the steady state value of Γ. Further

$$\mu = \frac{M_E + M_m}{M_m},\ \dot{\theta}_0^2 = \frac{GM_E}{r_0^3},\ \beta = GM_s\,(1 - q),\ K = \frac{GM_E\, J_2\, R_0^2}{\dot{\phi}^2\, r_0^5}$$

The amplitude A_i, B_i and C_i, D_i are functions of any or all the quantities, $\alpha_0, \alpha_1, \alpha_m$ $\gamma_0, J_2, J_2^{(2)}, \beta$ and the frequencies ω_i are linear combinations of $\dot{\theta}_0, \dot{\theta}_m, \dot{\phi}\ \dot{\psi}_m, \dot{\psi}_0$. All symbols have been defined in the list of notations.

For an unperturbed equatorial orbit, synchronous orbit is achieved when the satellite's orbital angular velocity is equal to the earth's spin rate. In that case the satellite will always remain stationary above a fixed point on the equator. But, if the satellite's orbital plane is inclined to the equator at an angle, α_G, the subsatellite point (the corresponding point on the earth) will no longer remain fixed since the latitude of the satellite will vary between, $+\ \alpha_G$, and $-\ \alpha_G$, during each completion of the orbit. But, if the orbital angular velocity of the satellite is again set equal to the angular velocity of the earth, the subsatellite point will trace out a fixed curve in the form of figure eight on the surface of the rotating earth. Besides, the inclination of the orbital plane, if perturbations which result in the regression of the orbital plane are taken, the value of the angular velocity of the satellite is modified in order that it takes care of the regression in the orbital plane.

Consider Fig. 1. Dotted lines indicate the initial orbital plane and X_1, the initial position of the subsatellite point. Suppose after n crossings of the reference plane by the satellite (after time t_n) the subsatellite point is at S_n and the corresponding point of the earth (which was initially at X_1) is at A.

Let $X_1\, A = \lambda_n, X_1\, S_n = \psi_n$, then $\lambda_n = \dot{\theta}_E\, t_n - 2\, \pi\, n,\ \psi_n = \dot{\psi}_0\, t_n,\ \dot{\theta}_0\, t_n = 2\, \pi\, n.$

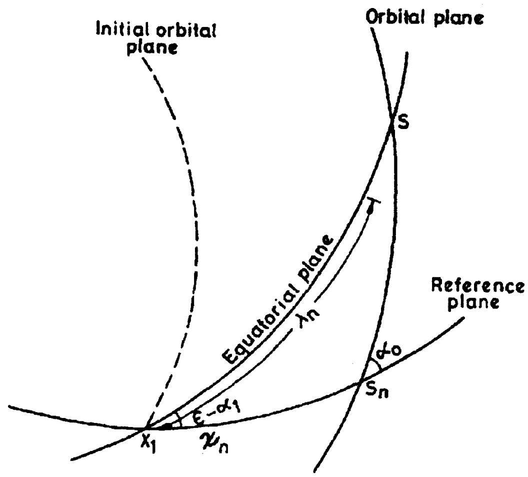

FIG. 1. Determination of condition for synchronism

For a synchronous satellite

$$\psi_n = \lambda_n$$

Therefore, $\dot{\theta}_0 = \dot{\theta}_E - \dot{\psi}_0$

With this angular velocity, $\dot{\theta}_0$, the ground trace is again figure eight though, its size and position relative to earth changes as the time elapses.

2. Ground Trace

Consider Fig. 2. Here λ and γ represent longitude and latitude respectively of the satellite X with respect to the equator at any time t_1 (supposing the satellite has already made n crossing of the

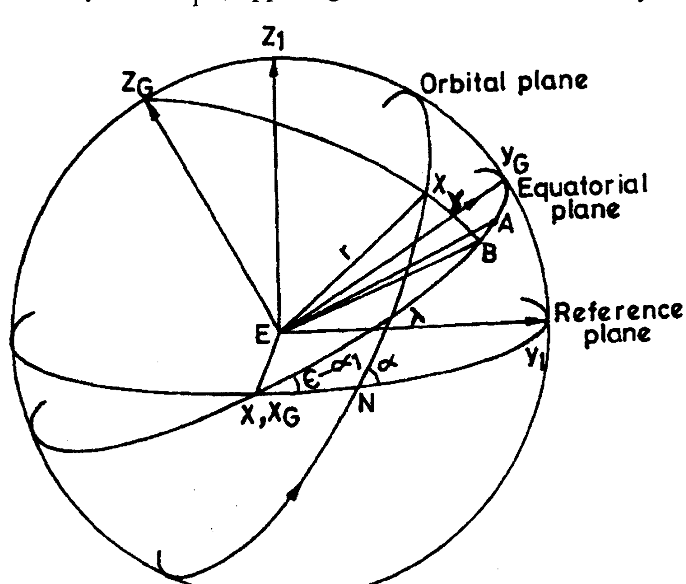

FIG. 2. Location of sub-satellite point A $X_1B = \lambda, BX = \gamma X_1N = \psi X_1A = \lambda_1, NX = \theta$

equator) and λ_1 the longitude of the corresponding subsatellite point on the equator. Further, let the initial position of the satellite and the subsatellite point on the equator be at X_1.

Then, $X_1 B = \lambda,$

$$XB = \gamma,$$

and $$X_1 A = \lambda_1.$$

The longitude, latitude of X and (λ, γ) and the longitude, latitude of A are (λ_1, O). The relative longitude and latitude of the satellite X with respect to the subsatellite point A are $(\lambda - \lambda_1, \gamma)$ or $(\Delta \lambda, \gamma)$. The locus of $(\Delta \lambda, \gamma)$ gives the required ground trace on the rotating earth λ, λ_1 and γ are determined as follows :

We have,

$$I = \hat{EX}$$
$$= \cos \gamma \cos \lambda I_G + \cos \gamma \sin \lambda J_G + \sin \gamma K_G.$$

Also, $I = a_x I_G + [b_x \cos(\varepsilon - \alpha_1) + c_x \sin(\varepsilon - \alpha_1)] J_G + [-b_x \sin(\varepsilon - \alpha_1) + c_x \cos(\varepsilon - \alpha_1)] k_G$

Equating like coefficients and simplification gives

$$\tan \lambda = \frac{b_x \cos(\varepsilon - \alpha_1) + c_x \sin(\varepsilon - \alpha_1)}{a_x}$$

and $\sin \gamma = b_x \sin(\varepsilon - \alpha_1)\, c_x \cos(\varepsilon - \alpha_1)$.

For calculating λ_1 consider Fig. 3. Suppose at time $t = 0$, the satellite crosses the equatorial plane at X_1 and after n crossing of the equatorial plane, suppose the time elapsed is T and it crosses the equatorial plane at S.

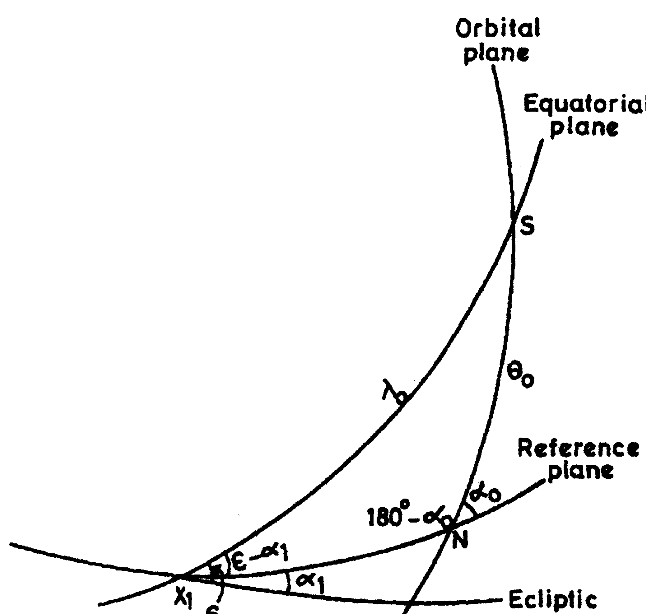

FIG. 3. Equatorial crossing X_1 = Initial equatorial crossing (t = 0) S = nth equatorial crossing (t = T)

Let $X_1 S = \lambda_0$. From the spherical triangle $X_1 SN$ at time T, we have

$$\tan \theta_0 = \frac{\sin \psi_0 \sin (\varepsilon - \alpha_1)}{\sin \alpha_0 \cos (\varepsilon - \alpha_1) - \cos \psi_0 \cos \alpha_0 \sin (\varepsilon - \alpha_1)}$$

and $$\psi_0 = \dot{\psi}_0 T.$$

Suppose, the nth crossing of the reference plane by the satellite takes place at a time earlier by a time t_0, that is, at time $T - t_0$. Then

$$NS = \theta_0 = \dot{\theta}_0 t_0$$

and $$\dot{\theta}_0 = \theta_0 / t_0.$$

Consider the general position of the satellite at time t_1 (Fig. 4) Then

$$t_1 = T - t_0 + t,$$

$$\psi = \dot{\psi}_0 t_1,$$

$$\theta = \dot{\theta}_0 t_1 - 2\pi n,$$

$$\lambda_1 = \dot{\theta}_E t_1 - 2\pi n,$$

$$\dot{\theta}_0 = \dot{\theta}_E - \dot{\psi}_0,$$

$$2\pi n = \dot{\theta}_0 (T - t_0),$$

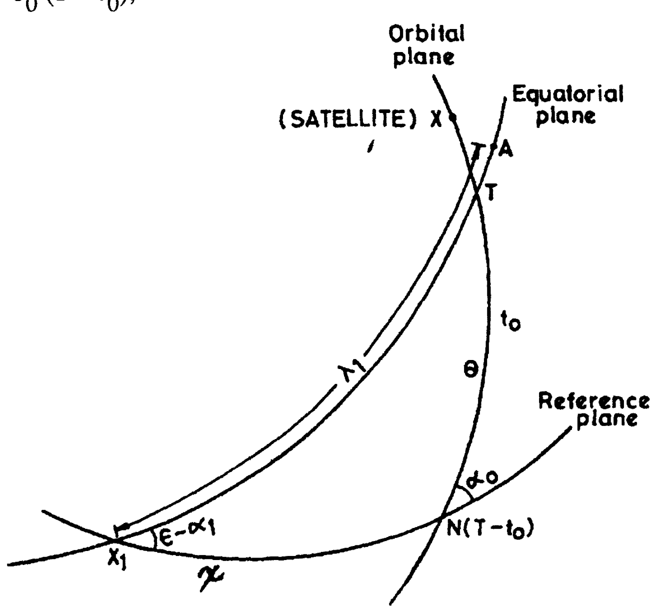

FIG. 4. Crossing of reference plane and equatorial plane N, Reference plane crossing point just prior equatorial crossing (Time = $T - t_0$)

$$\dot{\theta}_0 = \theta_0/t_0,$$

$$\lambda_1 = \dot{\theta}_E t_1 - 2\pi n = \theta + \psi,$$

$$\theta = \dot{\theta}_0 t,$$

$$\psi = \dot{\psi}_0 (T - t_0 + t).$$

Thus, the ground trace is given by the locus of $(\Delta\lambda, \gamma)$,

where,
$$\Delta\lambda = \lambda - \lambda_1, \quad \text{... (7)}$$

$$\gamma = \sin^{-1} [- b_x \sin(\varepsilon - \alpha_1) + c_x \cos(\varepsilon - \alpha_1)] \quad \text{... (8)}$$

and
$$\lambda = \tan^{-1} \left[\frac{b_x \cos(\varepsilon - \alpha_1) + c_x \sin(\varepsilon - \alpha_1)}{a_x} \right] \quad \text{... (9)}$$

$$\lambda_1 = \theta + \psi, \quad \text{... (10)}$$

$$a_x = \cos\theta \sin\psi - \sin\theta \sin\psi \cos\alpha,$$

$$b_x = \cos\theta \sin\psi + \sin\theta \cos\psi \cos\alpha$$

and
$$c_x = \sin\theta \sin\alpha.$$

For a given value of T (time after n crossings), eqs. (7), (8), (9), and (10) giving $\Delta\lambda$, γ in terms of t over one orbital period, give the ground trace. By taking various values of T over one regression period the phase of the regression can be shown for various initial orbital inclination α_0.

4. Special Cases of Ground Trace

The ground trace has been studied in the following cases :

(I) Orbit in the Reference Plane

(II) Orbit in the Equatorial Plane

Both cases have been discussed in th efollowing four cases

Case 1 — $J_2^{(2)} = 0, q = 1$ [see ref. 6]

Case 2 — $J_2^{(2)} \neq 0 \cdot q = 1.$

Case 3 — $J_2^{(2)} = 0 \cdot q \neq 1.$

Case 4 — $J_2^{(2)} \neq 0 \cdot q \neq 1.$

(I) Orbit in the Reference Plane

The latitude γ of the subsatellite point, it's change $\Delta\gamma$ and longitude difference

$\Delta\lambda = (\lambda - \lambda_1)$ are all measured in degrees and the time $t\,(0 \le t \le 24)$ in hours.

If the orbital plane coincides with the reference plane then $\alpha_0 = 0$, therefore,

$$a_x = \cos(\theta + \psi),$$

$$b_x = \sin(\theta + \psi),$$

$$c_x = 0,$$

and $$T = t_0$$

$$\tan\lambda = \cos(\varepsilon - \alpha_1)\tan(\theta + \psi),$$

$$\lambda_1 = \dot{\theta}_E t$$

Thus,

$$\tan\lambda = \cos(\varepsilon - \alpha_1)\tan\dot{\theta}_E t.$$

$$\sin\gamma = -\sin(\varepsilon - \alpha_1)\sin\dot{\theta}_E t.$$

Hence, the ground trace is given by $(\Delta\lambda, \gamma)$ where

$$\Delta\lambda = \lambda - \lambda_1 = \tan^{-1}[\cos(\varepsilon - \alpha_1)\tan\dot{\theta}_E t] - \dot{\theta}_E t.$$

and $$\gamma = \sin^{-1}[-\sin(\varepsilon - \alpha_1)\sin\dot{\theta}_E t].$$

The above equations are independent of T which means that the ground trace continues to repeat itself since this particular orbit remains fixed in inertial space. Time t is measured from the instant when the satellite crosses the reference plane.

Case 1 — $J_2^{(2)} = 0, q = 1.$

This case is represented in Fig. 5 which represents the relation $\gamma = \gamma(\Delta\lambda)$ for a synchronous altitude when $J_2^{(2)} = 0$, $q = 1$. The values of γ and $\Delta\lambda$ are worked out at values of $t\,(0^h \le t, \le 24^h)$ at an interval of one hour. It is seen that the graph takes the shape of Fig. eight crossing the equator twice. Maximum numerical value of γ is $(\varepsilon - \alpha_1)$ or 7^o 2944' at t = 6 and 18 hrs which is the inclination of the orbital plane to the equatorial plane and maximum numerical value of $\Delta\lambda$ is 0.2328^o at t = 3,9,15 and 21 hrs. The figure eight of the ground trace continues to repeat itself as this particular orbit remains fixed in inertial space.

Change in latitude γ only shall be considered in the subsequent three cases as the change in the longitude $\Delta\lambda$ is very small as can be seen from the Tables I, II & III.

TABLE I

Change in longitude $\Delta\lambda$ due to the earth's equatorial ellipticity alone ($|\Gamma_0| = 0^o$)

t (hrs)	0	2	4	6	8	10	12	14	16	18	20	22	24
Change in $\Delta\lambda$	0	– .0029	–0029	0	–.0100	0	0	–.0100	0	0	0	0	0

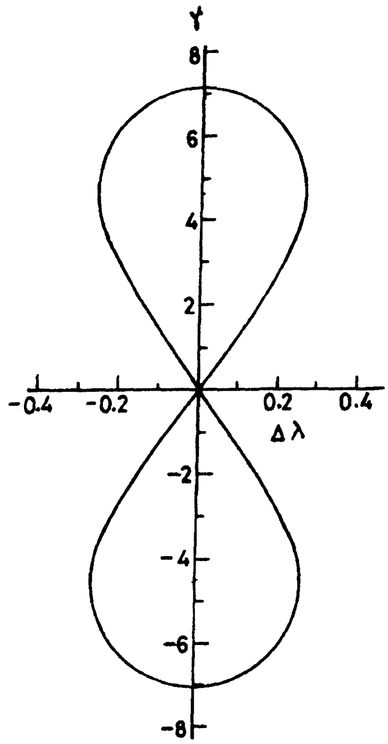

FIG. 5. Ground trace-orbit in reference plane (Frick case $J_2^{(2)} = 0, q = 1$)

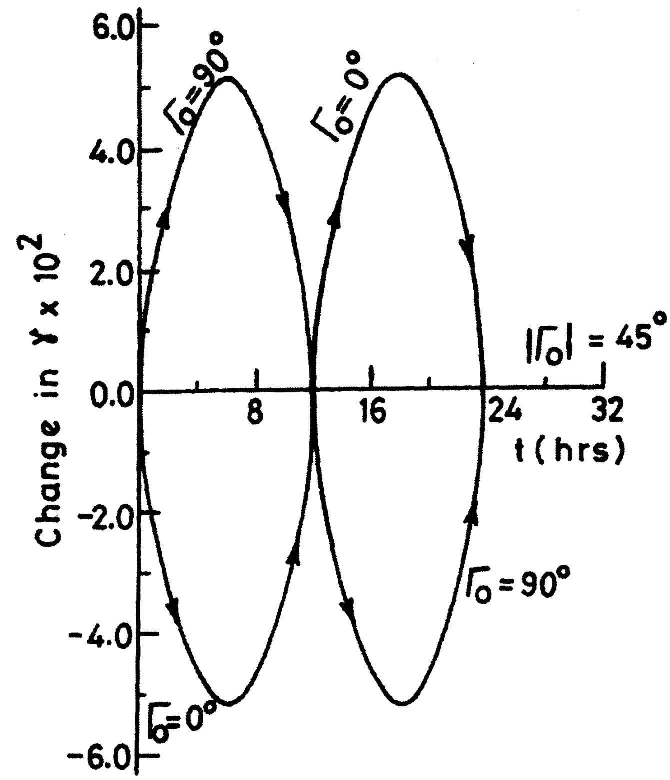

FIG. 6. Change in latitude of ground trace for an orbit in reference plane due to ellipticity ($J_2^{(2)} = 0, q = 1$)

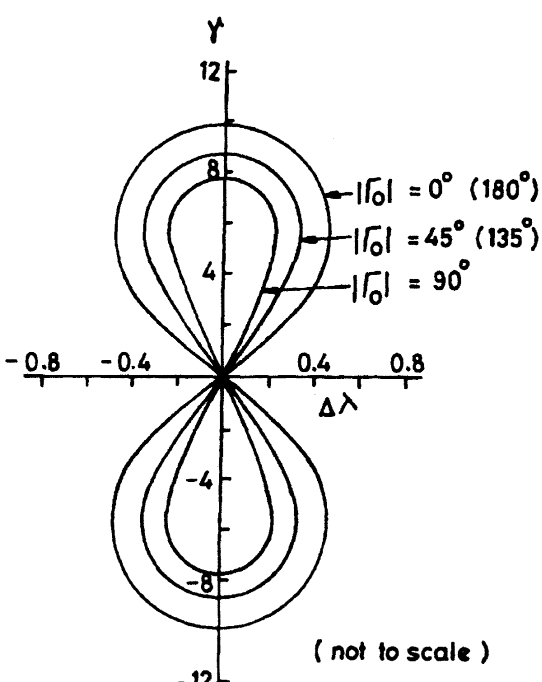

FIG. 7. Ground trace-orbit in reference plane (Frick case $J_2^{(2)} \neq 0, q = 1$) effect of ellipticity

Case 2 — $J_2^{(2)} \neq 0, q = 1$

This case is represented in Fig. 6 which represents the relation $\Delta\gamma = \Delta\gamma(t)$ for a synchronous altitude when $J_2^{(2)} \neq 0, q = 1$. $\Delta\lambda$means the change in the latitude γ of the ground trace versus $t\,(0^h \leq t \leq 24^h)$ due to the earth's equatorial ellipticity. Since γ is negative for $0^h \leq t \leq 12^h$ and positive for $12^h \leq t \leq 24^h$ and the change during these periods is also negative and positive respectively corresponding to $|\Gamma_0| = 0^0$, therefore the ground trace expands in the direction of γ as $|\Gamma_0|$ varies from 45^o to 0^0. For $|\Gamma_0| = 90^o$ the change in γ is exactly the image of what it is corresponding to $|\Gamma_0| = 0^o$. Similarly, it can be inferred that the ground trace shrinks in the direction of γ as $|\Gamma_0|$ varies from 45^o to 90^o. This fact is shown in Fig. 7. Corresponding to $|\Gamma_0| = 0^o$, 45^o , 90^o. It may be noted that the scale is not uniform. Also, the ground trace oscillates between the

TABLE II

Change in longitude $\Delta\lambda$ due to the solar radiation pressure alone (q = 0.9)

t (hrs)	0	2	4	6	8	10	12	14	16	18	20	22	24
Change in $\Delta\lambda$	0	.0093	.0093	0	–.0100	–.0100	0	.0100	.0100	0	–.0100	–.0100	0

TABLE III

Change in longitude $\Delta\lambda$ due to the combined effect of the earth's equatorial ellipticity and the solar radiation pressure ($|\Gamma_0| = 0^o, q = 0.9$)

t (hrs)	0	2	4	6	8	10	12	14	16	18	20	22	24
Change in $\Delta\lambda$	0	.0065	.0065	0	0	–.0100	0	0	.0100	0	–.0100	–.0100	0

two ground traces corresponding to $|\Gamma|_0 = 0^o$ and 90^o with it's mean position corresponding to $|\Gamma|_0 = 45^o$. The value of γ oscillates between 7.2419^o and 7.3464^o for $0^o \leq |\Gamma_0| \leq 180^o$.

Case 3 — $J_2^{(2)} = 0, q \neq 1$

This case is represented in Fig. 8 which represents the relation $\Delta\gamma = \Delta\gamma(t)$ for a synchronous altitude when $J_2^{(2)} = 0, q \neq 1$. $\Delta\gamma$ means the change in the latitude γ of the ground trace versus $t\,(0^h \leq t \leq 12^h)$ due to the solar radiation pressure alone corresponding to three fixed values of q namely q = 0.8, 0.8, 0.9. Since the value of γ is negative during $0^h \leq t \leq 12^h$ and positive during $12^h \leq t \leq 24^h$ and $\Delta\gamma$ is positive during $0^h \leq t \leq 12^h$ and negative during $12^h \leq t \leq 24^h$, the ground trace shrinks in the direction of γ. As q decreases, the ground trace further shrinks. The value of γ lies between 7.0725^o and 7.2938^o corresponding to q = 0.9, between 6.8999^o and 7.0014^o corresponding to q = 0.8 and between 6.7240^o and 6.8239^o corresponding to q = 0.7.

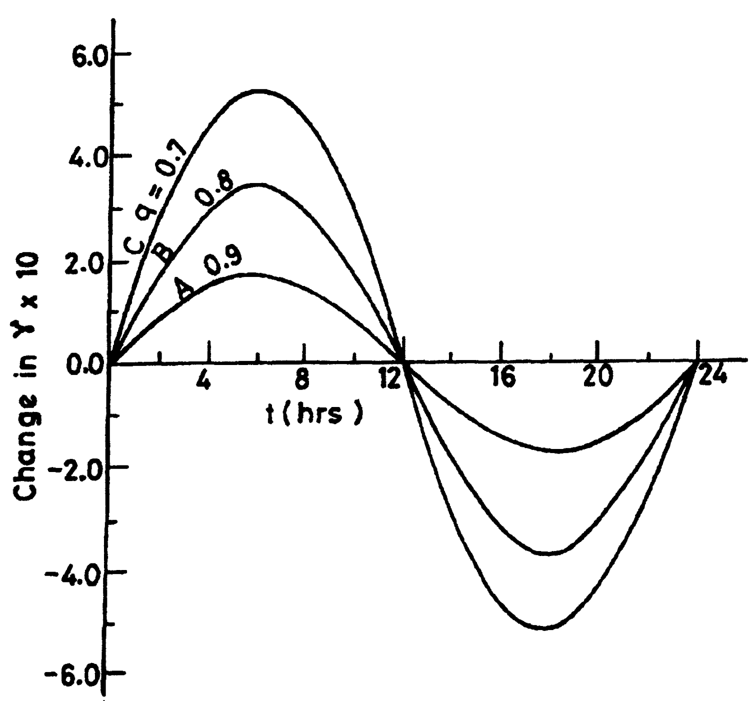

FIG. 8. Change in latitude of ground trace for an orbit in reference plane due to solar radiation pressure ($J_2^{(2)} = 0, q \neq 1$)

Case 4 — $J_2^{(2)} \neq 0, q \neq 1$

This case is represented the Fig. 9 and 10 which represent the relation $\Delta\gamma = \Delta\gamma(t)$ for a synchronous altitude when $J_2^{(2)} \neq 0, q \neq 1$ as compared to the case when both the effects are not there is when $J_2^{(2)} = 0$, $q = 1$.

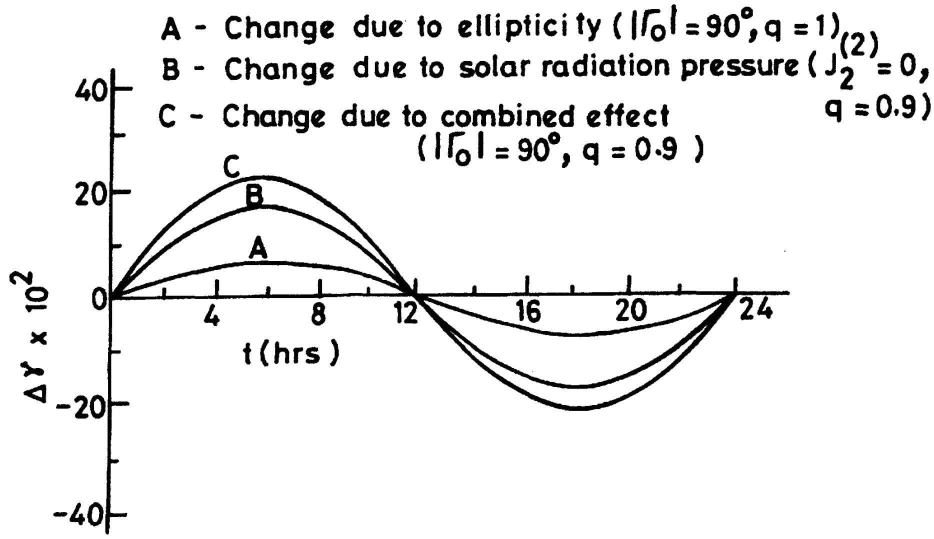

FIG. 9. Change in latitude of ground trace in reference plane

From Fig. 9 it is observed that due to earth's equatoria ellipticity ($|\Gamma_0| = 90^o$, $q = 1$) alone or the solar radiation pressure ($q = 0.9$) alone or taking into account both the effects $\Delta\gamma$ is positive for $0^h \leq t \leq 12^h$ and negative for $12^h \leq t \leq 24^h$ and therefore in each case, the ground trace will shrink itself in the direction of γ. Also, change in γ is more due to solar radiation pressure than due to the earth's equatorial ellipticity. The combined effect of both these phenomenon is accumulated in curve *C*. Maximum $\Delta\gamma$ occurs at $t = 6$ hours and minimum at $t = 18$ hours.

From Fig. 10, $\Delta\gamma$ nis more due to solar radiation pressure than due to the earth's equatorial ellipticity, so the over all effect of them is that for $0^h \leq t \leq 1\ 2$, $\Delta\gamma$ is positive and for $12^h \leq t \leq 24^h$ it is negative (curve C) and hence due to the combined effect the ground trace will shrink in the direction of γ. Due to the combined effect, the ground trace expands as $|\Gamma_0|$ varies from 0^o to 45^o, shrinks as $|\Gamma_0|$ varies from 45^o to 90^o, expands for 90^o to 135^o and shrinks for 135^o to 180^o. The numerical value of γ for $0^o \leq |\Gamma_0| \leq 180^o$, $q = 0.9$ lies between 7.0730^o and 7.2944^o. Also change in γ is more due to solar radiation pressure than due to earth's equatorial ellipticity or solar radiation pressure whether taken alone or together is extremely small.

Table IV gives latitude γ for ground for $J_2^{(2)} \neq 0$, $q = 0.8$ or different values of $|\Gamma_0|$

TABLE IV

Latitude γ for ground trace for $J_2^{(2)} = 0$, $q = .9$ (negative for $t = 6$ hour and positive for $t = 18$ hours)

$\lvert\Gamma_0\rvert$	0^o	45^o	90^o	135^o	180^o
γ	$\mp\ 7.1761^o$	$\mp\ 7.2955^o$	$\mp\ 7.030^o$	$\mp\ 7.2944^o$	$\mp\ 7.1761^o$

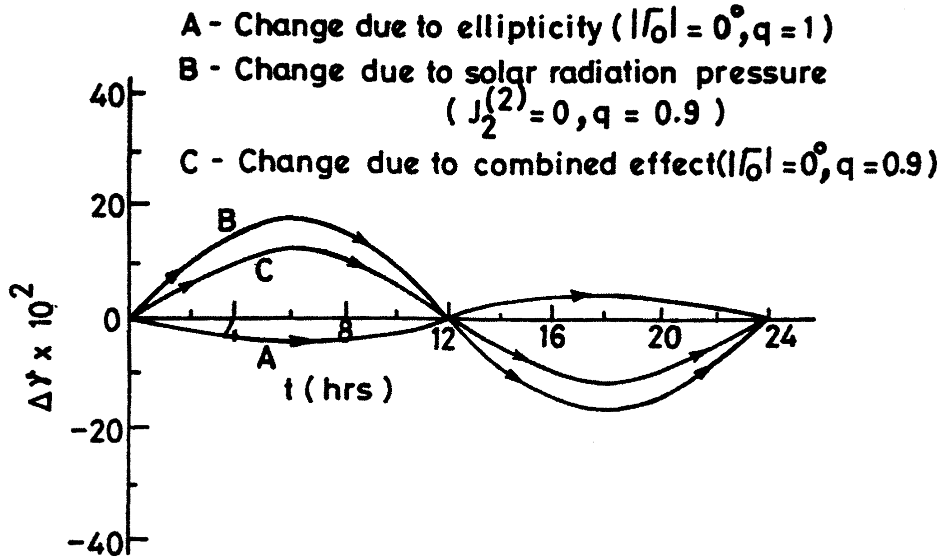

FIG. 10. Change in latitude of ground trace in reference plane

(II) Orbit in the Equatorial Plane

In this case the orbital plane is initially in the earth's equatorial plane and $\alpha_0 = (\varepsilon - \alpha_1)$. The condition for synchronous altitude is determined by equation $\dot{\theta}_0 = \dot{\theta}_E - \dot{\psi}_0$. On substituting the values of $\dot{\theta}_0$, $\dot{\theta}_E$ and $\dot{\psi}_0$, synchronous altitude can be determined for different values of orbital inclination α_0 ($0^\circ \le \alpha_0 \le 90^\circ$), solar radiation parameter q (1 and 0.9) and ellipticity parameter, Γ_0 (0° and 45°). It is seen that for different q and Γ_0 there is no change in the synchronous altitude. However, it increases with α_0 (Table V).

TABLE V
Synchronous altitude for different orbital inclination α_0

α_0 (days)	r_0 (km)
0°	42296.0
10°	42296.1
20°	42296.1
30°	42296.2
40°	42296.4
50°	42296.6
60°	42296.8
70°	42297.0
80°	42297.3
90°	42297.5

The regression period at synchronous altitude is given by

$$T_R = \frac{2\pi}{|\dot{\psi}_0|}$$

and regression $\dot{\psi}_0$ by eq. (5). The regression period is tabulated in Table VI. It is seen that the orbital regression varies between 52.49671 years and 53.50745 years.

TABLE VI
Regression period corresponding to different Γ_0 and q

$\lvert\Gamma_0\rvert$	q	T_R in years
90°	1	52.50745 years
	0.9	53.41937 years
	0.8	53.33023 years
	0.7	53.24002 years
45°	1	53.13046 years
	0.9	53.04328 years
	0.8	52.95506 years
	0.7	52.86579 years
0°	1	52.75867 years
	0.9	52.67237 years
	0.8	52.58506 years
	0.7	52.49671 years

The ground trace of the subsatellite point on the earth is determined by setting $(\varepsilon - \alpha_1) = \alpha_0$ that is, by the equations

$$\Delta\lambda = \lambda - \lambda_1,$$

$$\gamma \sin^{-1}[-b_x \sin\alpha_0 + c_x \cos\alpha_0],$$

$$\lambda = \tan^{-1}\left[\frac{b_x \cos\alpha_0 + c_x \sin\alpha_0}{a_x}\right]$$

$$\lambda_1 = \theta + \psi,$$

$$\theta \cong \dot{\theta}_E t,$$

$$t_0 = \frac{\theta_0}{\dot{\theta}_0}$$

$$\psi_0 = \dot{\psi}_0 T,$$

$$\psi = \dot{\psi}_0 (T - t_0 + t),$$

$$\tan\theta_0 = \frac{\sin\psi_0}{\cos\alpha_0 (1 - \cos\psi_0)}.$$

Since the trace changes as the orbit regresses, the computation is done at five years interval in T upto twenty five years (approximately half the regression period) that, is for T equal to 0, 5, 10, 15, 20 and 25 years in all the cases.

Case 1 — $J_2^{(2)} = 0,\ q = 1.$

This case is represented in Fig. 11 which represents the relation $\gamma = \gamma(\Delta\lambda)$ for a synchronous altitude when $J_2^{(2)} = 0,\ q = 1$. The values of γ and $\Delta\gamma$ are worked out at values of $t\,(0^h \le t \le 24^h)$ at an interval of one hour. It is seen that the graph takes the shape of figure eight for each T = 5, 10, 15, 20 and 25 years separately. It is observed that during the first half of the regression period the ground trace grows from a single point (0, 0) at T = 0 to figure eight which becomes maximum in size after half of one regression period. The dimension of the ground trace corresponding to T = 25 years are $\pm 14.456^{\circ}$ in latitude and $\pm$ 1.0081° in longitude. Not only the size of the ground trace changes but it's equatorial crossing also moves relative to the origin, which was the initial subsatellite point. In the first half of the regression period, the equatorial crossing moves to the east reaching a maximum of about $0.45^{\circ}$and then decreases to zero after half the regression period. In the second half of the regression period, the ground trace is the reverse of what it is in the firsty half but in this, it decreases in size and degenerates to a point at the end of one regression period. Here the equatorial crossing moves to the wet reaching a maximum displacement of 0.45° and returning to 0° at the end of the regression period. The behaviour of the ground trace during the second half of the regression cycle is a mirror image of that shown in Fig. 11.

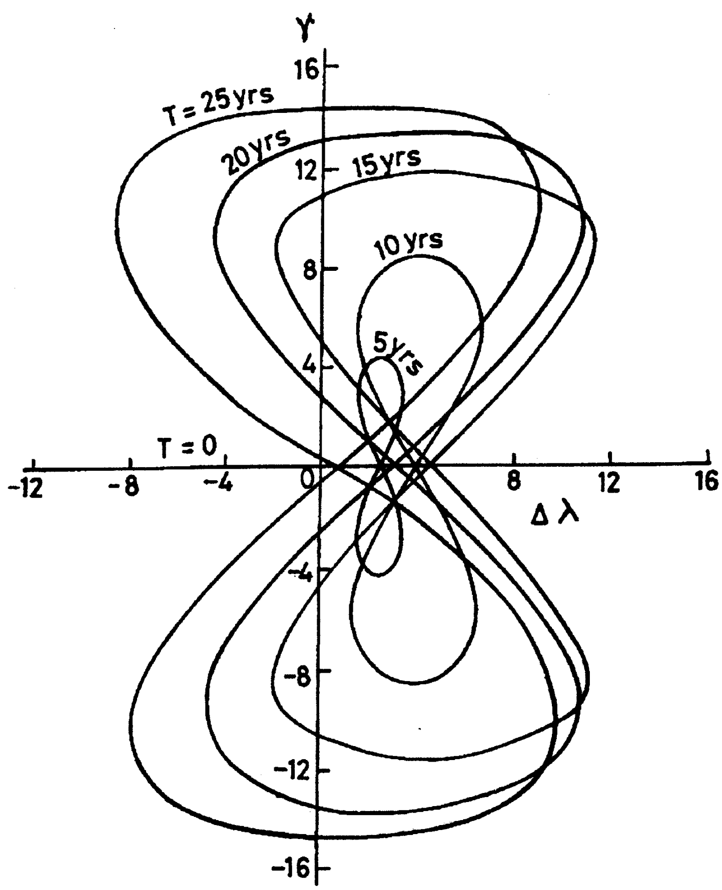

FIG. 11. Ground trace-orbit in equatorial plane (Frick case $J_2^{(2)} = 0, q = 1$)

Case 2 — $J_2^{(2)} \neq 0, q = 1$

In this case, the relation $\gamma = \gamma(\Delta\gamma)$ for a synchronous altitude when $J_2^{(2)} \neq 0$, $q = 1$ is represented by figure eight for each T = 5, 10, 15, 20 and 25 years separately. In Fig. 12, T = 25 years, the figure eight behaves exactly in the similar manner as it does in case 1 in one regression period. Also, when $|\Gamma_0| = 0^o$, the dimensions of the ground trace in one regression period change approximately by $\pm 0.2^o$ in latitude and $\pm$ 0.02^o in longitude and in the first half of the regression period, the equatorial crossing moves further towards the east with a maximum shift of 0.02^o.

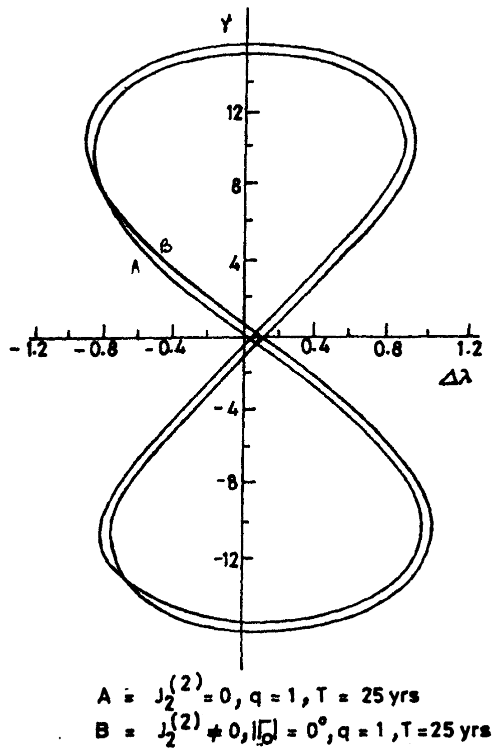

FIG. 12. Ground trace-orbit in equatorial plane for fixed T effect of ellipticity pressure

Case 3 — $J_2^{(2)} = 0$, $q = 0.9$

The relation $\gamma = \gamma(\Delta\gamma)$ for a synchronous altitude is represented by figure eight for each T = 5, 10, 15, 20 and 25 years separately.In Fig. 13, the Fig. 8 behaves exactly in the same manner as in case I in one regression period.

Due to the solar radiation pressure (q = 0.9) alone, the dimensions of the ground trace in one regression period change approximately by $\pm$ 0.3^o in latitude and $\pm$ 0.04^o in longitude and also in the first half of the regression period the equatorial crossing starts moving towards the west with a maximum shift of 0.02^o.

Case 4 — $J_2^{(2)} \neq 0, |\Gamma_0| = 0^o$, $q = 0.9$.

The relation $\gamma = \gamma(\Delta\lambda)$ for a synchronous altitude is represented by Fig. 8 for T = 25 years in Fig. 14. Behaviour of figure eight is same as in case I in one regression period. Due to the combined effect of the earth's equatorial ellipticity ($|\Gamma_0| = 0^o$) and the solar radiation pressure (q =

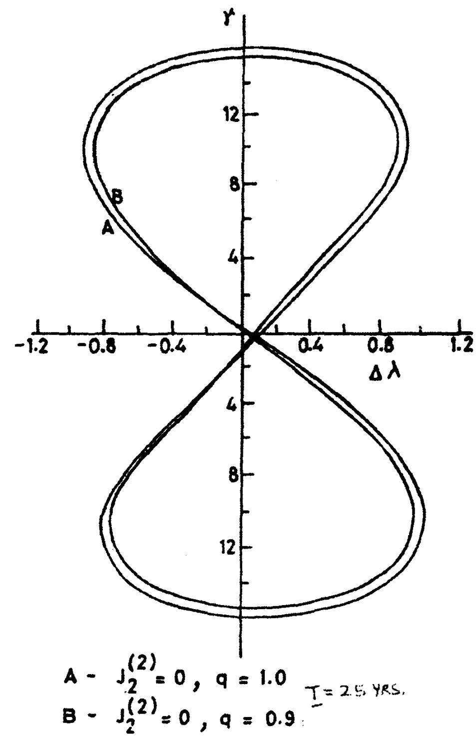

FIG. 13. Ground trace-orbit in equatorial plane for fixed T effect of solar radiation pressure

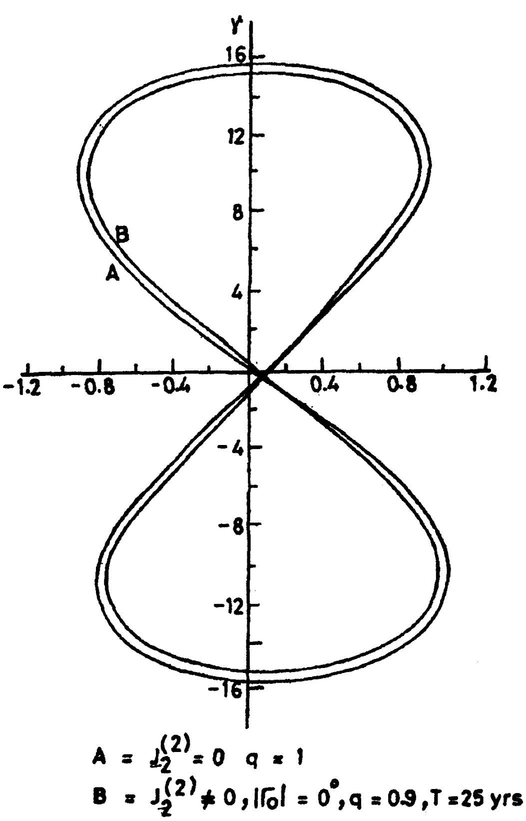

FIG. 14. Ground trace-orbit in equatorial plane for fixed T combined effect of solar ellipticity and solar radiation pressure

0.9), the dimensions of the ground trace in one regression period change approximately by $\pm$ 0.02^{o} in latitude and $\pm$ 0.02^{o} in longitude corresponding to T = 25 years. Further, due to the combined effect the equatorial crossing shifts very little that is, the change in the equatorial crossing is insignificant corresponding to T = 25 years.

CONCLUSION

The relative longitude and latitude of the satellite with respect to the subsatellite point give the required ground trace on the rotating earth. For an unperturbed equatorial orbit, synchronous orbit is achieved when $\dot{\theta}_0 = \dot{\theta}_E$. But, when perturbations are involved, synchronous orbit is achieved when $\dot{\theta}_0 = \dot{\theta}_E - \dot{\psi}_0$. With both these angular velocities, the ground trace is figure eight though it's size and position relative to the earth changes as the time elapses.

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