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SOME STRUCTURAL CONSIDERATIONS ON THE THEORY OF GRAVITATIONAL FIELD IN FINSLER SPACES — III

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The connection structure of the Finslerian gravitational field is considered by taking account of the fact that the Finslerian internal vector y shows its own intrinsic behaviour which is geometrically grasped by the newly introduced intrinsic connection δ_y, different from the conventional absolute differential Dy. Then, the connection structure in the case where y is reduced to a function of point x is considered, the reduction-process $y = y(x)$ itself being also likened to the averaging process with respect to y.

Key Words : Finsler Space; Intrinsic Connection; Internal Variable; Gravitational Field

1. INTRODUCTION

In the Finslerian gravitational field (Ikeda[1]), the internal vector $\boldsymbol{y}\,(=y^j; j = 1, 2, 3, 4)$ is attached to each point $\boldsymbol{x}\,(=x^i;\ i = 1, 2, 3, 4)$ as the independent internal variable. That is, the Finslerian gravitational field is regarded as one kind of nonlocal field nonlocalized by the vector $\boldsymbol{y}$.

The vector $\boldsymbol{y}$ shows, therefore, its own intrinsic behaviour and obeys its own inherent law.

From our physico-geometrial viewpoint, this intrinsic behaviour of $\boldsymbol{y}$ cannot be grasped by the ordinary conventional absolute differential of y (i.e., $\boldsymbol{Dy}$), but it should be represented by a newly introduced intrisic connection (or parallelism) of $\boldsymbol{y}$ (i.e., $\boldsymbol{\partial y} : \boldsymbol{\partial y} \neq \boldsymbol{Dy}$) (Ikeda[2]).

From this standpoint, in this paper, we shall reflect this $\boldsymbol{\partial}_{\boldsymbol{y}}$ in the connection structure of the Finslerian gravitational field, where the relation between $\boldsymbol{\partial y}$ and $\boldsymbol{Dy}$ is obtained. We shall also consider the case where y is given by a function of x, as in usual physical problems. In this case, the reduction-process $\boldsymbol{y} = y(x)$ itself is likened to the so-called averaging process with respect to $\boldsymbol{y}$.

2. CONNECTION STRUCTURES — I

First, let the conventional absolute differential of an arbitrary vector, say $\boldsymbol{X}\,(=X^i;\ i = 1, 2, 3, 4)$, be given by

$$\boldsymbol{D}X^i = dX^i + \Gamma^i_{jk} X^j\, dx^k + C^i_{jk} X^j\, dy^k, \qquad \dots (2.1)$$

where Γ^i_{jk} and C^i_{jk} denote the ordinary Finslerian connecti on factors. Then, as mentioned in Section 1, the intrinsic behaviour of the internal variable $\boldsymbol{y}$ cannot be grasped by $\boldsymbol{Dy}$ because of the vector $\boldsymbol{y}$ is the independent internal variable, not an arbitrary vector. Therefore, this intrinsic behaviour of $\boldsymbol{y}$ should be represented by a newly introduced intrinsic connection (or parallelism) of $\boldsymbol{y}$ in the form

$$\left.\begin{aligned} \delta y^i &= dy^i + \mathcal{H}^i_{jk}\, y^j\, dx^k + E^i_{jk}\, y^i\, dy^k \\ &\equiv N^i_k\, dx^k + M^i_k\, dy^k, \end{aligned}\right\} \qquad \text{... (2.2)}$$

where $(\Theta^i_{jk}, E^i_{jk})$ are different from $(\Gamma^i_{jk}, C^i_{jk})$, respectively, and we have put $N^i_k \equiv \mathcal{H}^i_{jk}\, y^j$ and $M^i_k \equiv \delta^i_k + E^i_{jk}\, y^j$. (Homogeneity conditions such as $E^i_{jk}\, y^j = 0$ and $C^i_{jk}\, y^j = 0$ are not assumed).

The intrinsic behaviour (2.2) is reflected in the connection structure of the Finslerian gravitational field by replacing dy^k with δy^k in (2.1) as follows :

$$DX^i = dX^i + F^i_{jk}\, X^i\, dx^k + H^i_{jk}\, X^j\, \delta y^k, \qquad \text{... (2.3)}$$

where $F^i_{jk} \equiv \Gamma^i_{jk} - N^l_k\, H^i_{jl}$ and $H^i_{jk} \equiv Q^l_k\, C^i_{jl}$, Q^l_k being the inverse of M^k_l. The quantity N^i_k plays the role of nonlinear connection for this case (Miron and Anastasiei[4]). From (2.3), the following covariant derivatives can be obtained :

$$\left.\begin{aligned} DX^i &= (X^i/k)\, dx^k + (X^i/k)\, \delta y^k; \\ X^i/k &= \frac{\delta X^i}{\delta x^k} + F^i_{jk}\, X^j, \\ X^i/k &= \frac{\delta X^i}{\delta y^k} + H^i_{jk}\, X^j, \end{aligned}\right\} \qquad \text{... (2.4)}$$

where $\dfrac{\delta}{\delta x^k} = \dfrac{\partial}{\partial x^k} - N^l_k \dfrac{\delta}{\delta y^l}$ and $\dfrac{\delta}{\delta y^k} = Q^l_k \dfrac{\partial}{\partial y^l}$. That is, $\left(\dfrac{\delta}{\delta x}, \dfrac{\delta}{\delta y}\right)$ is the dual frame of the adapted from $(dx, \delta y)$. In this case, the deflection tensors are given by $y^i/k = -N^l_k\, Q^i_l + F^i_{jk}\, y^j \neq 0$ and $y^i/k = Q^i_k + H^i_{jk}\, y^j \neq \delta^i_k$ (cf. Matsumoto[3], Miron and Anastasiei[4]). The nonlinear connection N^i_k cannot be determined by the condition $y^i/k = 0$.

As to the relation between $\boldsymbol{\delta y}$ and $\boldsymbol{Dy}$, we shall obtain this relation by taking account of the fact that the connection D is metrical for the metric tensor $g_{ij}(x, y)$ of the Finslerian gravitational field (i.e., $Dg_{ij} = 0$), but the connection δ is not metrical for $g_{ij}(x, y)$ (i.e., $\delta g_{ij} \neq 0$). Then, we can reconsider that the connection D is the metrical connection for g_{ij} (i.e., $Dg_{ij} = 0$) derived from the non-metrical connection δ for g_{ij} (i.e., $\delta g_{ij} \neq 0$). Under these situations, by applying the so-called Kawaguchi's theorem (Kawaguchi[5]), we can obtain the relation between $\boldsymbol{\delta y}$ and $\boldsymbol{Dy}$ as follows :

$$\boldsymbol{Dy}^i = \boldsymbol{\delta y}^i + \frac{1}{2}\, g^{il}\, (\delta g_{lj})\, y^j, \qquad \text{... (2.5)}$$

from which the following relations can also be obtained from (2.1) and (2.2) :

$$\Gamma^i_{jk} = \mathcal{H}^i_{jk} + \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} - \mathcal{H}^n_{lk} g_{xj} - \mathcal{H}^n_{jk} g_{ln} \right), \qquad \text{... (2.6)}$$

$$C^i_{jk} = E^i_{jk} + \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial y^k} - E^n_{lk} g_{nj} - E^n_{jk} g_{ln} \right).$$

(The relations between $(\Gamma^i_{jk}, C^i_{jk})$ and (F^i_{jk}, H^i_{jk}) cannot be obtained easily, because (F^i_{jk}, H^i_{jk}) contain Q^l_k or the inverse of $E^i_{jk} y^j$).

By the way, if the connection δ is made metrical for some metric tensor $h_{ij}(x, y)$ (i.e., $\delta h_{ij} = 0$), then the metrical structure of the total space of the tangent bundle should be given by

$$G = g_{ij}(x, y)\, dx^i \otimes dx^j + h_{ij}(x, y)\, \delta y^i \otimes \delta y^j. \qquad \text{... (2.7)}$$

Therefore, the connection coefficients Θ^i_{jk} and E^i_{jk} can be given by, for example, the following canonical forms (cf. Miron and Anastasiei[4]) :

$$\left.\begin{aligned} \Theta^i_{jk} &= \frac{1}{2} h^{il} \left(\frac{\partial h_{lj}}{\partial x^k} + \frac{\partial h_{kl}}{\partial x^j} - \frac{\partial h_{jk}}{\partial x^l} \right), \\ E^i_{jk} &= \frac{1}{2} h^{il} \left(\frac{\partial h_{lj}}{\partial yk} + \frac{\partial h_{kl}}{\partial y^j} - \frac{\partial h_{jk}}{\partial y^l} \right). \end{aligned}\right] \qquad \text{... (2.8)}$$

Thus, the connection structure of the Finslerian gravitational field is completely clarified by taking account of the intrinsic behaviour of $\boldsymbol{y}$.

3. Connection Structures - II

Now, we shall proceed to the case where the vector $\boldsymbol{y}$ is given by a function of point x, as in usual physical problems. In this case, $\boldsymbol{\delta y}$ (2.2) is reduced to, because of $\boldsymbol{dy}^i = \frac{\partial y^i}{\partial x^k} dx^k$,

$$\boldsymbol{\delta y}^i = N^i_k\, dx^k + M^i_k\, dy^k \equiv Y^i_k(x, y(x))\, dx^k, \qquad \text{... (3.1)}$$

where we have $Y^i_k \equiv N^i_k + M^i_l \frac{\partial y^l}{\partial x^k}$. Therefore, (2.3) is rewritten as

$$\begin{aligned} DX^i &= dX^i + F^i_{jk} X^j dx^k + (H^i_{jl} Y^l_k) X^j dx^k \\ &\equiv dX^i + \overline{F}^i_{jk} X^j dx^k, \end{aligned} \qquad \text{... (3.2)}$$

where $\overline{F}^i_{jk} \equiv F^i_{jk} + H^i_{jl} Y^l_k$. And only one kind of covariant derivative (X^i/k) is introduced from (2.4) as follows :

$$DX^i = (X^i//k)\, dx^k;$$

$$X^i//k = \frac{\delta X^i}{\delta x^k} + \overline{F}^i_{jk} X^j, \qquad \text{... (3.3)}$$

where $\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} + Y^l_k \frac{\partial}{\partial y^l}$. That is to say, in this case, the connection structure is represented by the connection coefficient $\overline{F}^i_{jk}$. (This connection structure is similar to the case of Y-Riemann connection (where $y^i = Y^i(x)$) considered by Matsumoto *et al.*[3 & 6])

By use of (3.3), the following curvature tensor $(\overline{\mathcal{R}}^i_{hjk})$ and torsion tensor $(\overline{T}^i_{jk})$ are introduced by means of the Ricci-identity :

$$\left.\begin{aligned}
&X^i_{//j//k} - X^i_{//k//j} = \overline{\mathcal{R}}^i_{hjk} X^h - \overline{T}^h_{jk} X^i_{//h};\\
&\overline{\mathcal{R}}^i_{hjk} = O_{jk}\left\{ \frac{\delta \overline{F}^i_{hj}}{\delta x^k} + \overline{F}^i_{lj} \overline{F}^l_{hk} \right\},\\
&\overline{T}^i_{jk} = O_{jk}\left\{ \overline{F}^i_{jk} \right\} = O_{jk}\left\{ F^i_{jk} + H^i_{jl} Y^l_k \right\}
\end{aligned}\right] \qquad \text{... (3.4)}$$

where the symbol O_{jk} means the interchange of j, k and subtraction. If the condition $\delta y^j = 0$ (i.e. $Y^j_k = 0$) is satisfied, namely, if the inherent law of y is satisfied, then those relations such as $\overline{F}^i_{jk} = F^i_{jk} (= \Gamma^i_{jk})$, $\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k}$, $\overline{T}^i_{jk} = O_{ij}\left\{ F^i_{jk} \right\}$, etc. hold good, because the connection coefficient $\overline{H}^i_{jk}$ does not appear above the surface from our physical viewpoint.

On the other hand, from our more physical point of view, if the reduction-process $y = y(x)$ itself is likened to the averaging process with respct to y, because only x appears above the surface, then the resulting averaged field becomes Riemannian, but the torsion $\overline{T}^i_{jk}(x)$ does not vanish, in general. In this averaged field, the following averaged quantities govern the whole spatial structure (cf. Kondo and Amari[7], Zalaletdinov[8]):

$$\left.\begin{aligned}
&\text{Metric:} && \bar{g}_{ij}(x) \equiv \dot{g}_{ij}(x, y(x)),\\
&\text{Connection :} && \overline{F}^i_{jk}(x) = \left\{ {i \atop jk} \right\} + \mathcal{T}^i_{jk}(T),
\end{aligned}\right] \qquad \text{... (3.5)}$$

Curvature : $\overline{\mathcal{R}}^i_{hjk}(x) = \mathcal{K}^i_{hjk}(\{\ \}) + \mathcal{L}^i_{hjk}(\mathcal{T})$.

Torsion : $\overline{T}^i_{jk}(x)$.

In (3.5), $\bar{g}_{ij}(x)$ is the Riemannian metric tensor, $\left\{ {i \atop jk} \right\}$ means the Christoffel symbol formed with $\bar{g}_{ij}(x)$ and $\mathcal{T}^{i}_{jk}$ the contortion tensor formed with the torsion $\bar{T}^{i}_{jk}(x)$, and $\mathcal{K}^{i}_{hjk}$ denotes the Riemannian curvature tensor formed with $\left\{ {i \atop jk} \right\}$ and $\mathcal{L}^{i}_{hjk}$ is defined as the rest formed with $\mathcal{T}^{i}_{jk}(x)$. Under these situations, the Einsteinian field equation can be given by

$$\bar{\mathcal{R}}_{ij} - \frac{1}{2}\bar{\mathcal{R}}\bar{g}_{ij} = \mathcal{K}_{ij} - \frac{1}{2}\mathcal{K}\bar{g}_{ij} + S_{ij} = 0, \qquad \text{... (3.6)}$$

where $\bar{\mathcal{R}}_{ij}\left(\equiv \bar{\mathcal{R}}^{k}_{ijk} \right)$ and $\mathcal{K}_{ij}\left(\equiv \mathcal{K}^{k}_{ijk} \right)$ are the Ricci-tensors, $\bar{\mathcal{R}}\left(\equiv \bar{\mathcal{R}}_{ij}\bar{g}^{ij} \right)$ and $\mathcal{K}\left(\equiv \mathcal{K}_{ij}\bar{g}^{ij} \right)$ are the scalar curvatures, and S_{ij} represents all the remaining terms consisting of $\mathcal{T}^{i}_{jk}(x)$ or $\bar{T}^{i}_{jk}(x)$ essentially. Namely,in (3.6), $\left(\mathcal{K}_{ij} - \frac{1}{2}\mathcal{K}\bar{g}_{ij} \right)$ is the purely Riemannian Einstein tensor and S_{ij} plays the role of material source (or sink) for the (Riemannian) gravitational field.

We should consider other interesting physical aspects underlying this kind of averaged field in future.

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