

## AN INTEGRAL METHOD FOR THE SOLUTION OF THE BOUNDARY LAYER EQUATION FOR POWER-LAW MHD FLUID

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The problem of a non-Newtonian power law conducting fluid past a semi-finite plate, in the presence of an extremal magnetic field, is studied. Integral method of Von-Karman-Pohlhausen's type is applied to obtain the analytic solution of this fundamental problem for two cases: Flow past a flat plate and flow near stagnation point.

**Key Words :** Integral Method; Boundary Layer Equation; Power-law MHD Fluid; Non-Newtonian Fluids

### 1. INTRODUCTION

Non-Newtonian fluids are of great importance in chemical engineering and technology. Many different types of such non-Newtonian fluids exist, but the simplest and most useful type is the power law fluid, with the two parameter nonlinear equation of state

$$\tau_{ij} = -P \delta_{ij} + k |e_{ij}|^{n-1} e_{ij}, \quad \dots (1.1)$$

where  $\tau_{ij}$  is the stress tensor,  $e_{ij}$  the rate of strain tensor,  $P$  is the pressure,  $\delta_{ij}$  is the Kronecker {italic delta},  $k$  the coefficient of consistency and  $n$  is the power flow behavior index of the fluid ( $n > 0$ ). The presence of the rheological properties in various degrees in a real material creates a growing need for investigating such flows in different circumstances. Recently, a number of research papers that investigate the boundary layer flows for non-Newtonian fluids have been published by many authors.<sup>1-11</sup>

An approximate solution for the predication of the power law fluid flow behaviour in the entrance region of a straight channel has been studied in [1 & 2]. This has been achieved by solving a hydrodynamically equivalent model of developing two-dimensional Newtonian flow in a channel by momentum integral method. The analysis leads to closed form expressions for the flow characteristics. In [3], publications are reviewed which show that both shear-thinning and other viscoelastic effects where stress over shoot appears must be taken into account in transient lubrication problems. More specifically, the progress in squeeze-film lubrication with non-Newtonian fluids is examined. In [4], they examined the fully developed flow of a generalized fluid of second grade between heated parallel plats due to a pressure gradient along the plate. A shear dependent viscosity with an exponent  $m$  replaces the constant coefficient of shear viscosity of a fluid of second grade, if the normal stress coefficients are set equal to zero; this model reduces to the standard power-law

model. Momentum and heat transfer in mixed-convective, power-law fluid flow over arbitrarily shaped two-dimensional or axisymmetric bodies are examined theoretically in [5]. The governing nonlinear equations, expressed in terms of a stream function and vorticity, were solved by finite differences for Reynolds numbers for various power-law indices;  $n$ . In [6], the cone and plate flow of an incompressible fluid of second grade is investigated. The exact solutions are presented generally as Fourier series in terms of the eigen functions of some suitable boundary value problems. In [7], the momentum and heat transfer occurring in the laminar boundary layer on a continuously moving and stretching two-dimensional surface in a non-Newtonian power law fluid is examined. The Merk-Chao series expansion is used to generate ordinary differential equations from the partial differential momentum equation in order to obtain universal velocity functions. For the problem of combined momentum and heat transfer in the boundary layer of the moving sheet, a general power series is used to describe the fluid's velocity and temperature.

Motivated by the above-mentioned works, the problem of magneto hydrodynamic (MHD) power law boundary layer equations has attracted the attention of many authors<sup>8-10</sup>. They have studied the laminar flow of power law conducting fluid past a semi-finite and infinite plate in the presence of a transverse magnetic field. Over the years, many analytical or similarity solutions have been obtained. The problem of a non-Newtonian power law conducting fluid past a semi-finite plate, in the presence of an external electromagnetic field, is studied in [8]. The electric conductivity is taken as a function of the velocity. The method of expansion for a small parameter is used to obtain the solution and for a special form of the free stream velocity, the method of similarity solutions is used to obtain the exact solutions of the boundary layer equations associated with this problem in a closed form. It was found that the presence of the electromagnetic field reduces the velocity of the fluid. In [9], the boundary layer in a power-law fluid flowing in the presence of a transverse variable magnetic field is investigated. Assuming the electric conductivity of the fluid is dependent on its velocity, Meksyn's method is used to get an analytical solution for the velocity field and the coefficient of friction. The effect of the magnetic field is then discussed. The problem of shear flow of non-Newtonian fluid, which is due to the motion of an infinite plate with velocity, depending on the magnetic field acting normally on the plate and when the plate is activated by constant impulse, is investigated in [10]. The governing equations are solved analytically, and the velocity in shear layer, the effect of the magnetic field on all physical characters and the thickness of the shear layer has been determined and discussed graphically. In [11-16], integral method of Von-Karman type is applied to obtain the analytical solution of the boundary layer equation for Newtonian and non-Newtonian fluid.

The main aim of this chapter was to obtain the approximate solution of the boundary layer equation following Karman-Pohlhausen's method for magnetohydrodynamic flow of power law fluids for two cases, namely, flow past a flat plate and flow near stagnation point, in the presence of a transverse magnetic field.

## 2. MATHEMATICAL FORMULATION

We shall consider a steady laminar flow of a non-Newtonian, electrically conducting and incompressible viscous fluid on a flat-plate. The  $x$ -axis corresponds to the direction of the flow whereas  $y$  and  $z$ -axes are normal and transverse directions to the flow. Uniform magnetic field is applied normal to the plate. The magnetic Reynolds number  $R_{em}$  is assumed small, the induced magnetic field is small compared with the external magnetic field and can be neglected.

The boundary layer equations for the steady two-dimensional flow of a power-law conducting fluid in the presence of magnetic field is given by [17].

*Continuity Equation*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \dots (2.1)$$

*Momentum Equation*

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)^n - \frac{\sigma B^2 u}{\rho}, \quad \dots (2.2)$$

subject to the boundary conditions

$$\left. \begin{aligned} u = 0, v = 0, \text{ at } y = 0, \\ u = U(x), \text{ as } y \rightarrow \infty, \end{aligned} \right\} \quad \dots (2.3)$$

where  $u, v$  are the velocity components along and normal to the plate,  $\nu = \frac{k}{\rho}$  is the kinematic viscosity,  $P$  is the pressure.

The pressure term in eq. (2.2) shall be expressed in terms of the free stream velocity  $U(x)$ . At large distance from the plate the velocity component  $u$  is a function of  $x$  only.

Eq. (2.2) will then become a generalized Bernoulli's equation

$$-\frac{1}{\rho} \frac{dP}{dx} = U \frac{dU}{dx} + \frac{\sigma B^2 U}{\rho}. \quad \dots (2.4)$$

By combining eqs. (2.2) and (2.4), we obtain

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)^n + \frac{\sigma B^2}{\rho} (U - u). \quad \dots (2.5)$$

We introduce the following non-dimensional quantities

$$\left. \begin{aligned} u' = \frac{u}{U_0}, U' = \frac{U}{U_0}, v' = \frac{v}{u_0} Re^{1+n}, \\ y' = \frac{y}{L} Re^{1+n}, x' = \frac{x}{L}, B' = \frac{B}{B_0}, \end{aligned} \right\} \quad \dots (2.6)$$

where  $U_0, B_0$  and  $L$  are the characteristics of velocity, magnetic field and length and  $Re = \frac{L^n U_0^{2-n}}{\nu}$  is the generalized Reynolds number.

In terms of the above non-dimensional variables, eq. (2.5) takes the form (dropping the primes for convenience) :

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = U \frac{dU}{dx} + \frac{\partial}{\partial y} \left[ \left( \frac{\partial u}{\partial y} \right)^n \right] + MB^2 (U - u), \quad \dots (2.7)$$

where  $M = \sigma B_0^2 L / \rho U_0$  is the induced coefficient of MHD.

The continuity equation takes the same form without change. By using the continuity equation, we can rewrite (2.7) in the form

$$\frac{\partial}{\partial x} u^2 + \frac{\partial}{\partial y} (uv) = U \frac{dU}{dx} + \frac{\partial}{\partial y} \left[ \left( \frac{\partial u}{\partial y} \right)^n \right] + MB^2 (U - u), \quad \dots (2.8)$$

also we have

$$\frac{\partial}{\partial x} (uU) + \frac{\partial}{\partial y} (vU) = u \frac{dU}{dx}, \quad \dots (2.9)$$

subtracting (2.8) from (2.9) we get

$$\frac{\partial}{\partial x} [u(U - u)] + \frac{\partial}{\partial y} [v(U - u)] + [MB^2(x) + U'] (U - u) = - \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)^n. \quad \dots (2.10)$$

Integrating eq. (2.10) across the boundary layer, we will be able to reduce the momentum equation to the same momentum integral equation for the boundary layer as Newtonian fluid in the form

$$U^2 \frac{d\theta}{dx} + (2\theta + \delta^*) U \frac{dU}{dx} + MB^2 U \delta^* = \tau_w, \quad \dots (2.11)$$

where  $\delta^*$ ,  $\theta$  and  $\tau_w$  are the displacement thickness, the momentum thickness and the shearing stress, which are defined in the form

$$\left. \begin{aligned} \delta^* &= \int_0^\delta \left( 1 - \frac{u}{U} \right) dy, \quad \theta = \int_0^\delta \left( \frac{u}{U} \right) \left( 1 - \frac{u}{U} \right) dy, \\ \text{and } \tau_w &= \left[ \left( \frac{\partial u}{\partial y} \right)^n \right]_{y=0} \end{aligned} \right\} \quad \dots (2.12)$$

where  $\delta(x)$  is the thickness of the boundary layer defined in the usual way.

### 3. APPROXIMATE SOLUTION

We investigate the problem by Pohlhausen's fourth degree velocity profile with the following boundary conditions :

$$\left. \begin{aligned} \text{at } y=0 : u=0, n \left[ \frac{\partial u}{\partial y} \right]^{n-1} \frac{\partial^2 u}{\partial y^2} &= - [U' + MB^2] U, \\ \text{at } y=\delta : u=U, \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} &= 0. \end{aligned} \right\} \quad \dots (3.1)$$

Let the profile of the velocity in the form :

$$\frac{u}{U} = f(\eta) = \sum_{i=0}^4 a_i \eta^i, \text{ where } \eta = \frac{y}{\delta} \quad \dots (3.2)$$

The boundary conditions become :

$$\left. \begin{aligned} \text{at } \eta = 0 : f = 0, |f'|^{n-1} f'' = -\lambda, \\ \text{at } \eta = 1 : f = 1, f' = 0, f'' = 0, \end{aligned} \right] \quad \dots (3.3)$$

Using the boundary conditions (3.3) in (3.2) we obtain for the velocity profile

$$\left. \begin{aligned} \frac{u}{U} = f(\eta) &= \left(2 + \frac{\lambda_0}{6}\right) \eta - \frac{\lambda_0}{2} \eta^2 + \left(\frac{\lambda_0}{2} - 2\right) \eta^3 + \left(1 - \frac{\lambda_0}{6}\right) \eta^4, \\ &= (2\eta - 2\eta^3 + \eta^4) + \lambda_0 \frac{\eta}{6} (1 - \eta)^3. \end{aligned} \right] \quad \dots (3.4)$$

where  $\lambda_0 \left[2 + \frac{\lambda_0}{6}\right]^{n-1} = \lambda, \lambda = \frac{\delta^{n+1}}{nU^{n-1}} [U^n + (MB^2)]. \quad \dots (3.5)$

To determine the range of  $\lambda_0$ , we have  $f'(\eta) = 0$  at  $\eta = 0$ , which give  $\lambda_0 = -12$ .

The monotonic increasing of the velocity throughout the boundary layer implies that :

$$2 + 4\eta + \frac{\lambda_0}{6} (1 - 4\eta) = 0, \eta \geq 1,$$

hence  $\lambda_0 \leq 12$  and so the range of values of

$$\lambda_0 \text{ is } -12 \leq \lambda_0 \leq 12. \quad \dots (3.6)$$

By using the velocity profile (3.4), the displacement thickness and the momentum thickness and the shear stress at the wall defined by (2.12) are given by

$$\theta = \delta H(\lambda_0), \delta^* = \delta F(\lambda_0), \tau_w = \frac{U^n}{\delta^n} C(\lambda_0) \quad \dots (3.7)$$

where  $\left. \begin{aligned} F(\lambda_0) &= \frac{3}{10} - \frac{\lambda_0}{120}, C(\lambda_0) = \left(2 + \frac{\lambda_0}{6}\right)^n, \\ H(\lambda_0) &= \frac{37}{315} - \frac{\lambda_0}{945} - \frac{\lambda_0^2}{9072}. \end{aligned} \right] \quad \dots (3.8)$

Introducing  $\theta, \delta^*$  and  $\tau_w$  in the eq. (2.11), gives

$$H \delta^n \frac{d\delta}{dx} + \left[ \frac{dH}{dx} + (2H + F) \frac{U'}{U} + \frac{MB^2}{U} F \right] \delta^{n+1} = U^{n-2} C (\lambda_0). \quad \dots (3.9)$$

#### 4. APPLICATIONS

We will consider the following problems

##### 4.1 Boundary Layer Flow of Newtonian Fluid Along a Flat Plate. (Blasius Problem)

For this problem, take  $U = a$ ,  $M = 0$  and  $n = 1$ .

From (3.5) and (3.8) we get

$$\lambda = \lambda_0 = 0, H = 0.12, F = 0.3 \text{ and } C = 2 \quad \dots (4.1)$$

To determine the thickness of the boundary layer, from (3.9) and (4.1) we obtain

$$\delta = 5.77 \sqrt{x/a} \quad \dots (4.2)$$

taking into account that  $\delta = 0$  at  $x = 0$ .

The displacement thickness, the momentum thickness and the shear stress on the wall, which are defined in (3.7), will take the values.

$$\left. \begin{aligned} \delta^* &= 0.3 \delta = 1.73 \sqrt{x/a}, \quad \theta = 0.12 \delta = 0.693 \sqrt{x/a}, \\ \tau_w &= \frac{a}{\delta} C = 0.346 \sqrt{a^3/x}. \end{aligned} \right\} \quad \dots (4.3)$$

The result obtained by Blasius as exact solution for the shear stress is

$$\tau_w = 0.333 \sqrt{a^3/x},$$

hence there is a very good agreement between the exact and approximates solution.

If we take into account the effect of the magnetic field *i.e.*  $M \neq 0$  in Blasius problem we get

$$\lambda = \lambda_0 \text{ from (3.5) and we will take } \lambda_0 = 1,$$

hence from (3.8) we obtain

$$H = 0.116, F = 0.292, C = 2.17.$$

To determine the boundary layer thickness, from (3.5) we obtain

$$\lambda = \delta^2 MB^2 = 1 \text{ i.e., } MB^2 = \frac{1}{\delta^2}, \quad \dots (4.4)$$

Eliminating  $MB^2$  between (4.4) and (3.9), the solution of the resulting equation upon the boundary condition ( $\delta = 0$  at  $x = 0$ ) is given by

$$\delta = 5.69 \sqrt{x/a}. \quad \dots (4.5)$$

Hence the displacement thickness, the momentum thickness and the shear stress on the plate will be obtained in the form

$$\left. \begin{aligned} \delta^* &= 0.292 \delta = 1.66 \sqrt{x/a}, \theta = 0.116 \delta = 0.66 \sqrt{x/a}, \\ \tau_w &= \frac{a}{\delta} C = 0.381 \sqrt{a^3/x}, \end{aligned} \right] \dots (4.6)$$

and the intensity of the magnetic field has the value

$$B = \frac{1}{\sqrt{M}} \cdot \frac{1}{\delta} = \frac{b}{\sqrt{x}}, \text{ } b \text{ is constant.}$$

This intensity will satisfy Maxwell equations at a large distance from the plate.

By comparing the results in the absence and existence of the magnetic field, we deduce that the existence of the magnetic field reduced the thickness of the boundary layer  $\delta$ , the thickness of the displacement and the thickness of the momentum, moreover the stress on the plate is increased.

4.2 Analysis of Steady Flow of a Power-Law Fluid in the Boundary Layer

It is convenient to assume that

$$U(x) = ax^m, \text{ } a > 0, m \geq 0 \text{ and } \lambda_0 = 2. \dots (4.7)$$

From (3.5) we get

$$MB^2 \delta^{n+1} = 2n \left( \frac{7a}{3} x^m \right)^{n-1} - am \delta^{n+1} x^{m-1}, \dots (4.8)$$

and from (3.8) we have

$$H = 0.115, F = 0.283, C = (7/3)^n. \dots (4.9)$$

Hence, inserting eq. (4.8) and (4.9) in (3.9) yields

$$\frac{d}{dx} \delta^{n+1} + \left( \frac{2m(n+1)}{x} \right) \delta^{n+1} = \left( \frac{7}{3} \right)^{n-1} (n+1) (20 - 4.9n) a^{n-2} \cdot x^{(n-2)m}.$$

The solution of this equation is

$$\delta(x) = \delta_0 \cdot x^{\left( \frac{mn+1-2m}{n+1} \right)}, \dots (4.10)$$

where

$$\delta_0 = \left[ \frac{(7/3)^{n-1} [20 - 4.9n] (n+1)}{3nm+1} a^{n-2} \right]^{\frac{1}{n+1}}. \dots (4.11)$$

To obtain the intensity of the acting magnetic field, substituting from (4.10) in (4.8) we get

$$B(x) = bx^{\left( \frac{m-1}{2} \right)}, \dots (4.12)$$

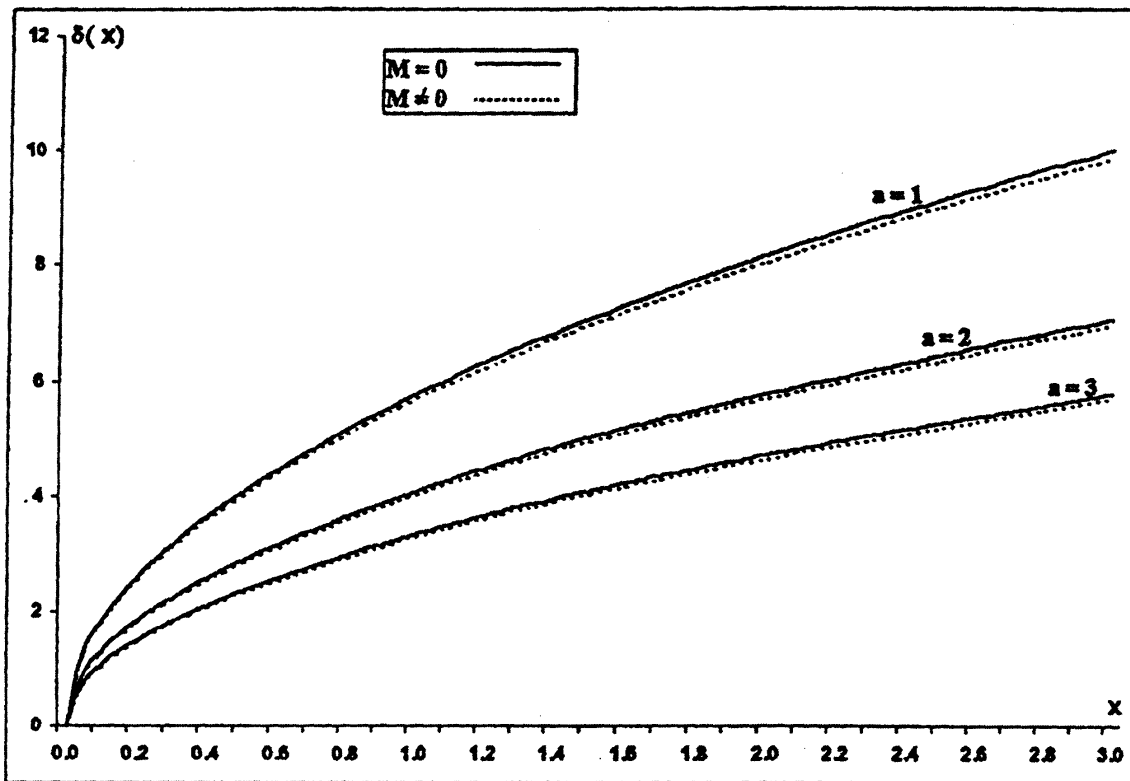


FIG. 1. Boundary layer thickness

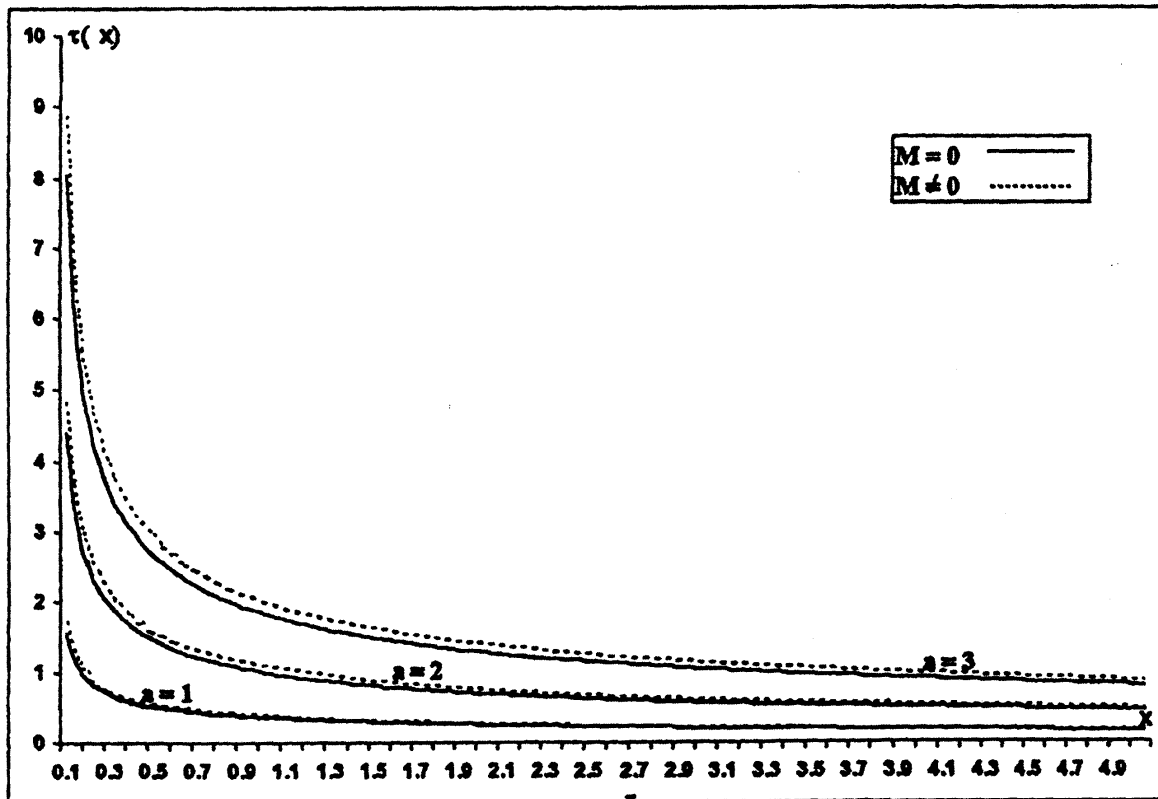


FIG. 2. Shear stress



where the constant  $b$  is given by

$$b = \left[ \left( \frac{a^{3-n}}{M} \right) \left( \frac{2n(3nm+1)}{(n+1)(20-4.9n)} \right) - \frac{am}{M} \right]^{\frac{1}{2}} \quad \dots(4.13)$$

Therefore, the displacement thickness, the momentum thickness and the shear stress on the wall are given as :

$$\left. \begin{aligned} \delta^* = \delta F = 0.283 \delta_0 x \left[ \frac{mn+1-2m}{n+1} \right], \theta = \delta H = 0.115 \delta_0 x \left[ \frac{mn+1-2m}{n+1} \right], \\ \tau_w = \left( \frac{U}{\delta} \right)^n C = \left( \frac{7a}{3\delta_0} \right)^n \cdot x^{\frac{n(3m-1)}{n+1}} \end{aligned} \right\} \quad \dots (4.14)$$

It is clear that the shear stress on the plate will increase with increasing  $x$  for  $m > 1/3$ , and decrease for  $m < 1/3$ . This result agrees with ref. [9].

4.3. Flow Near Stagnation point (at the Leading Edge)  $m = 1, U = ax$

For this case we get the boundary layer thickness, the displacement thickness, the momentum thickness and the shear stress, which take the form

$$\left. \begin{aligned} \delta(x) = \delta_0 x^{\frac{n-1}{n+1}}, \delta^*(x) = 0.283 \delta_0 x^{\frac{n-1}{n+1}}, \\ \theta(x) = 0.115 \delta_0 x^{\frac{n-1}{n+1}}, \tau_w = \left( \frac{7a}{3\delta_0} \right) \cdot x^{\frac{2n}{n+1}}, \end{aligned} \right\} \quad \dots (4.15)$$

where  $\delta_0$  is given in eq. (3.20) at  $m = 1$ .

From (4.15) it is obvious that for the Newtonian fluid ( $n = 1$ ), these thicknesses  $\delta(x), \delta^*(x)$  and  $\theta(x)$  are constants, while the shear stress takes the form of straight lines.

5. RESULTS AND DISCUSSION

The integral method of Von-Karman-Pohlhausen's types is adopted in this work for the solution of the flow of non-Newtonian power law conducting fluid past a semi-infinite plate. This enables us to conclude the following points :

(i) To see the influence of the magnetic field, the boundary layer thickness is plotted against  $x$  in Fig. (1). As seen from the figure, the increase of the magnetic field and the velocity of the free stream decreases the boundary layer thickness.

(ii) The boundary layer thickness  $\delta(x)$  is plotted vs.  $x$  in Figs. (3a, 3b) for different values of  $n = 0.5$  for pseudoplastic fluid,  $n = 1$  for Newtonian fluid,  $n = 1.5$  point dilatant fluid) and for uniform flow ( $m = 0$ ) and flow near the stagnation point ( $m = 1$ ). As is visible from these figures at  $m = 0$ , the thickness decreases with increasing  $n$ , while at  $m = 1$  the thickness decreases with

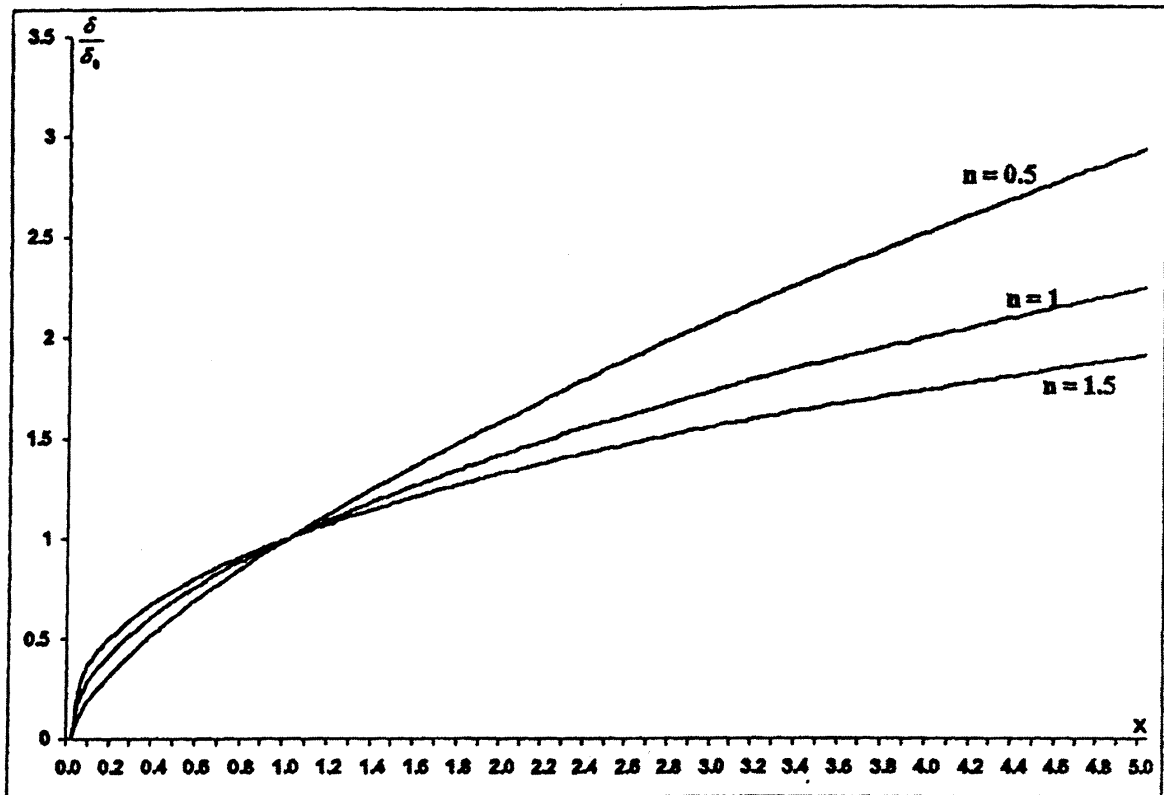


FIG. 3a. Boundary layer thickness for uniform flow ( $m = 0$ )

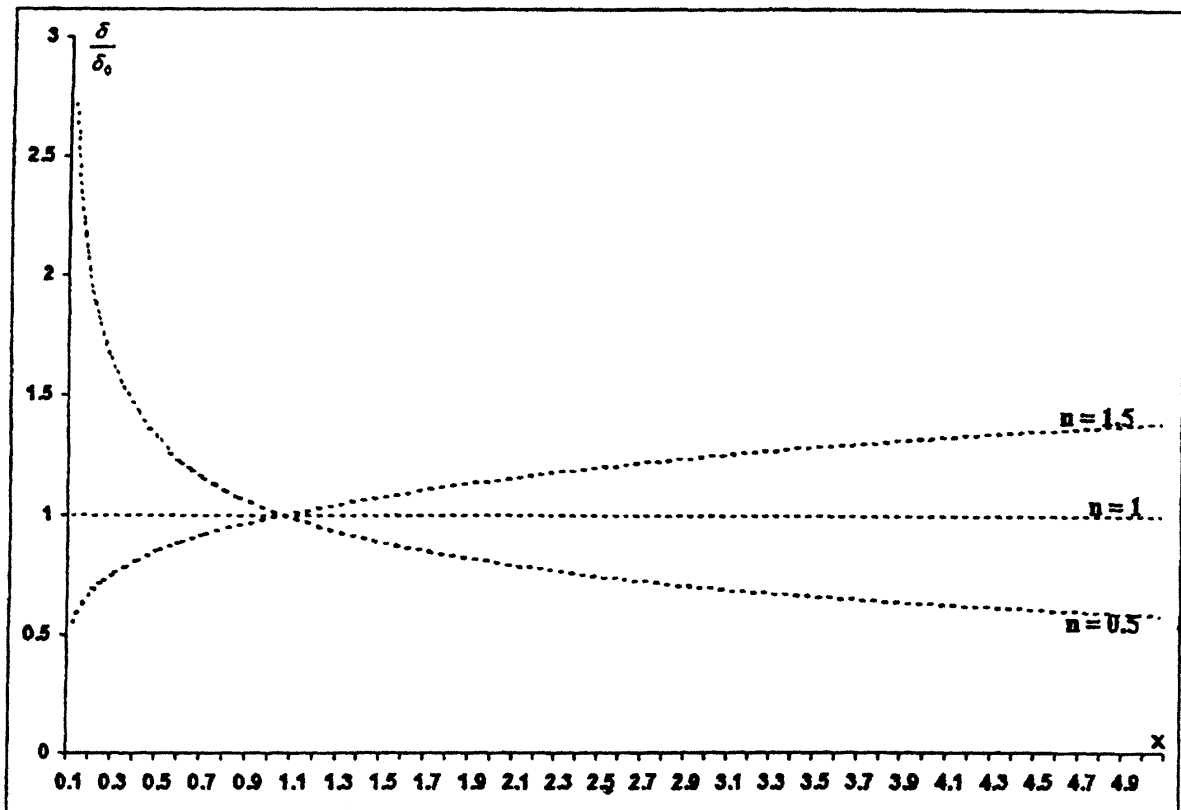


FIG. 3b. Boundary layer thickness about stagnation point ( $m = 1$ )

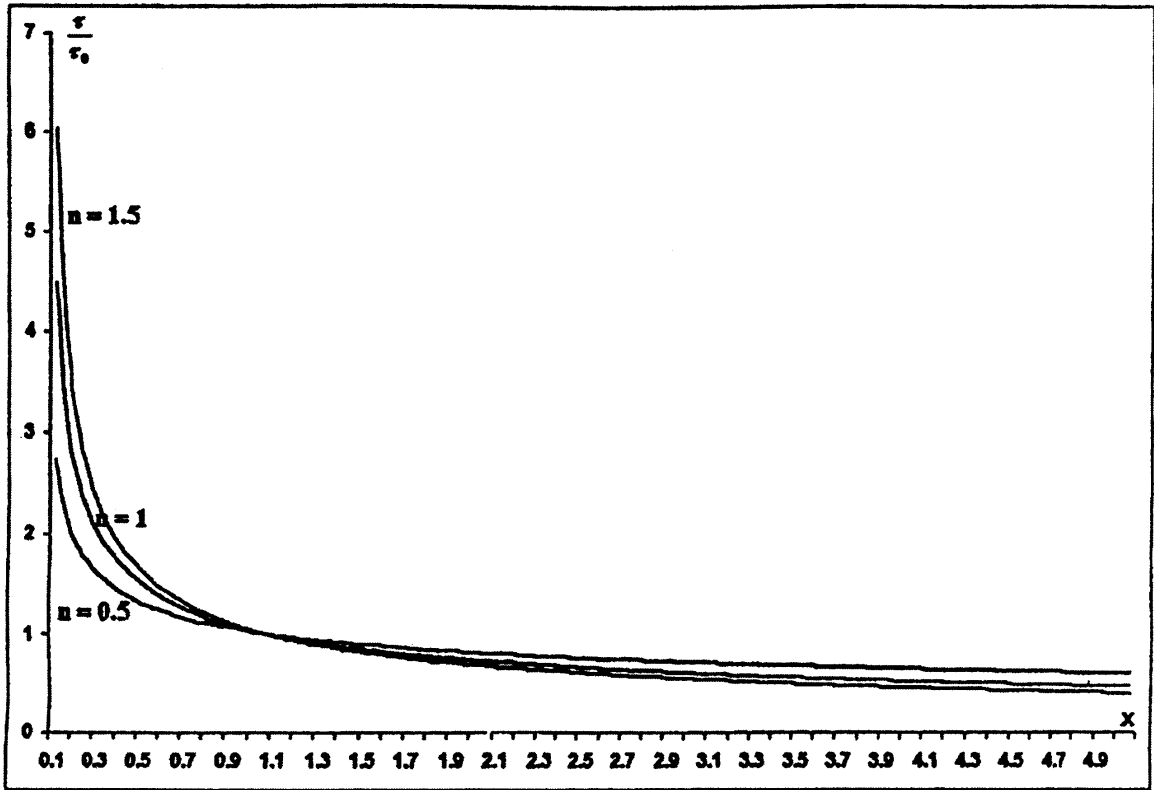


FIG. 4a. Shear stress at the plate ( $m = 0$ )

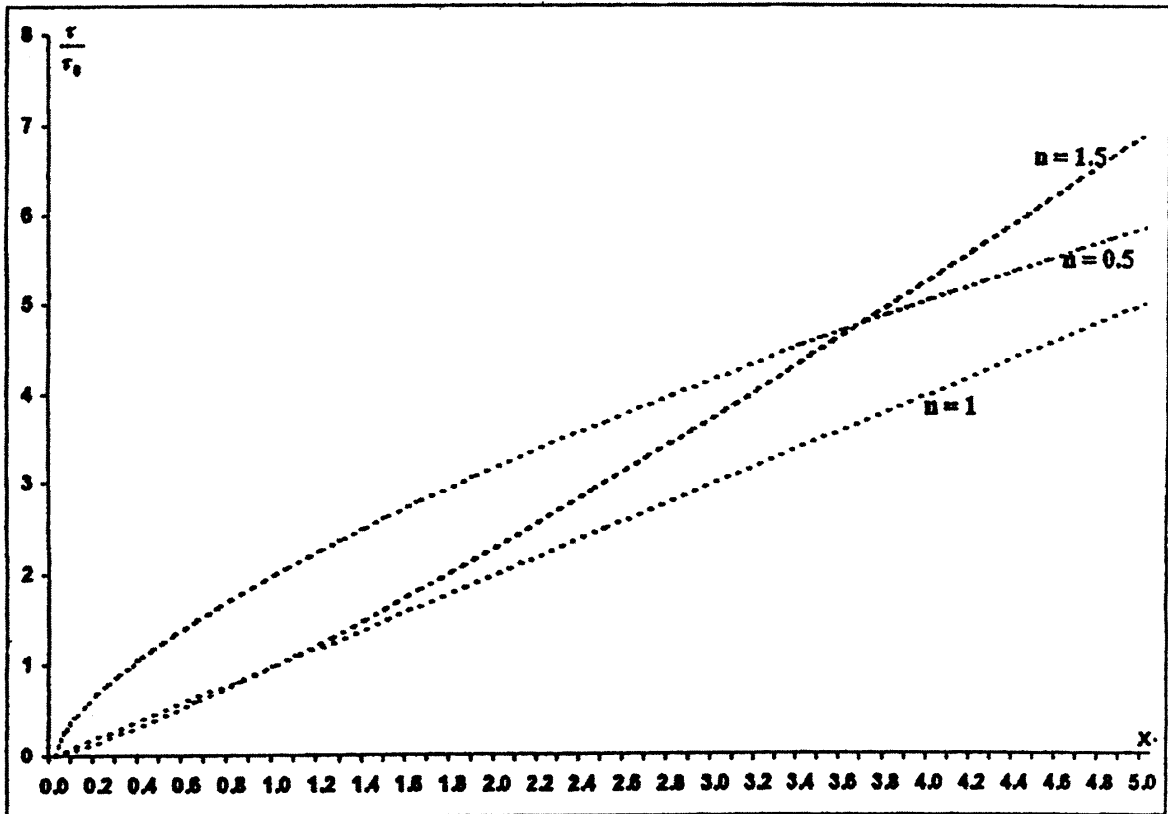


FIG. 4b. Shear stress about stagnation point ( $m = 1$ )

increasing  $n$ , while at  $m = 1$  the thickness starts from largest values at  $n = 1.5$  and increases, and for Newtonian fluid it takes constant values.

(iii) The shear stress  $\tau$  is plotted vs.  $x$  in Fig. 2 for different values of the velocity of free stream and  $M = 0$ ,  $M \neq 0$ . It can be seen from this figure that the existence of  $M$  leads to increasing  $\tau$ .

(iv) In figures (4a, 4b), the shear stress is plotted vs  $x$  for different values of  $n$  (0.5, 1, 1.5),  $m = 0$  and  $m = 1$ . These figures indicate that  $\tau$  increases with increasing values of the power-law fluid parameter  $n$ .

Finally, to conclude, it can be said that the results obtained by applying Karman-Pohlhausen's approximate method in this work agree with results obtained by Blasius in his exact solution (for Newtonian fluid  $n = 1$ ). Also the solution and figures obtained indicated that the effect of the magnetic field causes a decrease in the boundary layer thickness, the velocity and increases the shear stress on the plate.

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