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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A CLASS OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS

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This paper deals with the asymptotic behaviour of solutions of new class of second order nonlinear difference equations. The possible solutions of these equations are classified into several classes according to their asymptotic behaviour as $n \to \infty$ and then proved its existence with the help of fixed point techniques. Examples are also given to illustrate the results.

Key Words and Phrases : Asymptotic Behaviour; Nonlinear Difference Equations

1. INTRODUCTION

Consider the second order nonlinear difference equation

$$\Delta (a_n \Delta y_n) \pm f(n, y_n, \Delta y_n) = 0 \qquad \dots (E_{\pm})$$

where $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$. For eq. $(E_{\pm})$, the following conditions are assumed to hold :

(c_1) $\{a_n\}$ is a positive real sequence such that

$$R_n = \sum_{s=0}^{n-1} \frac{1}{a_s} \to \infty \text{ as } n \to \infty,$$

(c_2) $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R} \to (0, \infty)$ is a continuous and increasing in each of the last two variables.

A proto type of eq. $(E_{\pm})$ satisfying (c_1) and (c_2) is

$$\Delta^2 y_n \pm q_n e^{y_n} = 0, n \in \mathbb{N} \qquad \dots (E^1_{\pm})$$

or more generally

$$\Delta^2 y_n \pm q_n \exp(|y_n|^{\gamma-1} y_n + |\Delta y_n|^{\delta-1} \Delta y_n) = 0, n \in \mathbb{N}, \qquad \dots (E^2_{\pm})$$

where $\{q_n\}$ is a positive real sequence and γ and δ are positive constants.

By a solution of eq. $(E_\pm)$, we mean a real sequence $\{y_n\}$ satisfying eq. $(E_\pm)$ for all $n \in \mathbb{N}$. We consider only such solutions which are nontrivial for all large n.

The qualitative behaviour of solutions of special cases of eq. $(E_\pm)$, especially when $f(n, u, v) = f(n, u)$ has been examined by a number of authors with the assumption $uf(n, u) > 0$ for $u \neq 0$ and $n \in \mathbb{N}$, see for example [1-4, 6-10 & 12] and the references cited therein. However, it seems to us that no systematic study of the qualitative behaviour of solutions has so far been attempted even for the simple eq. $(E^1_\pm)$ or $(E^2_\pm)$ and this observation motivated the present work. Therefore, in this paper, we study the asymptotic behaviour of solutions of the difference eq. $(E_\pm)$.

On the basis of condition (c_2), we classify the set S^+ of all solutions $\{y_n\}$, $n \in \mathbb{N}$ of eq. (E_+) into the following four subsets according to the values of $\lim\limits_{n\to\infty} a_n \Delta y_n$:

$$A_+ = \left\{ y = \{y_n\} \in S^+ : \lim_{n\to\infty} a_n \Delta y_n = -\infty \right\};$$

$$B_+ = \left\{ y = \{y_n\} \in S^+ : \lim_{n\to\infty} a_n \Delta y_n = constant < 0 \right\};$$

$$C_+ = \left\{ y = \{y_n\} \in S^+ : \lim_{n\to\infty} a_n \Delta y_n = 0 \right\};$$

and

$$D_+ = \left\{ y = \{y_n\} \in S^+ : \lim_{n\to\infty} a_n \Delta y_n = constant > 0 \right\}.$$

If $y \in A_+ \cup B_+$, then $\{y_n\}$ is eventually decreasing and satisfies $\lim\limits_{n\to\infty} \frac{y_n}{R_n} = -\infty$ or $\lim\limits_{n\to\infty} \frac{y_n}{R_n}$ *constant* < 0 according as $y \in A_+$ or $y \in B_+$, if $y \in C_+ \cup D_+$, then $\{y_n\}$ is increasing for $n \in \mathbb{N}$ and satisfies $\lim\limits_{n\to\infty} \frac{y_n}{R_n} = 0$ or $\lim\limits_{n\to\infty} \frac{y_n}{R_n}$ = *constant* > 0 according as $y \in C_+$ or $y \in D_+$. Using fixed point techniques in Section 2 we establish criteria for the existence/non existence of solution of equation (E_+) in these classes.

Similarly, we classify the set S^- of all solutions $\{y_n\}$ of equation (E_-) into the following four subsets according to the values of $\lim\limits_{n\to\infty} a_n \Delta y_n$:

$$A_- = \left\{ y = \{y_n\} \in S^- : \lim_{n\to\infty} a_n \Delta y_n = +\infty \right\},$$

$$B_- = \left\{ y = \{y_n\} \in S^- : \lim_{n\to\infty} a_n \Delta y_n = constant > 0 \right\},$$

$$C_- = \left\{ y = \{y_n\} \in S^- : \lim_{n\to\infty} a_n \Delta y_n = 0 \right\}$$

and

$$D_- = \left\{ y = \{y_n\} \in S^- : \lim_{n\to\infty} a_n \Delta y_n = constant < 0 \right\}.$$

Again using fixed point techniques in Section 3 we present sufficient conditions for the existence of some or all of these classes of solution of (E_-). Examples are inserted to illustrate the results. The results established in this paper are partially motivated by that of in [5, 11].

2. EXISTENCE/NONEXISTENCE OF A_+, B_+, C_+ AND D_+ CLASS SOLUTION

In this section we study the structure of the different classes of solution of eq. (E_+). First we observe that the set $A_+ \cup B_+$ is always nonempty since the solution $\{y_n\}$ of (E_+) with initial conditions $y_{n_0} = \alpha, \Delta y_{n_0} = \beta \leq 0$ is decreasing for $n \geq n_0 \in I\!N$ and hence is a member of A_+ or B_+.

***Theorem* 2.1** — *With respect to the difference eq. (E_+), the conditions (c_1) and (c_2) hold. Then the class $A_+ \neq \phi$ if and only if*

$$\sum_{n=n_0}^{\infty} f\left(n, -kR_n, \frac{k}{a_n}\right) = \infty, \text{ for all } k > 0. \qquad \text{... (1)}$$

***Theorem* 2.2** — *With respect to the difference eq. (E_+), the conditions (c_1) and (c_2) hold. Then the class $B_+ \neq \phi$ if and only if*

$$\sum_{n=n_0}^{\infty} f\left(n, -kR_n, \frac{-k}{a_n}\right) < \infty, \text{ for some } k > 0. \qquad \text{... (2)}$$

PROOF OF THEOREM 2.2 : Let $y \in B_+$. Summation of eq. (E_+) shows that

$$\sum_{n=n_0}^{\infty} f(n, y_n, \Delta y_n) < \infty. \qquad \text{... (3)}$$

Since $\lim_{n\to\infty} \frac{y_n}{R_n} = \lim_{n\to\infty} a_n \Delta y_n = \textit{constant} < 0$, there exist a constant $k > 0$ and an integer $N \geq n_0 \in I\!N$ such that

$$y_n \geq -kR_n \text{ and } \Delta y_n \geq -\frac{k}{a_n}, \text{ for } n \geq N. \qquad \text{... (4)}$$

Combining (3) and (4) yields (2).

Conversely, suppose that (2) holds. Let $\alpha \in I\!R$ be fixed. In view of condition (c_2)

$$\sum_{n=n_0}^{\infty} f\left(n, \alpha - \lambda R_n, -\frac{\lambda}{a_n}\right)$$

is a non-increasing function of λ for $\lambda > k$, and so one can choose $l > k$ large enough so that

$$\sum_{n=n_0}^{\infty} f\left(n, \alpha - lR_{n,} -\frac{l}{a_n}\right) \le l. \qquad \dots (5)$$

Consider the space S of all real sequences $y = \{y_n\}$, $n \ge n_0$ with the topology of pointwise convergence. Let Y be the subset of S defined by

$$Y = \left\{ y \in S : \alpha\, 2lR_n \le y_n \le \alpha - lR_n, -\frac{2l}{n} \le \Delta y_n \le -\frac{l}{a_n}, n \ge n_0 \right\}$$

Clearly Y is a nonempty closed convex subset of S. Define the operator $T : Y \to S$ by

$$Ty_n = \alpha - 2lR_n + \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, y_t, \Delta y_t), n \ge n_0.$$

It can be easily checked using (5) that $TY \subset Y$ and satisfy all the conditions of Schauder-Tychonoff theorem. Hence, there exists $y \in Y$ such that $Ty = y$. That is,

$$y_n = \alpha - 2lR_n + \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, y_t, \Delta y_t)\; n \ge n_0.$$

It is easy to see that $\{y_n\}$ is a solution of eq. (E_+) such that

$$\lim_{n\to\infty} a_n \Delta y_n = -2l, \text{ that is, } \{y_n\} \in B_+.$$

PROOF OF THEOREM 2.1 : Let $y \in A_+$. Then, summing eq. (E_+) from n_0 to ∞, we have

$$\sum_{n=n_0}^{\infty} f(n, y_n, \Delta y_n) = \infty. \qquad \dots (6)$$

Since $\lim_{n\to\infty} \frac{y_n}{R_n} = \lim_{n\to\infty} Aa_n \Delta y_n = -\infty$, for any $k > 0$, there exists integer $N > n_0$ such that

$$y_n \le -kR_n \text{ and } \Delta y_n \le \frac{-k}{a_n}, n \ge N. \qquad \dots (7)$$

The relation (1) follows from (6) and (7).

Suppose now that (1) holds. Then Theorem 2.2 implies that $B_+ = \phi$ and so $A_+ \ne \phi$ by the remark made at the begining of this section. This completes the proof.

Remark 2.1 : In view of Theorems 2.1 and 2.2, we see that class A_+ is non empty if and only if class B_+ is empty; that is, classes A_+ and B_+ can not coexist.

Example 2.1 — Consider the difference eq.

$$\Delta^2 y_n + \exp\left\{ (n+1)^3 \left(\frac{n^3}{8} + 1 \right) \right\} \exp\left\{ y_n^3 + (\Delta y_n)^3 \right\} = 0, \quad n \geq 0. \quad \ldots (E_+^3)$$

With $R_n = n$, all conditions of Theorem 2.1 are satisfied and hence equation (E_+^3) has a solution $\{y_n\} \in A_+$. In fact (E_+^3) has a solution $\{y_n\} = \left\{ \frac{-n(n+1)}{2} \right\} \in A_+$.

Example 2.2 — Consider the difference eq.,

$$\Delta^2 y_n + \frac{4}{(n+2)(n+3)(n+4)} \exp\left(\frac{n(n+1)(n+3) + (+2)(n+3) - 2}{(n+2)(n+3)} \right) \exp$$

$$(y_n + \Delta y_n) = 0, \quad n \geq 0. \quad \ldots (E_+^4)$$

With $R_n = n$, all conditions of Theorem 2.2 are satisfied and hence equation (E_+^4) has a solution $\{y_n\} \in B_+$. In fact $\{y_n\} = \left\{ \frac{-n(n+1)}{n+2} \right\}$ is such a solution of (E_+^4).

Now we turn our attention to the study of classes C_+ and D_+. To ensure the existence of members of $C_+ \cup D_+$• we need the following additional condition

$$(c_3): \lim_{u \to -\infty} f(n, u, v) = 0 \text{ for } (n, v) \in I\!N \times I\!R.$$

Clearly (c_3) is satisfied for eqs. (E_+^1) and (E_+^2). In the next theorem, we discuss the existence of solutions of (E_+) in the class D_+.

Theorem 2.3 — *With reference to the difference eq.* (E_+) *assume conditions* $(c_1) - (c_3)$ *hold. Then the class* $D_+ \neq \phi$ *if and only if*

$$\sum_{n=n_0}^{\infty} f\left(n, kR_n, \frac{k}{a_n} \right) < \infty, \text{ for some } k > 0. \quad \ldots (8)$$

PROOF : Let $\{y_n\} \in D_+$. Then there exists a constant $k > 0$ and an integer $N > n_0 \in I\!N$ such that

$$y_n \geq kR_n \text{ and } \Delta y_n \geq \frac{k}{a_n}, \text{ for } n \geq N. \quad \ldots (9)$$

Combining (9) with (3) which also holds for $\{y_n\} \in D_+$, we see that condition (8) is satisfied.

Conversely, suppose that (8) holds. From $(c_2), (c_3)$ and (8) and the Lebesgue dominated convergence theorem, we have

$$\lim_{\alpha \to -\infty} \sum_{n=n_0}^{\infty} f\left(n, \alpha + kR_n, \frac{k}{a_n} \right) = 0,$$

and so there exists an $\alpha < 0$ such that

$$\sum_{n=n_0}^{\infty} f\left(n, \alpha + kR_n, \frac{k}{a_n}\right) \leq \frac{k}{2}.$$

Consider the set $Y \subset S$ and the mapping $T : Y \to S$ defined by

$$Y = \left\{ y \in S : \alpha + \frac{k}{2} R_n \leq y_n \leq \alpha + kR_n, \frac{k}{2a_n} \leq \Delta y_n \leq \frac{k}{a_n}, n \geq n_0 \right\} \text{ and}$$

$$Ty_n = \alpha + \frac{k}{2} R_n + \sum_{s=n_0}^{n-1} \frac{1}{a_s} \sum_{t=s}^{\infty} f(t, y_t, \Delta y_t), n \geq n_0.$$

Using an argument similar to the above, we can show that T satisfies the assumptions of the Schauder-Tychnoff fixed point theorem. Therefore, there exists an $y \in Y$ such that $Ty = y$, that is,

$$y_n = \alpha + \frac{k}{2} R_n + \sum_{s=n_0}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} f(t, y_t, \Delta y_t) \right), n \geq n_0.$$

Since

$$\Delta y_n = \frac{k}{2a_n} + \frac{1}{a_n} \sum_{t=n}^{\infty} f(t, y_t, \Delta y_t),$$

we have $a_n \Delta y_n \to \frac{k}{2}$ as $n \to \infty$ and hence $\{y_n\} \in D_+$. This completes the proof.

Example 2.3 — Consider the difference equation

$$\Delta^2 y_n + \frac{2}{(n+1)(n+2)(n+3)} \exp \left[-\left(\frac{n^3 (n+2)^3}{(n+1)^3} + \frac{(n+1)(n+2)+1}{(n+1)(n+2)} \right) \right] X$$

$$\exp(y_n^3 + \Delta y_n) = 0, n \geq 0. \qquad \ldots (E_+^5)$$

With $R_n = n$ all condition of Theorem 2.3 are satisfies and hence eq. (E_+^5) has a solution $y_n \in D_+$. In fact $\{y_n\} = \left\{ \frac{n(n+2)}{n+1} \right\}$ is such a solution of (E_+^5).

Next we examine the class C_+. This class consists of bounded solutions and unbounded solutions and therefore $C_+ = C_b \cup C_u$ where

$$C_b = \left\{ y \in C_+ : \lim_{n \to \infty} y_n = constant \right\},$$

$$C_u = \left\{ y \in C_+ : \lim_{n \to \infty} y_n = \infty \right\}.$$

Theorem 2.4 — *(i) Suppose conditions* (c_1) *and* (c_2) *hold. If* $C_b \neq \phi$ *then there is a constant* k *such that*

$$\sum_{n=n_0}^{\infty} R_n f(n, k, 0) < \infty. \qquad \text{... (10)}$$

(ii) Suppose the conditions $(c_1)-(c_3)$ *hold. If there are constants* k *and* $l > 0$ *such that*

$$\sum_{n=n_0}^{\infty} R_n f\left(n, k, \frac{l}{a_n} \right) < \infty, \qquad \text{... (11)}$$

then $C_b \neq \phi$.

PROOF : Let $\{y_n\} \in C_b$. Summing eq. (E_+) from n to ∞ we have

$$a_n \Delta y_n = \sum_{s=n}^{\infty} f(s, y_s, \Delta y_s), \; n \geq n_0.$$

Dividing the last equation by a_n and then summing from n_0 to $n-1$ we obtain

$$y_n = y_{n_0} + \sum_{s=n_0}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} f(t, y_t, \Delta y_t) \right), n \geq n_0$$

which implies that

$$\sum_{s=n_0}^{\infty} \left(\frac{1}{a_s} \left(\sum_{t=s}^{\infty} f(t, y_t, \Delta y_t) \right) \right) = \sum_{t=n_0}^{\infty} \left(\sum_{s=n_0}^{t} \frac{1}{a_s} \right) f(t, y_t, \Delta y_t) < \infty,$$

that is,

$$\sum_{n=n_0}^{\infty} R_n f(n, y_n, \Delta y_n) < \infty. \qquad \text{... (12)}$$

In view of $\Delta y_n > 0$ and $y_n \geq y_{n_0}$ for all $n \geq n_0$, we see from (12) that (10) holds.

(ii) Suppose that (11) holds for some k and $l > 0$. Choose α so that $\alpha = min\ \{k, -l\}$ and

$$\sum_{n=n_0}^{\infty} \quad max\ \{1, R_n\}\ f\left(n, \alpha, \frac{l}{a_n} \right) \leq l$$

and $Y \subset S$ denote the set

$$Y = \left\{ y \in S : 2\,\alpha \le y_n \le \alpha, 0 \le \Delta y_n \le \frac{l}{a_n}, n \ge n_0 \right\}$$

and define the mapping $T : Y \to S$ by

$$Ty_n = 2\,\alpha + \sum_{s=n_0}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} f(t, y_t, \Delta y_t) \right), n \ge n_0.$$

It is easy to show that T satisfies all conditions of Schauder-Tychnoff fixed point theorem. Hence, T has a fixed point $y \in Y$, which is a solution of (E_+) satisfying $\lim_{n \to \infty} a_n \Delta y_n = 0$ and $\lim_{n \to \infty} y_n = constant \in [2\,\alpha, \alpha]$. Thus $C_b \ne \phi$. The proof is now complete.

Remark 2.2 : Suppose f satisfies $(c_1) - (c_3)$ and

$f(n, u, v) = f(n, u)$, then a necessary and sufficient condition for the existence of a member of C_b is that

$$\sum_{n=n_0}^{\infty} R_n f(n, k) < \infty, \text{ for some consant } k. \qquad \ldots (13)$$

Remark 2.3 : Consider the particular eq. (E_+)

$$\Delta^2 y_n + \phi_n f(y_n)\, g(\Delta y_n) = 0, \qquad \ldots (14)$$

where $\{\phi_n\}$ is a positive real sequence; $f, g : I\!R \to (0, \infty)$ are continuous and increasing and $\lim_{u \to -\infty} f(u) = 0$. Clearly conditions (10) and (11) are equivalent for eq. (14) and reduce to

$$\sum_{n=n_0}^{\infty} n\,\phi_n < \infty. \qquad \ldots (15)$$

It follows that eq. (14) has a solution of class C_b if and only if (15) holds.

Example 2.4 — Consider the difference equation

$$\Delta^2 y_n + \frac{2}{(n+1)(n+2)(n+3)} \exp\left[-\left(\frac{n^3 (n+1)^2 (n+2)^5 + 1}{(n+1)^5 (n+2)^5} \right) \right] X$$

$$exp\ (y_n^3 + \Delta (y_n)^5) = 0, n \ge 0. \qquad \ldots (E_+^6)$$

With $R_n = n$, all conditions of Theorem 2.4 are satisfied and hence eq. (E_+^6) has a class

C_b solution. In fact (E_+^6) has a solution $\{y_n\} = \left\{\frac{n}{n+1}\right\}$ belongs to the class C_b.

It is very difficult to find sufficient conditions which ensure that $C_u \neq \phi$, however, a simple necessary condition for $C_u \neq \phi$ is given in the following theorem.

***Theorem* 2.5** — *Assume conditions* (c_1) *and* (c_2) *hold. If* $C_u \neq \phi$ *for the eq.* (E_+), *then*

$$\sum_{n=n_0}^{\infty} f(n, k, O) < \infty, \text{ for all } k > 0. \quad \dots (16)$$

PROOF : If $\{y_n\} \in C_u$ then clearly (3) holds. Since $\lim_{n\to\infty} a_n \Delta y_n = 0$ and $\lim_{n\to\infty} y_n = \infty$, for any $k > 0$ there is an integer $N > n_0$ such that $y_n \geq k$ and $\Delta y_n > 0$ for $n \geq N$. Using these inequalities in (3), gives (16).

Corollary 2.6 — Assume conditions (c_1) and (c_2) are satisfied. Then $C_+ \cup D_+ = \phi$, if

$$\sum_{n=n_0}^{\infty} f(n, k, 0) = \infty, \text{ for every constant } k. \quad \dots (17)$$

PROOF : From Theorem 2.5, the condition (17) implies $C_u = \phi$. Since condition (17) also implies

$$\sum_{n=n_0}^{\infty} R_n f(n, k, 0) = \infty$$

for all k, part (*i*) of Theorem 2.4 shows that $C_b = \phi$. Noting that (17) also implies

$$\sum_{n=n_0} f\left(n, kR_n, \frac{k}{a_n}\right) = \infty$$

for all $k > 0$, we see from Theorem 2.3 that $D_+ = \phi$. This completes the proof.

The condition (17) ensuring that $C_+ \cup D_+ = \phi$ can be strengthened under more restrictive condition on the nonlinearity of eq. (E_+).

***Theorem* 2.7** — *Consider the difference equation*

$$\Delta(a_n \Delta y_n) + \phi_n f(y_n, \Delta y_n) = 0, \quad \dots (18)$$

where $\{a_n\}$ is defined as before, and $\{\phi_n\}$ is a positive sequence and $f: I\!R \times I\!R \to (0, \infty)$ is continuous. Suppose $f(u, v)$ is non decreasing in u and v and

$$\int_{\delta}^{\infty} \frac{du}{f(u,0)} < \infty, \text{ for all } \delta. \qquad \text{... (19)}$$

If
$$\sum_{n=n_0}^{\infty} R_n \phi_n = \infty, \qquad \text{... (20)}$$

then $C_+ \cup D_+ = \phi$. for eq. (18).

PROOF : Assume that eq. (18) has a solution $\{y_n\} \in C_+ \cup D_+$.

Summing eq. (18) from n to ∞ and noting that $\{y_n\}$ is increasing, we obtain

$$a_n \Delta y_n \geq \sum_{s=n}^{\infty} \phi_s f(y_s, \Delta y_s) \geq \sum_{s=n+1}^{\infty} \phi_s \, (f(y_s, \Delta y_s) \geq f(y_{n+1}, 0) \sum_{s=n+1}^{\infty} \phi_s, \; n \geq n_0$$

and hence

$$\sum_{s=n_0}^{n-1} \frac{1}{a_s} \left(\sum_{t=s+1}^{\infty} \phi_t \right) \leq \sum_{s=n_0}^{n-1} \frac{\Delta y_s}{f(y_{s+1}, 0)} \leq \int_{y_{n_0}}^{y_n} \frac{du}{f(u,0)}.$$

Letting $n \to \infty$ in the last inequality we obtain

$$\sum_{n=n_0}^{\infty} \left(\sum_{s=n_0}^{n-1} \frac{1}{a_s} \right) \phi_s \leq \int_{y_{n_0}}^{\infty} \frac{du}{f(u,0)} < \infty,$$

a contradiction. This completes the proof.

Corollary 2.8 — Consider the difference equation

$$\Delta (a_n \Delta y_n) + \phi_n f(y_n) = 0, \qquad \text{... (21)}$$

where $\{a_n\}$ and $\{\phi_n\}$ are as in theorem 2.7, and $f : I\!R \to (0, \infty)$ is a continuous nondecreasing function such that $\lim_{u \to -\infty} f(u) = 0$ and

$$\int_{\delta}^{\infty} \frac{du}{f(u)} < \infty. \qquad \text{... (22)}$$

Then, for equation (21), $C_+ \cup D_+ = \phi$ if and only if condition (20) holds.

Corollary 2.9 — Consider eq. (14) satisfying the conditions as in Remark 2.3. Moreover, condition (22) is satisfied. Then for eq. (14), $C_+ \cup D_+ = \phi$ and only if

$$\sum_{n=n_0}^{\infty} n\,\phi_n = \infty. \qquad \text{... (23)}$$

We conclude this section with an example.

Example 2.5 — Consider the difference eq.

$$\Delta^2 y_n + q_n \exp\left(|y_n|^{\gamma-1} y_n + |\Delta y_n|^{\delta-1} \Delta y_n\right) = 0 \qquad \text{... } (E_+^2)$$

where γ and δ are positive constants. Applying our results to eq. (E_+^2) yields the following :

i) Eq. (E_+^2) has a solution $\{y_n\}$ such that $\lim_{n\to\infty} \frac{y_n}{n} = -\infty$ (that is, $A_+ \neq \phi$) if and only if

$$\sum_{n=n_0}^{\infty} q_n \exp(-k_n^{\gamma}) = \infty, \text{ for all } k > 0. \qquad \text{... (24)}$$

ii) Eq. (E_+^2) has a solution $\{y_n\}$ such that $\lim_{n\to\infty} \frac{y_n}{n} = -\infty$ (that is, $A_+ \neq \phi$) if and only if

$$\sum_{n=n_0}^{\infty} q_n \exp(-kn^{\gamma}) < \infty, \text{ for some } k > 0. \qquad \text{... (25)}$$

iii) Eq. (E_+^2) has a solution $\{y_n\}$ such that $\lim_{n\to\infty} y_n$ = *constant* (that is, $C_b \neq \phi$) if and only if

$$\sum_{n=n_0}^{\infty} nq_n < \infty. \qquad \text{... (26)}$$

iv) Eq. (E_+^2) has a solution $\{y_n\}$ such that $\lim_{n\to\infty} \frac{y_n}{n}$ = *constant* > 0 (that is, $D_+ \neq \phi$) if and only if

$$\sum_{n=n_0}^{\infty} q_n \exp(kn^{\gamma}) < \infty \qquad \text{... (27)}$$

for some $k > 0$.

v) If (27) holds (for examples $q_n = \exp(-mn^{\alpha}), m > 0$ and $\alpha \geq \gamma$) then each of B_+, C_+ and D_+ has a member and we have $S_+ = B_+ \cup C_+ \cup D_+$ for eq. (E_+^2).

vi) If (26) holds but (27) does not (for example $q_n = mn^{-\alpha}, m > 0$ and $\alpha > 2$), then both

B_+ and C_+ have members and $S_+ = B_+ \cup C_+$ for eq. (E_+^2).

vii) If (25) holds and (26) does not (for example $q_n = \exp(mn^{\alpha})$, $m > 0$ and $\alpha \le \gamma$), then all solutions of eq. (E_+^2) are members of B_+, that is $S_+ = B_+$.

viii) If (25) does not hold (for example, $q_n = \exp(mn^{\alpha})$, $m > 0$ and $\alpha > \gamma$), then all solutions of eq. (E_+^2) are members of A_+, that is $S_+ = A_+$.

3. Existence/Nonexistence of A_-, B_-, C_- and D_- Class Solutions

In this section we give explicit sufficient conditions for existence of some of these classes of solutions of eq. (E_-). We begin by giving a condition under which eq. (E_-) has solutions belong to D_- class. Since the proofs of the theorem given here can be modelled as that of in Section 2 and the details are omitted.

Theorem 3.1 — *Assume that there exists a constant $k > 0$ such that*

$$\sum_{n=n_0}^{\infty} f\left(n, -kR_n, -\frac{k}{a_n}\right) < \infty. \qquad \dots (28)$$

Then for any $b \in (k, \infty)$ and $\gamma \in I\!R$, eq. (E_-) has a solution $\{y_n\} \in D_-$ satisfying

$$y_{n_0} = \gamma \quad \text{and} \quad \lim_{n \to \infty} a_n \Delta y_n = -b. \qquad \dots (29)$$

Proof : From (28) and (c_2) we have

$$\sum_{n=n_0}^{\infty} f\left(n, \gamma - bR_n, -\frac{b}{a_n}\right) < \infty.$$

Let S be the space defined in Theorem 2.2 and let $Y \subset S$ be defined by

$$Y = \left\{ y \in S : \gamma - bR_n - \sum_{s=n_0}^{n-1} \left(\frac{1}{a_s} \sum_{t=s}^{\infty} f\left(t, \gamma - bR_t, -\frac{b}{a_t}\right)\right) \le y_n \le \gamma - bR_n, \right.$$

$$\left. -b - \sum_{s=n}^{\infty} f\left(s, \gamma - bR_s, -\frac{b}{a_s}\right) \le a_n \Delta y_n \le -b, \; n \ge n_0 \right\}$$

Clearly, Y is a non empty, closed convex subset of S. Define the operator $T : Y \to S$ by

$$Ty_n = \gamma - bR_n - \sum_{s=n_0}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} f(t, y_t, \Delta y_t) \right), \; n \ge n_0.$$

It is easy to verify that $TY \subset Y$ and satisfies all conditions of Schauder-Tychonoff fixed point theorem. Hence T has a fixed point $y = \{y_n\}$ in Y. This fixed point $\{y_n\}$ is a D_- class solution of eq. (E_-) satisfying the condition (29).

Remark 3.1 : If eq. (E_-) has a solution belongs to D_- class then (28) holds for some $k > 0$.

Next we examine the class C_- of eq. (E_-) which does not contain the term Δy_n :

$$\Delta (a_n \Delta y_n) - h(n, y_n) = 0, n \in \mathbb{N}, \qquad \text{... (30)}$$

where $\{a_n\}$ satisfies condition (c_1) and $h : \mathbb{N} \times \mathbb{R} \to (0, \infty)$ is continuous and nondecreasing in the second argument.

Theorem 3.2 — *With respect to the difference eq. (30), assume that*

$$\sum_{n=n_0}^{\infty} h(n, -kR_n) < \infty \text{ for all } k > 0$$

and

$$\sum_{n=n_0}^{\infty} R_n\, h(n, k) < \infty \text{ for some } k \in \mathbb{R}.$$

Then $C_- \neq \phi$.

PROOF : Choose $l \geq 0$ large enough so that

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} h(s, -l) \leq l.$$

Consider the set

$$Y = \{y \in S : -2l \leq y_n \leq -l, n \leq n_0\}$$

and define the operation $T : Y \to S$ by

$$Ty_n = -2l + \sum_{s=n}^{\infty} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} h(t, y_t) \right), n \geq n_0.$$

Then, from the Schauder-Tychnoff fixed point theorem T has a fixed point $y = \{y_n\}$ in Y. Clearly this $y = \{y_n\}$ is a solution of eq. (30) belongs to the class C_-. This completes the proof.

Finally, we obtain conditions for the existence of B_- class solutions of equation (E_-).

Theorem 3.3 — *Assume that there exists a constant $k > 0$ such that*

$$\sum_{n=n_0}^{\infty} f\left(n, kR_n, \frac{k}{a_n}\right) < \infty. \qquad \text{... (31)}$$

Then for any $b \in (0, k)$ and any $\gamma \in I\!R$, eq. (E_-) has a solution belongs to B_- class satisfying

$$y_{n_0} = \gamma \text{ and } \lim_{n \to \infty} a_n \Delta y_n = b. \qquad \text{... (32)}$$

PROOF : First note that

$$\sum_{n=n_0}^{\infty} f\left(n, \gamma + bR_n, \frac{b}{a_n}\right) < \infty.$$

Let Y denote the set of all $y \in S$ satisfying the inequalities :

$$Y = \left\{ y \in S : \gamma + bR_n - \sum_{s=n_0}^{n-1} \frac{1}{a_s}\left(\sum_{t=s}^{\infty} f\left(t, \gamma + bR_t, \frac{b}{a_t}\right)\right) \le y_n \le \gamma + bR_n, \right.$$

$$\left. b - \sum_{s=n}^{\infty} f\left(s, \gamma + bR_s, \frac{b}{a_s}\right) \le a_n \Delta y_n \le b, n \ge n_0 \right\}$$

Then the operator $T : Y \to S$ defined by

$$Ty_n = \gamma + bR_n - \sum_{s=n_0}^{n-1} \frac{1}{a_s}\left(\sum_{t=s}^{\infty} f(t, y_t, \Delta y_t)\right), n \ge n_0$$

satisfies all conditions of Schauder-Tychanoff fixed point theorem and hence it has a fixed point $y = \{y_n\}$ in Y. This fixed point gives the desired B_- class solution of (E_-) satisfying (32).

We conclude this paper with the following remarks.

Remark 3.2 : (*i*) If eq. (E_-) has a B_- type solution, then condition (31) holds for some $k > 0$.

ii) As in Section 2, one can construct examples to illustrate the results obtained in this section and hence the details are omitted.

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