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AN EXTENSION OF MARKOV-KAKUTANI'S FIXED POINT THEOREM

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Mappings defined on a star-shaped domain of a topological linear space are investigated in this paper. Some interesting results are derived. Our main result is a fixed point theorem for a commutative family of continuous affine mappings defined on a non-convex domain that extends the classical Markov-Kakutani Theorem.

Key Words : Common Fixed Point; Star-Shaped Domain; Star-Core; Affine Mappings

1. INTRODUCTION

In 1910, Brouwer[3] proved that every continuous function which maps a non-empty compact convex subset $K \subseteq E^n$ into itself has a fixed point in K. Later in 1922, Banach[1] proved that every contraction mapping of a non-empty, complete metric space M into itself has a unique fixed point in M. Due in part to the immediate applications of these two fundamental theorems to the theory of ordinary differential equations, many mathematicians were attracted to the study of fixed point theory and numerous important results were subsequently obtained. Two of the more significant results developed in the period 1930-40 were the Schauder-Tychonoff Theorem [6, p. 456] and the Markov-Kakutani Theorem [6, p. 456]. The Schauder-Tychonoff Theorem states that a continuous function $f: K \to K$ from a compact convex subset $K \subseteq X$ of a locally convex topological linear space X into itself has a fixed point. And Marov-Kakutani Theorem states that every commutative family of continuous affine mappings of a compact convex set of a topological linear space into itself has a common fixed point. Naturally, mathematicians have tried to obtain the same conclusion of the above-mentioned theorems under weaker assumptions, and in fact important fixed point theorems for non-linear mappings with non-compact domains have been discovered. Often, the assumption that the domain is convex has proved to be essential for these results. Indeed, it has been shown (see Bing[2]) that there exists a continuous mapping of a compact star-shaped subset K of E^3 into itself that fails to have a fixed point. Whether Markov-Kakutani Theorem is still valid if the domain of the mappings is merely assumed to be star-shaped had been an open question for decades. It is our primary purpose in this paper to give an affirmative answer to the above-mentioned open problem. Also, we remark that some of the recent papers might be of interest to the readers[4, 5 & 9].

2. BASIC DEFINITIONS AND NOTATIONS

Let V be a topological linear space. A subset $A \subseteq V$ is said to be star-shaped if there exists an element $x \in K$ such that $tx + (1-t)\,y \in K$ for all $t \in [0, 1]$ and for all $y \in K$. Such an element x is called a star-point of K. The set of all star-points of K is called the star-core of K. The algebraic-segment joining two points x and y will be denoted by $[x, y]$. That is

$$[x, y] = \{z \mid z - \alpha x + (1-\alpha) y \text{ for some } \alpha \in [0, 1]\}$$

A mapping $T: K \to V$ that has a star-shaped domain K is said to be affine on K if for any segment $[x, y] \subseteq K$, we have $T[\alpha x + (1-\alpha) y] = \alpha Tx + (1-\alpha) Ty$ for all $\alpha \in [0, 1]$.

3. Main Results

We shall first establish some interesting facts that are essential for our main theorem. We shall begin with

Theorem 1 — *Let K be a compact star-shaped subset of a topological linear space X. Then every decreasing chain of non-empty compact star-shaped subsets of K has a nonempty intersection that is compact and star-shaped.*

PROOF : Let $\{A_\alpha : \alpha \in I\}$ be a decreasing chain of nonempty compact star-shaped subsets of K. It follows then from the compactness of K that $A = \bigcap_{\alpha \in I} A_\alpha$ is nonempty and compact. It remains so show that A is star-shaped. For that purpose, let K_α be the corresponding star-core of A_α and let $x_\alpha \in K_\alpha \subseteq A_\alpha$. Then by compactness of K, the net $\{x_\alpha : \alpha \in I\}$ has a convergent subset $\{x_{\alpha'} : \alpha' \in I' \subseteq I\}$ and a point x in K such that $x_{\alpha'}$ converges to x. We shall now show that $x \in A = \bigcap_{\alpha \in I} A_\alpha$. Indeed, since $\{x_{\alpha'} : \alpha' \in I'\}$ is a subnet, for each $\alpha \in I$, there exists an $\beta \in I'$ with $\beta > \alpha$. From the fact that $\{A_\alpha : \alpha \in I\}$ is a decreasing chain, we have $\{x_{\alpha'} : \alpha' > \beta\} \subseteq A_\alpha$. Since A_α is compact, $x = \lim_{\alpha'} x_{\alpha'} \in A_\alpha$. That is, $x \in A_\alpha$ for each $\alpha \in I$ and hence $x \in \bigcap A_\alpha = A$. Next, we shall show that x is a star-point of A. For that purpose, we let y be an arbitrary element in $A = \bigcap A_\alpha$ and $0 \leq \lambda \leq 1$ be arbitrary too. Since $x_{\alpha'} \in K_{\alpha'}$, we have $\lambda x_{\alpha'} + (1-\lambda) y \in A_{\alpha'}$. Also, since addition and scalar multiplication are continuous, $x_{\alpha'} \to x$ implies $\lambda x_{\alpha'} + (1-\lambda) y \to \lambda x + (1-\lambda) y$. By similar reasoning as in the previous paragraph, for each $\alpha \in I$, we may choose a $\beta \in I'$ with $\beta > \alpha$. Then $\{\lambda x_{\alpha'} + (1-\lambda) y : \alpha' > \beta\} \subseteq A_\alpha$. Compactness of A_α then implies $\lambda x + (1-\lambda) y \in A_\alpha$ for each $\alpha \in I$. Consequently, $\lambda x + (1-\lambda) y \in A = \bigcap A_\alpha$

We shall make use of the following lemma which has been noted in [7, 8] and whose proof is straight forward and thus omitted.

Lemma 1 — Suppose K is a compact star-shaped subset of a topological linear space V and A s the corresponding star-core of K. Then A is a compact convex subset of K.

Lemma 2 — Suppose K is a star-shaped subset of a topological linear space V and $T: K \to K$ is a surjective mapping that is affine on K. Then the star-core of K is invariant under T.

PROOF : Let C be the star-core of K and $x_0 \in C$ a star-point of K. We need to show that $Tx_0 \in C$. For that purpose, let $y \in K$ and $0 \leq \lambda \leq 1$ be arbitrary. Since T is surjective, there exists some $x \in K$ such that $Tx = y$. Since x_0 is a star-point of K, we have $\lambda x_0 + (1-\lambda) x \in K$. It follows then from the facts that T is affine and K is invariant under T that $\lambda T x_0 + (1-\lambda) y = \lambda Tx_0 + (1-\lambda) Tx = t[\lambda x_0 + (1-\lambda) x] \in T(K) \subset K$. Hence Tx_0 is star-point of K. Thus C is invariant under T and the proof is complete.

We are now ready to state and prove our main result as the following

Theorem 2 — *Let K be a compact star-shaped subset of a topological linear spave V. Suppose F is a commutative family of continuous affine mappings of K into itself. Then F has a common fixed point in K.*

PROOF : We may use Theorem 1 and Zorn's lemma to obtain a set $M \subseteq K$ such that M is minimal with respect to being nonempty, compact, star-shaped and invariant under each $f \in F$. We claim then $f(M) = M$ for each $f \in F$. We claim then $f(M) = M$ for each $f \in F$. Assume the contrary and let $g \in F$ be sucth that $g(M) \subsetneq M$. Let $g(M) = N$. Since g is affine and star-shapedness is preserved by affine mappings, it follows that N is star-shaped. Also, N is obviously nonempty and compact. Suppose now $x \in N = g(M)$. Then there exists some $y \in M$ such that $x = g(y)$. Since F is commutative, we have $f(x) = f(g(y)) = g(f(y)) \in g(M) = N$ for each $f \in F$. Consequently, $f(N) \subseteq N$ for all $f \in F$. Thus $N \subsetneq M$ is a nonempty compat star-shaped subset of K that is invariant under each $f \in F$. That is a contradiction to the minimality of M. Hence, $f(M) = M$ for each $f \in F$. Let C be the star-core of M. It follows then from the lemmas and that C is a nonempty compact convex subset of K that is invariant under each $f \in F$. It follows then from the Markov-Kakutani Theorem that the family F has a common fixed point in $C \subseteq M \subseteq K$ and the proof is complete.

4. EXAMPLES

First, recall that in Theorem 1, we have assumed that $\{A_\alpha : \alpha \in I\}$ is a decreasing chain of nonempty compact star-shaped subsets. One might be tempted to think that the compactness is used to ensure a nonempty intersection only and has nothing to do with the star-shapedness. He might, therefore, make the following conjecture analogous to Theorem 1.

Conjecture — Let K be a star-shaped subset of a topological linear space X. Then every decreasing chain of star-shaped subsets of K has an intersection that is also star-shaped (might be empty).

We give the following example to show that the conjecture above is false.

Example 1 — Let $z_1 = (-1, 0), z_2 = (1, 0), \upsilon_n = (0, 1/n) \in \mathbb{R}^2$ for $n = 1, 2, 3, \ldots$, and $M = (z_1, z_2)$ is the open line-segment joining the two points z_1 and z_2. Suppose D_n is the closed triangular region with vertices z_1, z_2 and υ_n, and $K_n = D_n \setminus M$ (i.e. K_n is D_n with the open line-segment joinint z_1 and z_2 deleted). It follows that each υ_n is a star-point of K_n and $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$. However, $\bigcap_{n=1}^{\infty} K_n = \{z_1, z_2\}$ is a set consisting of two isolated points and is not star-shaped.

Next, we given an example to show that the surjectivity condition of the mapping T as in Lemma 2 is not superfluous.

Example 2 — Let $A_1 = \{(x, 0) : 0 \le x \le 1\}$, $A_2 = \{(x, y) : 0 \le x \le 1, y = x\}$ and let $K = A_1 \cup A_2$. Then K is star-shaped and its star-core is $\{(0, 0)\}$.

Define $T : K \to K$ by $T : (x, y) \to (1/2, 0)$ if $(x, y) \in A_2$,

$$T : (x, y) \to (x/2 + 1/2, 0) \text{ if } (x, y) \in A_1.$$

Then it is easily seen that T is affine on K but the star-core of K is not invariant under T.

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