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GENERALIZED NONLINEAR VARIATIONAL INCLUSIONS FOR FUZZY MAPPINGS

RAIS AHMAD SALAHUDDIN AND S. HUSSAIN*

Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, *India*
*(E-mail: mmr*013*@amu.up.nic.in Salahuddin*12*@mailcity.com*

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In this paper, we consider the generalized nonlinear variational inclusions for fuzzy mappings and develop an iterative algorithm. We prove the existence of solutions for the generalized nonlinear variational inclusions for fuzzy mappings and the convergence of iterative sequences generated by the algorithm.

Key Words : Variational Inclusions; Fuzzy Mappings; Iterative Algorithm; Existence; Convergence

1. INTRODUCTION

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ respectively. Let $\mathcal{F}(H)$ be a collection of all fuzzy sets over H. A mapping F from H into $\mathcal{F}(H)$ is called a Fuzzy mapping on H. If F is a Fuzzy mapping on H, then $F(x)$ (denote it by F_x, in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x. Let $B \in F_x(H)$, $m \in [0, 1]$, then the set $(B)_m = \{x \in H : B(x) \geq m\}$ is called a m-cut set of B. Further, let $T L H \to \mathcal{F}(H)$ be fuzzy mapping such that there exist a real number $a \in [0, 1]$, suchthat for all $x \in H$, the set $(T_x)_a$ belongs to $CB(H)$, where $CB(H)$ denotes the family of all nonempty bounded closed subset of H.

Now we define the set-valued mapping $\tilde{T} : H \to \mathcal{F}(H)$ by for any $x \in H$, $\tilde{T}(x) = (T_x)_a$. In the sequel, $\tilde{T}$ is called the set-valued mapping induced by the fuzzy mapping T. Let $\phi : H \to R \bigcup \{+\infty\}$ be a proper convex lower semicontinuous function and $\partial \phi$ be the subdifferential of ϕ. Give a fuzzy mapping $T : H \to \mathcal{F}(H)$, and the single-valued mappings $f, g : H \to H$, with Im $(g) \bigcap$ dom $(\partial \phi) \neq \phi$, then we consider the following generalized nonlinear variational inclusion problem for fuzzy mappings (GNVIPFM) :

(GNVIPFM) : Find $x, w \in H$ such that $g(x) \bigcap \text{dom } (\partial \phi) \neq \phi$, and $T_x(w) \geq a$

$$\langle g(x) - f(w), y - g(x) \rangle \geq \phi(g(x)) - \phi(y), \ \forall \ y \in H. \qquad \text{... (1.1)}$$

Inequality (1.1) is called the generalized nonlinear variational inclusion for fuzzy mappings.

*Department of Applied Mathematics, ZHCOE&T, Aligarh Muslim University, Aligarh 202 002, India

In particular, if $T: H \to 2^H$ is a classical set-valued mapping and $\phi \equiv \delta_K$, the indicator function of closed convex set K in H defined by

$$\phi(x) = \begin{cases} 0, & x \in K \\ +\infty, & x \notin K, \end{cases}$$

then (GNVIPFM) reduces to the following generalized variational inequality problem (GVIP) considered by Verma[1].

(GVIP) : Find $x \in H, w \in T(x)$ such that $g(x) \in K$, and

$$\langle g(x) - f(w), y - g(x) \rangle \geq 0, \ \forall \ y \in K. \qquad \text{... (1.2)}$$

Remark 1.1 : For appropriate and suitable choice of the mappings f, g and T and the function a, the variational inclusion (1.1) includes a number of known classes of variational inclusions and variational inequalities considered in 2-6.

2. Iterative Algorithm

In this section, we first establish the equivalence of the generalized nonlinear variational inclusion for fuzzy mappings (1.1) to a nonlinear equation. Then we suggest an iterative algorithm for finding the approximate solution of (1.1).

Lemma 2.1 — x and w are solutions of problem (1.1) if and only if there exists $w \in \tilde{T}(x)$ such that

$$g(x) = J_\alpha^\phi (g(x) - \alpha (g(x) - f(w))), \qquad \text{... (2.1)}$$

where $\alpha > 0$ is a constant and $J_\alpha^\phi = (I + \alpha \partial \phi)^{-1}$ is the so-called proximal mapping on H.

PROOF : From the definition of the proximal mapping J_α^ϕ, one has

$$g(x) - \alpha (g(x) - f(w)) \in g(x) + \alpha \partial \phi (g(x)).$$

and hence

$$f(w) - g(x) \in \partial \phi (g(x)).$$

From the definition of $\partial \phi$, we have

$$\phi(y) \geq \phi(g(x)) + \langle f(w) - g(x), y - g(x) \rangle, \ \forall \ y \in H.$$

Thus x and w are solutions of (GNVIPFM).

For finding the approximate solution of (1.1), we can apply a successive approximation method to the problem of solving

$$x \in F(x) \qquad \text{... (2.2)}$$

where

$$F(x) = x - g(x) + J_{\alpha}^{\phi}(g(x) - \alpha(g(x_n) - f(\tilde{T}(x)))).$$

Based on (2.1) and (2.2), we suggest the following iterative algorithm.

Let $T: H \rightarrow \mathcal{F}(H)$ be a closed fuzzy mapping such that there exist a real number $a \in [0, 1]$ such that $(T_x)_a \in B(H)$, for all $x \in H$ and $\tilde{T}$ be the set-valued mapping induced by the fuzzy mapping T. For given $x_0 \in H$, let $w_0 \in \tilde{T}(x_0)$ and

$$x_1 = x_0 - g(x_0) + J_{\alpha}^{\phi}(g(x_0) - \alpha(g(x_0) - f(w_0))),$$

By [7], there exist $w_1 \in T(x_1)$ such that

$$\| w_1 - w_0 \| \leq (1 + 1) H(\tilde{T}(x_1), \tilde{T}(x_0)),$$

where $\hat{H}$ is the Hausdorff metric on $CB(H)$.

Algorithm 2.1 — Let T be a closed fuzzy mapping such that there exist a real number $a \in [0, 1]$ such that $(T_x)_a \in CB(H)$, $\tilde{T}$ be the set-valued mapping induced by the fuzzy mapping T, and $f, g: H \rightarrow H$. For given $x_0 \in H$, compute x_{n+1} by the rule

$$\left.\begin{array}{l} x_{n+1} = x_n - g(x_n) + J_{\alpha}^{\phi}(g(x_n) - \alpha(g(x_n) - f(w_n))) \\ w_n \in T(x_n), \| w_{n+1} - w_n \| \leq (1 + (1 + n))^{-1}) \hat{H}(T(x_{n+1}), \tilde{T}(x_n)), \end{array}\right] \quad \ldots (2.3)$$

where $\alpha > 0$ is a constant and $n = 0, 1, 2...$

3. Existence and Convergence

We need the following concepts and result to prove the main result of this paper.

Definition 3.1 — A mapping $g: H \rightarrow H$ is said to be

(*i*) Strongly monotone if there exists $r > 0$ such that

$$\langle g(x_1) - g(x_2), x_1 - x_2, \rangle \geq r \| x_1 - x_2 \|^2, \ \forall \ x_1, x_2 \in H,$$

(*ii*) Lipschitz continuous if there exists $s > 0$ such that

$$\| g(x_1) - g(x_2) \| \leq s \| x_1 - x_2 \|, \ \forall \ x_1, x_2 \in H.$$

Definition 3.2 — A set-valued mapping $T: H \rightarrow 2^H$ is said to be $\hat{H}$-Lipschitz continuous if there exists $\mu > 0$ such that

$$\hat{H}(T(x_1), T(x_2)) \leq \mu \| x_1 - x_2 \|, \ \forall \ x_1, x_2 \in H.$$

Lemma 3.1[8] — Let ϕ be a proper convex lower semicontinuous function. Then

$J_{\alpha}^{\phi} = (I + \alpha \partial \phi)^{-1}$ is nonexpansive, that is

$$\| J_{\alpha}^{\phi}(x) - J_{\alpha}^{\phi}(y) \| \leq \| x - y \|, \ \forall \ x, y \in H.$$

Now we prove the main result of this paper.

***Theorem* 3.1** — *Let $T : H \to \mathcal{F}(H)$ be the closed fuzzy mapping such that there exists a real number $a \in [0, 1]$, such that for all $x \in H$ the set $(T_x)_a$ belongs to $CB(H)$, T be the set-valued mapping induced by the fuzzy mapping T. Suppose that $g : H \to H$ is strongly monotone and Lipschitz continuous with corresponding constants $r > 0$ and $s > 0$, and $f : H \to H$ be Lipschitz continuous with constant $t > 0$. Let $T : H \to \mathcal{F}(H)$ is $\hat{H}$ -Lipschitz continuous with constant $\mu > 0$. If the following conditions hold*

$$\left| \alpha - \frac{(1-q)\, t\, \mu - r}{(t^2 \mu^2 - s^2)} \right| < \frac{\sqrt{(r - (1-q)\, t\, \mu)^2 - (t^2 \mu^2 - s^2)\, q(q-2)}}{(t^2 \mu^2 - s^2)}, \qquad \text{...(3.1)}$$

$$r > (1-q)\, t\, \mu + \sqrt{(t^2 \mu^2 - s^2)\, q(q-2)} \ \text{ and } \ q < 1,$$

where $$q = 2\sqrt{1 - 2r + s^2},$$

then there exists $x, w \in H$, which are the solution of problem (1.1).

Moreover, $x_n \to x$, $w_n \to w$, $n \to \infty$, where $\{x_n\}$, $\{w_n\}$ are defined in algorithm (2.1).

PROOF : From (2.3), we have

$$\| x_{n+1} - x_n \| = \| x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) + J_{\alpha}^{\phi}(h(x_n)) - J_{\alpha}^{\phi}(h(x_{n-1})) \|, \qquad \text{... (3.2)}$$

where $h(x_n) = g(x_n) - \alpha(g(x_n) - f(w_n))$. Also we have

$$\| J_{\alpha}^{\phi}(h(x_n)) - J_{\alpha}^{\phi}(h(x_{n-1})) \| \leq \| h(x_n) - h(x_{n-1}) \| \leq \| x_n - x_{n-1} - \alpha(g(x_n) - g(x_n)) \|$$

$$+ \| x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) \| + \alpha \| f(w_n) - f(w_{n-1}) \|. \qquad \text{... (3.3)}$$

From (3.2) and (3.3), we get

$$\| x_{n+1} - x_n \| \leq 2 \| x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) \| +$$

$$\| x_n - x_{n-1} - \alpha(g(x_n) - g(x_{n-1})) \| + \alpha \| (f(w_n) - f(w_{n-1}) \|. \qquad \text{... (3.4)}$$

By Lipschitz continuity and strong monotonicity of g, we obtain

$$\| x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) \|^2 \leq (1 - 2r + s^2) \| x_n - x_{n-1} \|^2, \qquad \text{... (3.5)}$$

and $$\| x_n - x_{n-1} - \alpha(g(x_n) - g(x_{n-1})) \|^2 \leq (1 - 2\alpha r + \alpha^2 s^2) \| x_n - x_{n-1} \|^2. \qquad \text{... (3.6)}$$

Since T is $\hat{H}$-Lipschitz continuous and f is Lipschitz continuous, we have

$$\alpha \| f(w_n) - f(w_{n-1}) \| \leq \alpha t (1 + n^{-1}) \mu \| x_n - x_{n-1} \|. \qquad \text{... (3.7)}$$

So by combining (3.4) to (3.7) and denoting

$$\theta_n = 2\sqrt{1-2r+s^2} + \sqrt{1-2\,\alpha\, r+\alpha^2 s^2} + \alpha\, t(1+n^{-1})\,\mu,$$

we get $$\| x_{n+1} - x_n \| \le \theta_n \| x_n - x_{n-1} \|.$$

Letting $\theta = 2\sqrt{1-2r+s^2} + \sqrt{1-2\,\alpha\, r+\alpha^2 s^2} + \alpha\, t\, \mu.$

We know that $\theta_n \to \theta$. It follows from (3.1) that $\theta < 1$. Hence $\theta_n < 1$, for n sufficiently large. Therefore, $\{x_n\}$ is a Cauchy sequence and we can suppose that $x_n \to x \in H$.

Now we prove that

$$w_n \to w \in \tilde{T}(X).$$

Infact, it follows from the algorithm 2.1 that

$$\| w_n - w_{n-1} \| \le (1+n^{-1})\,\mu \| x_n - x_{n-1} \|$$

i.e., $\{w_n\}$ is a cauchy sequence. Let $w_n \to w$. Further, we have

$$\begin{aligned} d(w, T(x)) &= \inf \{\| w - z \| : z \in T(x)\} \\ &\le \| w - w_n \| + d(w_n, T(x)) \\ &\le \| w - w_n \| + H(T(x_n), \tilde{T}(x)) \\ &\le \| w - w_n \| + \mu \| x_n - x \| \to 0. \end{aligned}$$

Hence $w \in \tilde{T}(x)$.

This completes the proof.

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