Indian J. pure appl. Math., 32(6) : 903-914, June 2001

UNIQUENESS THEOREMS FOR MEROMORPHIC FUNCTIONS CONCERNING SMALL FUNCTIONS*

HONG-XUN YI

Department of Mathematics, Shandong University, Jinan 250100, *People's Republic of China, (E-mail: hxyi@sdu.edu.cn)*

(*Received* 21 *February* 2000; *accepted* 17 *October* 2000)

In this paper, we deal with the problem of uniqueness of meromorphic functions concerning small functions, and obtain several results which improve and extend some theorems given by Nevalinna, Shirosaki, Toda, Zhang, Li, Li, Ishizaki and Toda, Li and Qiao, and other authors.

Key Words and Phrases : Meromorphic Function; Small Function; Uniqueness Theorem

1. INTRODUCTION

By a meromorphic function we shall always mean a meromorphic function in the complex plane $\mathbb{C}$. Let $f(z)$ be a nonconstant meromorphic function. We use with their usual definitions the Nevanlinna functions $T(r, f)$, $N(r, f)$, etc.; $S(r, f)$ is a function, not necessarily the same at each occurrence, that is $o(T(r, f))$ as $r \to \infty$ outside some set of finite linear measure that depends on the context and may be empty (see [1]). A meromorphic function $a(z)$ that satisfies $T(r, a) = S(r, f)$ is called a small function with respect to f. Let $S(f)$ be the set of meromorphic functions which are small functions with respect to f. Note that $\mathbb{C} \in S(f)$ and $S(f)$ is a field (see [2]).

Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions and let $a(z) \in \{S(f) \cap S(g)\} \cup \{\infty\}$. We denote by $\overline{N}_0(r, a, f, g)$ the counting function of common zeros of $f(z) - a(z) = 0$ and $g(z) - a(z) = 0$ (ignoring multiplicities), each point in this counting function is counted only once, where $f(z) - \infty$ means $1/f(z)$. Let

$$\overline{N}_{12}(r, a, f, g) = \overline{N}(r, a, f) + \overline{N}(r, a, g) - 2\,\overline{N}_0(r, a, f, g). \qquad \text{... (1.1)}$$

Thus, $\overline{N}_{12}(r, a, f, g)$ denotes the counting function of different zeros of $f(z) - a(z) = 0$ and $g(z) - a(z) = 0$. Set

$$\lambda(a, f, g) = 1 - \limsup_{r \to \infty} \frac{\overline{N}_{12}(r, a, f, g)}{T(r, f) + T(r, g)}. \qquad \text{... (1.2)}$$

Obviously, $0 \le \lambda(a, f, g) \le 1$ (see [3]).

*Project supported by the NSFC (No. 19871050) and the RFDP (No. 98042209)

If $\overline{N}_{12}(r, a, f, g) = 0$, we say $f(z)$ and $g(z)$ share $a(z)$ IM. If

$$\overline{N}_{12}(r, a, f, g) = S(r, f) + S(r, g), \qquad \dots (1.3)$$

we say $f(z)$ and $g(z)$ share $a(z)$ "IM" (see [4]).

In 1929, R. Nevanlinna proved the following well-known theorem.

Theorem A (see [3]) — *If f and g are meromorphic functions sharing a_j **IM** for $j = 1, 2, \dots, 5$, where $a_1, a_2, \dots, a_5$ are five distinct elements in $\mathbb{C} \cup \{\infty\}$, then $f \equiv g$.*

In [3], R. Nevanlinna ask whether it is possible to extend this theorem to the case when $a_1, a_2, \dots, a_5$ belong to $\{S(f) \cap S(g)\} \cup \{\infty\}$. In the recent years, several results on this problem are known (see [4-9]).

Recently, Y. Li and J. Qiao proved the following theorem, which answered the above problem in the affirmative:

Theorem B (see [10]) — *If f and g are nonconstant meromorphic functions sharing a_j **IM** ($j = 1, 2, \dots, 5$), where a_j ($j = 1, 2, \dots, 5$) are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$, then $f \equiv g$.*

In this paper, we shall deal with the problem of uniqueness of meromorphic functions concerning small functions, and obtain several results which improve and extend some theorems given by Nevalinna[3], Shirosaki[5], Toda[6], Zhang[7], Li[8], Li[9], Ishizaki and Toda[2], Li and Qiao[10], and other authors[4]. The following results are two of them.

Theorem 1.1 — *Let f and g be nonconstant meromorphic functions and let a_j ($j = 1, 2, \dots, 5$) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$, Suppose that f and g share a_j ($j = 1, 2, 3$) "**IM**". If $f \not\equiv g$, then*

$$2T(r, f) + 2T(r, g) \le 9\,\overline{N}_{12}(r, a_4, f, g) + 9\,\overline{N}_{12}(r, a_5, f, g) + S(r, f). \qquad \dots (1.4)$$

Theorem 1.2 — *Let f and g be nonconstant meromorphic functions and let a_j ($j = 1, 2, \dots, 5$) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_j ($j = 1, 2, 3$) "**IM**", and*

$$\lambda(a_4, f, g) + \lambda(a_5, f, g) > \frac{16}{9}, \qquad \dots (1.5)$$

then $f \equiv g$.

Remark : In Theorem 1.1 and Theorem 1.2, we do not suppose that f and g always share a_4 and a_5 "**IM**". In Theorem B, we suppose that f and g share a_4 and a_5 **IM**, and hence $\overline{N}_{12}(r, a_4, f, g) + \overline{N}_{12}(r, a_5, f, g) = 0$ and $\lambda(a_4, f, g) + \lambda(a_5, f, g) = 2$, from this we know that Theorem 1.1 and Theorem 1.2 are improvements of Theorem B.

2. Some Lemmas

Lemma 2.1 (see [1]) — Let $f(z)$ be a nonconstant meromorphic function and let a_1, a_2, a_3 be three distinct elements in $S(f) \cup \{\infty\}$. Then

$$T(r,f) \le \sum_{j=1}^{3} \overline{N}(r, a_j, f) + S(r,f).$$

Lemma 2.2 — Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions sharing a_j (j = 1, 2, 3) "**IM**", where a_j (j = 1, 2, 3) are three distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. Then

$$S(r, g) = S(r, f). \qquad \text{... (2.1)}$$

PROOF : Noting that f and g share a_j (j = 1, 2, 3) "**IM**", by Lemma 2.1, we have

$$T(r, g) \le \sum_{j=1}^{3} \overline{N}(r, a_j, g) + S(r, g) = \sum_{j=1}^{3} \overline{N}(r, a_j, f) + S(r,f) + S(r, g)$$

$$\le 3T(r,f) + S(r,f) + S(r, g),$$

from this we obtain the conclusion of Lemma 2.2.

Lemma 2.3 — Let h be a nonconstant meromorphic function, and let $a \in S(h), a \not\equiv 0$ and $a \not\equiv 1$. Then

$$m\left(r, \frac{a'\, h(h-1) - a(a-1)\, h'}{(h-1)\,(h-a)}\right) = S(r, h), \qquad \text{... (2.2)}$$

and

$$m\left(r, \frac{a'\, h(h-1) - a(a-1)\, h'}{h\,(h-1)\,(h-a))}\right) = S(r, h). \qquad \text{... (2.3)}$$

PROOF : Noting that

$$\frac{a'\, h(h-1) - a(a-1)\, h'}{(h-1)\,(h-a)} = a' + \frac{ah'}{h-1} - \frac{a(h'-a')}{h-a}, \qquad \text{... (2.4)}$$

and

$$\frac{a'\, h(h-1) - a(a-1)\, h'}{h\,(h-1)\,(h-a)} = \frac{ah'}{h-1} - \frac{(a-1)\, h'}{h} - \frac{h'-a'}{h-a}, \qquad \text{... (2.5)}$$

from this we obtain the conclusion of Lemma 2.3.

Lemma 2.4 (see [7]) — Let $f(z)$ be a nonconstant meromorphic function and let $a_1, \dots, a_5$ be five distinct elements in $S(f) \cup \{\infty\}$. Then

$$2T(r,f) \le \sum_{j=1}^{5} \overline{N}(r, a_j, f) + S(r,f). \qquad \text{... (2.6)}$$

Lemma 2.5 (see [2]) — Let f and g be nonconstant meromorphic functions such that f and g share a_j (j = 1, 2, 3, 4) "**IM**", where a_j (j = 1, 2, 3, 4) are four distinct elements in $S(f) \cap S(g)$. If

$$\overline{N}(r,f) \le u\, T(r,f) + S(r,f),\ \ \overline{N}(r, g) \le v\, T(r, g) + S(r, g) \qquad \text{... (2.7)}$$

for some $u, v \in (0, 1)$, then

$$(1-u)\,T(r,f) \le T(r,g) + S(r,f),\ (1-v)\,T(r,g) \le T(r,f) + S(r,g). \qquad \text{... (2.8)}$$

3. Preparatory Theorems

***Theorem* 3.1** — *Let f and g be nonconstant meromorphic functions. Suppose that a_j ($j = 1, 2, ..., 5$) are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If $f \not\equiv g$, and f and g share a_1, a_2 "IM", then*

$$\overline{N}_0(r, a_5, f, g) \le \sum_{j=3}^{4} \overline{N}_{12}(r, a_j, f, g) + S(r,f) + S(r,g). \qquad \text{... (3.1)}$$

PROOF : If $\overline{N}_0(r, a_5, f, g) = S(r,f) + S(r,g)$, (3.1) holds obviously. In the following we suppose

$$\overline{N}_0(r, a_5, f, g) \ne S(r,f) + S(r,g). \qquad \text{... (3.2)}$$

Set

$$L(w) = \frac{(w-a_2)(a_1-a_3)}{(w-a_3)(a_1-a_2)}. \qquad \text{... (3.3)}$$

We have $L(a_1) = 1, L(a_2) = 0, L(a_3) = \infty$. Let $F(z) = L(f(z)), G(z) = L(g(z)), a = L(a_4), b = L(a_5)$. Then

$$T(r,F) = T(r,f) = S(r,f),\ T(r,G) = T(r,g) + S(r,g) \qquad \text{... (3.4)}$$

and $a, b \in S(F) \cap S(G)$. Noting $f \not\equiv g$, we have

$$F \not\equiv G. \qquad \text{... (3.5)}$$

Since f and g share a_1, a_2 "IM", F and G share 1, 0 "IM". From (3.2) and (3.4), we have

$$\overline{N}_0(r, b, F, G) \ne S(r,F) + S(r,G). \qquad \text{... (3.6)}$$

Let

$$H = \frac{F'(a'G(G-1) - a(a-1)G')(F-G)}{F(F-1)G(G-1)(G-a)} - \frac{G'(a'F(F-1) - a(a-1)F')(F-G)}{G(G-1)F(F-1)(F-a)}. \qquad \text{... (3.7)}$$

Then we have

$$H = \frac{(F-G)Q}{F(F-1)(F-a)G(G-1)(G-a)}, \qquad \text{... (3.8)}$$

where
$$Q = F'\{a' G(G-1) - a(a-1) G'\} (F - a)$$
$$- G' \{a' F(F-1) - a(a-1) F'\} (G - a)$$
$$= -a(a-1)(F-G) F' G' + a' FF' G(G-1) - a' F(F-1) GG'$$
$$- aa'F' G(G-1) + aa' F(F-1) G'. \qquad \text{... (3.9)}$$

Suppose that $H \equiv 0$. From (3.5) and (3.7) we obtain

$$\frac{G'(a' F(F-1) - a(a-1) F')}{F-a} \equiv \frac{F'(a' G(G-1) - a(a-1) G')}{G-a}. \qquad \text{... (3.10)}$$

If a is a constant, from (3.10) we get $F \equiv G$, which contradicts (3.5). Thus, $a' \not\equiv 0$. From (3.10), we have

$$\frac{G'(a' F(F-1) - a(a-1) F')}{F'(a'G(G-1) - a(a-1) G')} - 1 \equiv \frac{F-a}{G-a} - 1.$$

From this we get

$$\frac{F' - G'}{F-G} \equiv \frac{(F+G-1) F'}{F(F-1)} - \frac{F'(a' G(G-1) - a(a-1) G')}{a' F(F-1)(G-a)}. \qquad \text{... (3.11)}$$

By (3.6), we know that there is a point z_0 such that z_0 is a common zero of $F - b$ and $G - b$ that is not a zero or a pole of $a', b, b-1, b-a$. It is obvious that z_0 is a pole of the left-hand side of (3.11), and not a pole of the right-hand side of (3.11), which is a contradiction. Thus,

$$H \not\equiv 0. \qquad \text{... (3.12)}$$

By Lemma 2.3, from (3.7) we obtain

$$m(r, H) = S(r, F) + S(r, G). \qquad \text{... (3.13)}$$

Next we make an estimation on $N(r, H)$. By (3.7), we know that the poles of H occur possibly only from the zeros of F, G, $F-1$, $G-1$, $F-a$ and $G-a$, and the poles of F, G and a. Noting that F and G share 0, 1 "IM", we discuss the following ten cases.

Case 1 — Suppose that z_1 is a zero of F of order p_1 and G of order q_1 that is not a zero or a pole of a. From (3.9), we know that z_1 is a zero of Q of order at least $p_1 + q_1 - 1$. Again from (3.8), we know that z_1 is not a pole of H.

Case 2 — Suppose that z_2 is a zero of $F - 1$ of order p_2 and $G - 1$ of order q_2 that is not a zero or a pole of $1 - a$. From (3.9), we know that z_2 is a zero of Q of order at least $p_2 + q_2 - 1$. Again from (3.8), we know that z_2 is not a pole of H.

Case 3 — Suppose that z_3 is a common zero of $F - a$ and $G - a$ that is not a zero or a pole of a and $a - 1$. From (2.5), (3.7), we know that z_3 is not a pole of H.

Case 4 — Suppose that z_4 is a pole of F of order p_4 and G of order q_4 that is not a pole of a. From (3.9), we know that z_4 is a pole of Q of order at most $2p_4 + 2q_4 + 1$. Again from (3.8), we know that z_4 is not a pole of H.

Case 5 — Suppose that z_5 is a zero of $F - a$, which is neither a zero of G, $G - 1$, $G - a$, a and $a - 1$ nor a pole of G and a. From (2.5), (3.7), we know that z_5 is a pole of H of order at most 1.

Case 6 — Suppose that z_6 is a zero of $G - a$, which is neither a zero of F, $F - 1$, $F - a$, a and $a - 1$ nor a pole of F and a. From (2.5), (3.7), we know that z_6 is a pole of H of order at most 1.

Case 7 — Suppose that z_7 is a pole of F, which is neither a zero of G, $G - 1$ and $G - a$ nor a pole of G and a. From (3.7), we know that z_7 is a pole of H of order at most 1.

Case 8 — Suppose that z_8 is a pole of G, which is neither a zero of F, $F - 1$ and $F - a$ nor a pole of F and a. From (3.7), we know that z_8 is a pole of H of order at most 1.

Case 9 — Suppose that z_9 is a zero of $F - a$ and a pole of G, which is not a zero or a pole of a and $a - 1$. From (2.5), (3.7), we know that z_9 is a pole of H of order at most 2.

Case 10 — Suppose that z_{10} is a zero of $G - a$ and a pole of F, which is not a zero or a pole of a and $a - 1$. From (2.5), (3.7), we know that z_{10} is a pole of H of order at most 2.

From the above, we obtain

$$N(r, H) \le \overline{N}_{12}(r, a, F, G) + \overline{N}_{12}(r, \infty, F, G) + S(r, F) + S(r, G). \qquad \text{... (3.14)}$$

Combining (3.13) and (3.14) we get

$$T(r, H) \le \overline{N}_{12}(r, a, F, G) + \overline{N}_{12}(r, \infty, F, G) + S(r, F) + S(r, G). \qquad \text{... (3.15)}$$

By (3.7), (3.12) and (3.15), we have

$$\overline{N}_0(r, b, F, G) \le N(r, 0, H) \le T(r, H) + O(1)$$

$$\le \overline{N}_{12}(r, a, F, G) + \overline{N}_{12}(r, \infty, F, G) + S(r, F) + S(r, G).$$

From this we obtain (3.1), and Theorem 3.1 is thus proved.

Theorem 3.2 — *Let f and g be nonconstant meromorphic functions. Suppose that a_j ($j = 1, 2, \ldots, 5$) are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If $f \not\equiv g$, then*

$$\overline{N}_0(r, a_5, f, g) \le \sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g), \qquad \text{... (3.16)}$$

$$\overline{N}(r, a_5, f) + \overline{N}(r, a_5, g) \le 2 \sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + \overline{N}_{12}(r, a_5, f, g)$$
$$+ S(r, f) + S(r, g). \qquad \text{... (3.17)}$$

PROOF : If $\overline{N}_0(r, a_5, f, g) = S(r, f) + S(r, g)$, (3.16) holds obviously. In the following we suppose

$$\overline{N}_0(r, a_5, f, g) \ne S(r, f) + S(r, g). \qquad \text{... (3.18)}$$

Let $L(w)$ be given by (3.3), and let $F(z)=L(f(z))$, $G(z)=L(g(z))$, $a=L(a_4)$, $b=L(a_5)$. Then, we may obtain (3.4), (3.5) and (3.6). Let H be given by (3.7). From (3.5), (3.6), (3.7), we may obtain (3.12), (3.13). Next we make an estimation on $N(r, H)$. By (3.7), we know that the poles of H occur possibly only from the zeros of F, G, $F-1$, $G-1$, $F-a$ and $G-a$, and the poles of F, G and a. In the following, suppose that $b_1=1, b_2=0, b_3=\infty$ and $b_4=a, f-b_3$ means $1/f$. We discuss the following three cases.

Case A — Suppose that z_1^* is a common zero of $F-b_j$ and $G-b_j$ (j = 1, 2, 3, 4) that is not a zero or a pole of a and $a-1$. In the same manner as Case 1, ..., Case 4 in the proof of Theorem 3.1, we known that z_1^* is not a pole of H.

Case B — Suppose that z_2^* is a zero of $F-b_j$ (j = 1, 2, 3, 4), which is neither a zero of $G-b_k$ (k = 1, 2, 3, 4), a and $a-1$ nor a pole of a or z_2^* is a zero of $G-b_j$ (j = 1, 2, 3, 4), which is neither a zero of $F-b_k$ (k = 1, 2, 3, 4), a and $a-1$ nor a pole of a. In the same manner as Case 5, ..., Case 8 in the proof of Theorem 3.1, we know that z_2^* is a pole of H of order at most 1.

Case C — Suppose that z_3^* is a zero of $F-b_j$ (j = 1, 2, 3, 4) and of $G-b_k$ (k = 1, 2, 3, 4, $k \neq j$), which is not a pole of a. In the same manner as Case 9 and Case 10 in the proof of Theorem 3.1, we know that z_3^* is a pole of H of order at most 2.

From the above, we obtain

$$N(r, H) \leq \sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G). \quad \text{... (3.19)}$$

Combining (3.13) and (3.19) we get

$$T(r, H) \leq \sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G). \quad \text{... (3.20)}$$

By (3.7), (3.12) and (3.20), we have

$$\overline{N}_0(r, b, F, G) \leq N(r, 0, H) \leq T(r, H) + O(1)$$

$$\leq \sum_{j=1}^{4} \overline{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G).$$

From this we obtain (3.16). Noting that

$$\overline{N}(r, a_5, f) + \overline{N}(r, a_5, g) = 2\overline{N}_0(r, a_5, f, g) + \overline{N}_{12}(r, a_5, f, g), \quad \text{... (3.21)}$$

from this and (3.16) we obtain (3.17), and Theorem 3.2 is thus proved.

Let $f(z)$ be a nonconstant meromorphic function and let a_1, a_2, a_3, a_4 be four distinct elements in $S(f) \cup \{\infty\}$. The cross ratio of a_1, a_2, a_3, a_4 is defined by

$$(a_1, a_2, a_3, a_4) = \frac{(a_1 - a_3)(a_2 - a_4)}{(a_2 - a_3)(a_1 - a_4)}.$$

It is easy to prove that if (a_1, a_2, a_3, a_4) is a constant, then $(a_{j1}, a_{j2}, a_{j3}, a_{j4})$ is also a constant, where $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$.

If (a_1, a_2, a_3, a_4) is a constant, from (3.3) we know that $L(a_4) = (a_4, a_1, a_2, a_3)$, and hence $a = L(a_4)$ is also a constant. From (3.7) we have

$$H = \frac{a(1-a) F' G' (F-G)^2}{F(F-1)(F-a) G(G-1)(G-a)}. \qquad \text{... (3.22)}$$

Using (3.22) and proceeding as in the proof of Theorem 3.2, we get the following theorem.

Theorem 3.3 — *Let f and g be nonconstant meromorphic functions. Suppose that a_j (j = 1, 2, ..., 5) are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$ such that (a_1, a_2, a_3, a_4) is a constant, and (a_1, a_2, a_3, a_5) is not a constant. If $f \not\equiv g$, then*

$$2\overline{N}_0(r, a_5, f, g) \le \sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g), \qquad \text{... (3.23)}$$

$$\overline{N}(r, a_5, f) + \overline{N}(r, a_5, g) \le \sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g). \qquad \text{... (3.24)}$$

4. MAIN RESULTS

Theorem 4.1 — *Let f and g be nonconstant meromorphic functions and let a_j (j = 1, 2, ..., 5) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If $f \not\equiv g$ then*

$$2T(r, f) + 2T(r, g) \le 9 \sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g). \qquad \text{... (4.1)}$$

PROOF : Using Theorem 3.2, from (3.17) we obtain for k = 1, 2, ..., 5

$$\overline{N}(r, a_k, f) + \overline{N}(r, a_k, g) \le 2 \sum_{j=1}^{5} \overline{N}_{12}(r, a_j, f, g)$$

$$- \overline{N}_{12}(r, a_k, f, g) + S(r, f) + S(r, g). \qquad \text{... (4.2)}$$

By Lemma 2.4, from (4.2) we get (4.1).

Theorem 4.2 — *Let f and g be nonconstant meromorphic functions and let a_j (j = 1, 2, ..., 5) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If*

$$\sum_{j=1}^{5} \lambda(a_j, f, g) > \frac{43}{9}, \qquad \ldots (4.3)$$

then $f \equiv g$.

PROOF : Suppose that $f \not\equiv g$. By Theorem 4.1, we have (4.1). From (4.1), we get

$$\sum_{j=1}^{5} \lambda(a_j, f, g) \leq \frac{43}{9}, \qquad \ldots (4.4)$$

which contradicts (4.3). Thus $f \equiv g$.

5.1 Results on Sharing Three Small Functions

***Theorem* 5.1** — *Let f and g be nonconstant meromorphic functions and let a_j (j = 1, 2, ..., 5) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. Suppose that f and g share a_j (j = 1, 2, 3) "**IM**". If $f \not\equiv g$, then*

$$\overline{N}_0(r, a_5, f, g) \leq \overline{N}_{12}(r, a_4, f, g) + S(r, f). \qquad \ldots (5.1)$$

PROOF : Noting that f and g share a_j (j = 1, 2, 3) "**IM**", by Lemma 2.2, we have

$$S(r, g) = S(r, f). \qquad \ldots (5.2)$$

Using Theorem 3.2 and noting that f and g share a_j (j = 1, 2, 3) "**IM**", from (3.16) and (5.2) we obtain (5.1).

5.2 Proof of Theorem 1.1

Using Theorem 4.1 and noting that f and g share a_j (j = 1, 2, 3) "**IM**", from (4.1) and (5.2) we get (1.4).

5.3. Proof of Theorem 1.2.

Suppose that $f \not\equiv g$. By Theorem 1.1, we have (1.4). From (1.4), we get $\lambda(a_4, f, g) + \lambda(a_5, f, g) \leq 16/9$, which contradicts (1.5). Thus, $f \equiv g$.

6. Results on Sharing Four Small Functions

By Theorem 5.1, we immediately obtain the following theorem :

***Theorem* 6.1** — *Let f and g be nonconstant meromorphic functions and let a_j (j = 1, 2, ..., 5) be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. Suppose that f and g share a_j (j = 1, 2, 3, 4) "**IM**". If*

$$\overline{N}_0 (r, a_5, f, g) \neq S(r, f), \qquad \text{... (6.1)}$$

then $f \equiv g$.

By Theorem 1.1, we immediately obtain the following theorem.

Theorem 6.2 — *Let f and g be nonconstant meromorphic functions and let* a_j $(j = 1, 2, \ldots, 5)$ *be five distinct elements in* $\{S(f) \cap S(g)\} \cup \{\infty\}$. *Suppose that f and g share* a_j $(j = 1, 2, 3, 4)$ *"IM". If* $f \not\equiv g$, *then*

$$2T(r, f) + 2T(r, g) \leq 9\, \overline{N}_{12} (r, a_5, f, g) + S(r, f). \qquad \text{... (6.2)}$$

By Theorem 6.2, we immediately obtain the following theorem.

Theorem 6.3 — *Let f and g be nonconstant meromorphic functions and let* a_j $(j = 1, 2, \ldots, 5)$ *be five distinct elements in* $\{S(f) \cap S(g)\} \cup \{\infty\}$. *If f and g share* a_j $(j = 1, 2, 3, 4)$ *"IM", and*

$$\lambda (a_5, f, g) > \frac{7}{9}, \qquad \text{... (6.3)}$$

then $f \equiv g$.

Remark : In Theorem 6.2 and Theorem 6.3, we do not suppose not that f and g always share a_5 "IM". In Theorem B, we suppose that f and g share a_5 **IM**, and hence, $\overline{N}_{12} (r, a_5, f, g) = 0$ and $\lambda (a_5, f, g) = 1$, from this we know that Theorem 6.2 and Theorem 6.3 are improvements of Theorem B.

Theorem 6.4 — *Let f and g be nonconstant meromorphic functions and let* a_j $(j = 1, 2, \ldots, 5)$ *be five distinct elements in* $\{S(f) \cap S(g)\} \cup \{\infty\}$. *Suppose that f and g share* a_j $(j = 1, 2, 3, 4)$ *"IM". If* $f \not\equiv g$, *and* $\overline{N} (r, a_5, g) = S(r, g)$, *then*

$$2T (r, f) \leq 5\, \overline{N} (r, a_5, f) + S(r, f). \qquad \text{... (6.4)}$$

PROOF : Noting that $\overline{N} (r, a_5, g) = S(r, g)$, from (1.1) and (5.2) we obtain

$$\overline{N}_{12} (r, a_5, f, g) = \overline{N} (r, a_5, f) + S(r, f). \qquad \text{... (6.5)}$$

Using Theorem 3.2 and noting that the condition of Theorem 6.4, from (3.17) and (6.5) we obtain for $k = 1, 2, 3, 4$

$$\overline{N} (r, a_k, f) + \overline{N} (r, a_k, g) \leq 2\, \overline{N} (r, a_5, f) + S(r, f),$$

that is

$$\overline{N} (r, a_k, f) \leq \overline{N} (r, a_5, f) + S(r, f). \qquad \text{... (6.6)}$$

By Lemma 2.4, from (6.6) we get (6.4).

7. On the Results of Ishizaki and Toda

In 1998, K. Ishizaki and N. Toda proved the following theorem.

Theorem C *(see [2]) — Let f and g be transcendental meromorphic functions such that f and g share a_j (j = 1, 2, 3, 4)* ***IM****, where a_j (j = 1, 2, 3, 4) are four distinct elements in $S(f) \cap S(g)$. If $\overline{N}(r,f)$ and $\overline{N}(r,g)$ satisfy one of the following conditions (a), (b), (c) and (d):*

(a) $\overline{N}(r,f) = S(r,f)$ *and* $\overline{N}(r,g) = S(r,g)$;

(b) $\overline{N}(r,g) = S(r,g), \overline{N}(r,f) \neq S(r,f)$ *and* $\overline{N}(r,f) \leq uT(r,f) + S(r,f)$ *for some* $u \in (0, 1/19)$;

(c) $\overline{N}(r,f) = S(r,f), \overline{N}(r,g) \neq S(r,g)$ *and* $\overline{N}(r,g) \leq \upsilon T(r,g) + S(r,g)$ *for some* $\upsilon \in (0, 1/19)$;

(d) $\overline{N}(r,f) \neq S(r,f), \overline{N}(r,g) \neq S(r,g)$ *and*

$\overline{N}(r,f) \leq u\, T(r,f) + S(r,f),\ \overline{N}(r,g) \leq \upsilon\, T(r,g) + S(r,g)$

for some $u, \upsilon \in (0, 1)$ *satisfying either*

(i) $0 < u < 1/19$ *and* $0 < \upsilon < (2 - 19u)/(20 - 19u)$ *or*

(ii) $0 < \upsilon < 1/19$ *and* $0 < u < (2 - 19\upsilon)/(20 - 19\upsilon)$,

then $f \equiv g$.

In this section, we shall use a different method to prove the following theorems, which are improvements of Theorem C.

Theorem 7.1 — *Let f and g be nonconstant meromorphic functions such that f and g share a_j (j = 1, 2, 3, 4) "****IM****", where a_j (j = 1, 2, 3, 4) are four distinct elements in $S(f) \cap S(g)$. If $\overline{N}(r,f)$ and $\overline{N}(r,g)$ satisfy one of the following conditions (a), (b) and (c) :*

(a) $\overline{N}(r,g) = S(r,g), \overline{N}(r,f) \leq u\, T(r,f) + S(r,f)$ *for some* $u \in [0, 2/5)$;

(b) $\overline{N}(r,f) = S(r,f), \overline{N}(r,g) \leq \upsilon\, T(r,g) + S(r,g)$ *for some* $\upsilon \in [0, 2/5)$;

(c) $\overline{N}(r,f) \neq S(r,f), \overline{N}(r,g) \neq S(r,g)$ *and*

$\overline{N}(r,f) \leq u\, T(r,f) + S(r,f),\ \overline{N}(r,g) \leq \upsilon\, T(r,g) + S(r,g)$

for some $u, \upsilon \in (0, 1)$ *satisfying either*

(i) $0 < u < 2/9$ *and* $0 < \upsilon < (4-9u)/(11 - 9u)$ *or*

(ii) $0 < \upsilon < 2/9$ *and* $0 < u < (4 - 9\upsilon)/(11 - 9\upsilon)$,

then $f \equiv g$.

Proof : By Lemma 2.2, we have

$$S(r,g) = S(r,f),\ S(r,f) = S(r,g). \qquad \text{... (7.1)}$$

We discuss the following three cases.

Case (a) — Suppose that $\overline{N}(r,g) = S(r,g)$, and $\overline{N}(r,f) \leq uT(r,f) + S(r,f)$ for some $u \in [0, 2/5)$.

If $f \not\equiv g$, by Theorem 6.4 we have

$$2T(r,f) \leq 5\,\overline{N}(r,f) + S(r,f) \leq 5u\, T(r,f) + S(r,f),$$

this is a contradiction. Thus $f \equiv g$.

Case (b) — Suppose that $\overline{N}(r,f) = S(r,f)$, and $\overline{N}(r,g) \leq \upsilon\, T(r,g) + S(r,g)$ for some $\upsilon \in [0, 2/5)$. Similar to the case (a), we have $f \equiv g$.

Case (c) — Suppose that $\overline{N}(r,f) \neq S(r,f), \overline{N}(r,g) \neq S(r,g)$ and

$$\overline{N}(r,f) \leq uT(r,f) + S(r,f),\ \overline{N}(r,g) \leq v\,T(r,g) + S(r,g)$$

for some $u, v \in (0, 1)$. We discuss the following two subcases.

Subcase (i). Suppose that $0 < u < 2/9$ and $0 < v < (4-9u)/(11-9u)$.

By (1.1) we have

$$\overline{N}_{12}(r, \infty, f, g) \leq \overline{N}(r,f) + \overline{N}(r,g). \qquad \text{... (7.2)}$$

If $f \not\equiv g$, by Theorem 6.2, we have from (7.2)

$$2T(r,f) + 2T(r,g) \leq 9\,\overline{N}(r,f) + 9\,\overline{N}(r,g) + S(r,f) + S(r,g). \qquad \text{... (7.3)}$$

By Lemma 2.5, we have (2.8). From (2.8), (7.1) and (7.3) we get

$$(4-9u)\,T(r,g) \leq (11-9u)\,v\,T(r,g) + S(r,g),$$

which contradicts $0 < v < (4-9u)/(11-9u)$. Thus $f \equiv g$.

Subcase (ii). Suppose that $0 < v < 2/9$ and $0 < u < (4 - 9\,v)/(11-9\,v)$. Similar to the Subcase (i), we have $f \equiv g$. This completes the proof of Theorem 7.1.

By Theorem 7.1, we immediately obtain the following corollary.

Corollary 7.1 — Let f and g be nonconstant meromorphic functions satisfying

$$\overline{N}(r,f) \leq uT(r,f) + S(r,f) \text{ and } \overline{N}(r,g) \leq v\,T(r,g) + S(r,g)$$

for some constants u and v satisfying $(u, v) \in [0, 2/9] \times [0, 2/9)$ or $(u, v) \in [0, 2/9) \times [0, 2/9]$. If there exist four distinct functions $a_j \in S(f) \cap S(g)$ $(j = 1, 2, 3, 4)$ such that f and g share a_j $(j = 1, 2, 3, 4)$ "**IM**" then $f \equiv g$.

REFERENCES

1. W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
2. K. Ishizaki and N. Toda, *Kodai Math. J.* **21** (1998) 365-71.
3. R. Nevanlinna, *Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes*, Gauthier-Villars, Paris, 1929.
4. H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Pure and Applied Math. Monographs No. 32, Science Press, Beijing, 1995.
5. M. Shirosaki, *Tohoku Math. J.* **45** (1993) 491-97.
6. N. Toda, *Proc. Japan Acad. Ser. A. Math. Sci.* **69** (1993) 61-65.
7. Q. D. Zhang, *Acta Math. Sinica* **36** (1993) 826-33.
8. B. Q. Li, *Amer. J. Math.* **119** (1997) 841-58.
9. Y. H. Li, *Acta Math. Sinica* **41** (1998) 249-60.
10. Y. H. Li and J. Y. Qiao, *Science in China (A)* **43** (2000) 581-90.