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SPECTRAL SINGULARITIES OF THE KLEIN-GORDON *S*-WAVE EQUATION

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In this paper, we investigate the spectral singularities and the eigenvalues of the operator L generated in $L_2(\boldsymbol{R})$ by the Klein-Gordon s-wave equation

$$y'' + [\lambda - p(x)]^2 y = 0, x \in \boldsymbol{R} = (-\infty, \infty),$$

where the potential p is a complex valued function and λ is a spectral parameter.

Key Words : Spectral Singularity; Klein-Gordon Equations; *s*-Wave Equations; Differential Operators

1. INTRODUCTION

The study of the spectral analysis of a non-selfadjacent Schrödinger operator with continuous and discrete spectrum was begun by Naimark[11]. He showed the existence of spectral singularities in the continuous spectrum of the Schrödinger operators. The effect of spectral singularities in the spectral expansion of the Schrödinger operators in terms of principal functions was considered by Lyance[10]. Spectral singularities of the Schrödinger operators with rapidly decreasing potential were investigated in the papers[5/12]. In a series of papers, the spectral analysis of quadratic pencil of the Schrödinger and Dirac operators with spectral singularities have been studied[1-3, 6 & 9]

Let L denote the operator generated in $L_2(\boldsymbol{R})$ by the equation

$$y'' + [\lambda - p(x)]^2 y = 0, x \in \boldsymbol{R} = (-\infty, \infty), \qquad \text{... (1.1)}$$

where p is a complex valued function and λ is a spectral parameter. Note that in relativistic quantum mechanics, (1.1) is called the Klein-Gordon s-wave equation for a particle of zero mass with static potential p. In this paper, the authors have investigated the spectral singularities and the eigenvalues of L using the analytic continuation and the uniqueness theorems of analytic functions. Some problems of spectral analysis of the Klein-Gordon s-wave equation in a semi-axis have been discussed previously[4].

2. JOST SOLUTIONS OF (1.1)

Let us suppose that p is continuously differentiable on $\boldsymbol{R}$ and

$$\lim_{x\to\pm\infty} p(x)=0, \quad \int_{-\infty}^{\infty} (1+|x|)\,|p'(x)|\,dx<\infty. \qquad \text{... (2.1)}$$

In a similar way to the Schrödinger equation, we will denote the Jost solutions of (1.1) satisfying

$$\lim_{x\to\infty} e^{+}(x,\lambda)\,e^{-i\lambda x}=1,\ \lambda\in \mathbf{C}_{+} := \{\lambda:\lambda\in \mathbf{C}, \operatorname{Im}\lambda\geq 0\},$$

$$\lim_{x\to-\infty} e^{-}(x,\lambda)\,e^{i\lambda x}=1,\ \lambda\in \mathbf{C}_{+},$$

$$\lim_{x\to\infty} E^{+}(x,\lambda)\,e^{i\lambda x}=1,\ \lambda\in \mathbf{C}_{-} := \{\lambda:\lambda\in \mathbf{C}, \operatorname{Im}\lambda\geq 0\}$$

and

$$\lim_{x\to-\infty} E^{-}(x,\lambda)\,e^{i\lambda x}=1,\ \lambda\in \mathbf{C}_{-},$$

by $e^{+}(x,\lambda)$, $e^{-}(x,\lambda)$, $E^{+}(x,\lambda)$ and $E^{-}(x,\lambda)$, respectively. Under the condition (2.1) the Jost solution of (1.1) have the representations

$$\left.\begin{aligned} e^{+}(x,\lambda) &= e^{i\alpha^{+}(x)+i\lambda x}+\int_{x}^{\infty} K^{+}(x,t)\,e^{i\lambda t}\,dt,\\ \text{and}\quad e^{-}(x,\lambda) &= e^{i\alpha^{-}(x)-i\lambda x}+\int_{-\infty}^{x} K^{-}(x,t)\,e^{-i\lambda t}\,dt, \end{aligned}\right] \qquad \text{... (2.2)}$$

for $\lambda\in \mathbf{C}_{+}$ and

$$\left.\begin{aligned} E^{+}(x,\lambda) &= e^{i\alpha^{+}(x)+i\lambda x}+\int_{x}^{\infty} Q^{+}(x,t)\,e^{-i\lambda t}\,dt,\\ \text{and}\quad E^{-}(x,\lambda) &= e^{-i\alpha^{-}(x)-i\lambda x}+\int_{-\infty}^{x} Q^{-}(x,t)\,e^{i\lambda t}\,dt, \end{aligned}\right] \qquad \text{... (2.3)}$$

for $\lambda\in \mathbf{C}_{-}$, where

$$\alpha^{+}(x)=\int_{x}^{\infty} p(t)dt,\ \alpha^{-}(x)=\int_{-\infty}^{x} p(t)dt.$$

Moreover, the kernels $K^{\pm}(x,t)$ and $Q^{\pm}(x,t)$ may be expressed in terms of p. $K^{\pm}(x,t)$, $Q^{\pm}(x,t)$ are continuously differentiable with respect to their arguments and

$$|K^{\pm}(x,t)|,\ Q^{\pm}(x,t)| \le C\,\xi^{\pm}\left(\frac{x+t}{2}\right)\exp\{\eta^{\pm}(x)\}, \quad \dots (2.4)$$

$$|K^{\pm}_{x_i}(x_1,x_2)|,\ |Q^{\pm}_{x_i}(x_1,x_2)| \le C\left[\xi^{\pm}\left(\frac{x_1+x_2}{2}\right)+\Theta\left(\frac{x_1+x_2}{2}\right)\right],\ i=1,2, \quad \dots (2.5)$$

where
$$\xi^{+}(x)=\int_x^{\infty}\left[|p(t)|^2+|p'(t)|\right]dt,\ \eta^{+}(x)=\int_x^{\infty}\left[(t-x)|p(t)|^2+2|p(t)|\right]dt,$$

and
$$\xi^{-}(x)=\int_{-\infty}^{x}\left[|p(t)|^2+|p'(t)|\right]dt,\ \eta^{-}(x)=\int_{-\infty}^{x}\left[(x-t)|p(t)|^2+2|p(t)|\right]dt,$$

$$\Theta(x)=\frac{1}{4}\left[|p(x)|^2+|p'(x)|\right],$$

and $C>0$ is a constant (Jaulent and Jean[8]). Therefore, the solutions $e^{\pm}(x,\lambda)$ and $E^{\pm}(x,\lambda)$ are analytic with respect to λ in $\boldsymbol{C}_{+}=\{\lambda:\lambda\in C, \operatorname{Im}\lambda>0\}$ and $\boldsymbol{C}_{-}=\{\lambda:\lambda\in \boldsymbol{C}, \operatorname{Im}\lambda<0\}$, respectively, and continuous up to the real axis.

According to (2.2)-(2.5), the Wronskian of the solutions $e^{\pm}(x,\lambda)$ and $E^{\pm}(x,\lambda)$ are

$$W[e^{\pm}(x,\lambda),E^{\pm}(x,\lambda)]=\mp 2i\lambda,$$

for $\lambda\in\boldsymbol{R}$. So the pairs $e^{+}(x,\lambda), E^{+}(x,\lambda)$ and $e^{-}(x,\lambda), E^{-}(x,\lambda)$ form two fundamental systems of solutions of (1.1) for $\boldsymbol{R}^{*}:\boldsymbol{R}\setminus\{0\}$. The following relations hold :

$$e^{+}(x,\lambda)=s_{11}(\lambda)e^{-}(x,\lambda)+s_{12}(\lambda)E^{-}(x,\lambda),$$

and
$$E^{+}(x,\lambda)=s_{21}(\lambda)e^{-}(x,\lambda)+s_{22}(\lambda)E^{-}(x,\lambda),$$

for $\lambda\in\boldsymbol{R}^{*}$, where

$$s_{11}=\frac{W[e^{+}E^{-}]}{2i\lambda},\ s_{12}=-\frac{W[e^{+},e^{-}]}{2i\lambda},$$

$$s_{21}=\frac{W[E^{+},E^{-}]}{2i\lambda},\ s_{22}=\frac{W[e^{-},E^{+}]}{2i\lambda}.$$

The functions $s_{ij}(\lambda)\ i,j=1,2$ are defined for all $\lambda\in\boldsymbol{R}^{*}$. Moreover $s_{12}(\lambda)$ and $s_{21}(\lambda)$ admit analytic continuation to $\boldsymbol{C}_{+}$ and $\boldsymbol{C}_{-}$, respectively.

3. Discrete Spectrum of L

Let us denote the eigenvalues and the spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definition of the eigenvalues and the spectral singularities it follows that

$$\sigma_d(L) = \{\lambda : \lambda \in C_+, a^+(\lambda) = 0\} \cup \{\lambda : \lambda \in C_-, a^-(\lambda) = 0\}, \qquad \dots (3.1)$$

and
$$\sigma_{ss}(L) = \{\lambda : \lambda \in R^*, a^+(\lambda) = 0\} \cup \{\lambda : \lambda \in R^*, a^-(\lambda) = 0\}, \qquad \dots (3.2)$$

where
$$a^+(\lambda) = -2i\lambda s_{12}(\lambda), a^-(\lambda) = 2i\lambda s_{21}(\lambda).$$

From (3.1) and(3.2) we see that to investigate the structure of the eigenvalues and the spectral singularities of L, we need to discuss the structure of the zeros of the functions $a^+(\lambda)$ and $a^-(\lambda)$ in $\overline{C}_+$ and $\overline{C}_-$, respectively. For the sake of simplicity, we will consider only the zeros of $a^+(\lambda)$ in $\overline{C}_+$.

Definition 1 — The multiplicity of a zero $a^+(\lambda)$ (or $a^-(\lambda)$) in $\overline{C}_+$ (or $\overline{C}_-$) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of L.

Theorem — *If (2.1) holds, then the function $a^+(\lambda)$ is analytic in C_+, continuous in $\overline{C}_+$ and*

$$a^+(\lambda) = -2i\lambda e^{i\alpha} + \beta + \int_0^\infty f(t)\, e^{i\lambda t}\, dt, \qquad \dots (3.3)$$

where α and β are constants and $f \in L_1(R_+), R_+ = (0, \infty)$.

PROOF : By the definition of $a^+(\lambda)$ we write

$$a^+(\lambda) = W[e^+, e^-] = e^+(0, \lambda)\, e_x^-(0, \lambda) - e_x^+(0, \lambda)\, e^-(0, \lambda). \qquad \dots (3.4)$$

From (2.2) and (3.4) we get (3.3) where α, β and f defined by the following :

$$\alpha = \int_{-\infty}^{\infty} p(x)\, dx, \beta = 2ip(0)\, e^{i\alpha} + 2K^-(0,0)\, e^{i\alpha^+(0)} + 2K^+(0,0)\, e^{i\alpha^-(0)},$$

and
$$f(t) = K^-(0,0)\, K^+(0,t) + K^+(0,0)\, K^-(0,-t) + e^{i\alpha^+(0)} K_x^-(0,-t) + e^{i\alpha^-(0)} K_x^+(0,t)$$
$$+ ip(0)\, e^{i\alpha^-(0)} K^+(0,t) + ip(0)\, e^{i\alpha^+(0)} K^-(0,-t) + e^{i\alpha^-(0)} K_t^+(0,t) - e^{i\alpha^+(0)} K_t^-(0,-t)$$
$$+ \int_0^\infty K^+(0,u)\, K_x^-(0,u-t)\, du - \int_0^\infty K_x^+(0,u)\, K^-(0,u-t)\, du. \qquad \dots (3.5)$$

Using (2.4), (2.5) and (3.5) we obtain $f \in L_1(R_+)$. □

Let us define

$$M_1^{\pm} = \{\lambda : \lambda \in X_{\pm}, a^{\pm}(\lambda) = 0\},\ M_2^{\pm} = \{\lambda : \lambda \in R\, a^{\pm}(\lambda) = 0\}.$$

From (3.1) and (3.2) we have

$$\sigma_d(L) = M_1^+ \cup M_1^-, \ \sigma_{ss}(L) = \{M_2^+ \cup M_2^-\} \setminus \{0\}.$$

***Theorem* 2** — *Under the condition (2.1)*

(i) The set of eigenvalues of L is bounded, is at most countable, and its limit points can lie only in a bounded subinterval of the real axis.

(ii) The set of spectral singularities of L is compact and its linear Lebesgue measure is zero.

PROOF : From Theorem 1 we obtain $a^+(\lambda)$ is analytic in $C_\pm$ and continuous in $\overline{C}_+$ and

$$a^+(\lambda) = -2i\lambda e^{i\alpha} + \beta + 0(1), \lambda \in \overline{C}_+, |\lambda| \to \infty. \quad \text{... (3.6)}$$

Using (3.6) and uniqueness theorem of analytic functions we have (i) and (ii) (Dolzhenko[7]). □

***Theorem* 3** — *If*

$$\lim_{x \to \pm\infty} p(x) = 0, \ \sup_{x \in R} (e^{\varepsilon|x|} p'(x)|) < \infty, \varepsilon > 0, \quad \text{... (3.7)}$$

then the operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

PROOF : From (2.4), (2.5), (3.3), (3.5) and (3.7) we find that $a^+(\lambda)$ has analytic continuation to the half-plane $Im\lambda > -\frac{\varepsilon}{4}$. Since the limit points of its zeros in $\overline{C}_+$ can not lie in $\boldsymbol{R}$. Therefore using Theorem 2, we have the finiteness of zeros of $a^+(\lambda)$ in $\overline{C}_+$. Similarly, we find the finiteness of zeros of $a^-(\lambda)$ in $\overline{C}_-$. □

It is seen that the condition (3.7) guarantees of the analytic continuation of $a^+(\lambda)$ and $a^-(\lambda)$ from the real axis to lower and upper half-planes, respectively. So the finiteness of eigenvalues and spectral singularities of L are obtained as a result of this analytic continuations.

Now let us suppose that

$$\lim_{x \to \pm\infty} p(x) = 0, \ \sup_{x \in R} (e^{\varepsilon|x|^\delta} |p'(x)|) < \infty, \varepsilon > 0, \frac{1}{2} \le \delta < 1, \quad \text{... (3.8)}$$

which is weaker than (3.7). It is evident that under the condition (3.8) the function $a^+(\lambda)$ is analytic in C_+ and infinitely differentiable on the real axis.

But $a^+(\lambda)$ does not have an analytic continuation from the real axis to lower half-plane. Similarly $a^-(\lambda)$ does not have an analytic continuation from the real axis to upper half-plane. Therefore under the condition (3.8) the finiteness of eigenvalues and spectral singularities of L can not be shown in a way similar to Theorem 3.

Let us denote the set of limit points of M_1^+ by M_3^+ and the set of all zeros of $a^+(\lambda)$ with infinitely multiplicity in $\overline{C}_+$ by M_4^+.

It is evident that

$$M_1^+ \cap M_4^+ = \phi,\ M_3^+ \subset M_2^+, M_4^+ \subset M_2^+,$$

and the linear Lebesgue measures of M_3^+ and M_4^+ are zero. Using of the continuity of all derivatives of $a^+(\lambda)$ on the real axis we have

$$M_3^+ \subset M_4^+. \qquad \text{... (3.9)}$$

Lemma 1 — If (3.8) holds, then $M_4^+ = \phi$.

PROOF : It is clear that the function $a^+(\lambda)$ is analytic in C_+ all of its derivatives are continuous up to the real axis, and there exist $T > 0$ such that

$$\left| \frac{d^m}{d\lambda^m} a^+(\lambda) \right| \le A_m^+,\ m = 0, 1, \ldots, \lambda \in \overline{C}_+, |\lambda| < 2T. \qquad \text{... (3.10)}$$

From Theorem 2, we get that

$$\left| \int_{-\infty}^{-T} \frac{\ln |a^+(\lambda)|}{1+\lambda^2} d\lambda \right| < \infty,\ \left| \int_{T}^{\infty} \frac{\ln |a^+(\lambda)|}{1+\lambda^2} d\lambda \right| < \infty, \qquad \text{... (3.11)}$$

hold. Since the function $a^+(\lambda)$ is not equal to zero identically, then by the Pavlov's Theorem[9&12], M_4^+ satisfies

$$\int_0^h \ln F(s)\, d\mu(M_{4,s}^+) > -\infty, \qquad \text{... (3.12)}$$

where $F(s) = \inf_m \frac{A_m^+ s^m}{m!}$, $\mu(M_{4,s}^+)$ is the linear Lebeshue measure of s-neighbourhood of M_4^+, the constants A_m^+, $m = 0, 1, \ldots$, are defined by (3.10) and $h > 0$ is a constant. Using (2.4), (2.5), (3.3) and (3.5) we obtain

$$A_m^+ = 2^m C \int_0^\infty x^m e^{-\varepsilon x^\delta} dx \le D d^m\, m!\, m^{m\frac{1-\delta}{6}}, \qquad \text{... (3.13)}$$

where D and d are constats depending ε, δ and C. Substituting (3.13) in the definition of $F(s)$ we arrive at

$$F(s) = \inf_m \frac{A_m^+ s^m}{m!} \le D \exp\left\{ -\frac{1-\delta}{\delta} e^{-\frac{1}{1-\delta}} d^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}} \right\}.$$

By (3.12), we find

$$\int_0^h s^{-\frac{\delta}{1-\delta}} d\mu (M_{4,s}^+) < \infty. \qquad \text{... (3.14)}$$

So $\frac{\delta}{1-\delta} \geq 1$. Then (3.14) holds for arbitrary s if and only if $\mu (M_{4,s}^+) = 0$ or $M_4^+ = \phi$. □

***Theorem* 4** — *Under the condition (3.8) the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

PROOF : To be able to prove the theorem we have to show that the functions $a^+(\lambda)$ and $a^-(\lambda)$ have a finite number of zeros with finite multiplicities in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. We will prove it only for $a^+(\lambda)$. The case of $a^-(\lambda)$ is similar.

From Lemma 1 and (3.9) we find that $M_4^+ = \phi$. So the bounded set M_1^+ have no limit point i.e., the function $a^+(\lambda)$ has only a finite number of zeros in $\mathbb{C}_+$. Since $M_4^+ = \phi$ these zeros are of finite multiplicity. □

From Theorem 4 it is seen that the weakest condition which guarantees the finiteness of eigenvalues and spectral singularities of L is

$$\lim_{x \to \pm\infty} p(x) = 0, \quad \sup_{x \in R} (e^{\varepsilon \sqrt{|x|}}) < \infty, \ \varepsilon > 0.$$

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