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THEOREM OF ALTERNATIVE FOR A CLASS OF QUASICONVEX N-SET FUNCTIONS AND ITS APPLICATIONS TO MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS

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In this paper, we introduce a class of r-quasiconvex n-set functions. A new theorem of alternative is proved for this class of functions. This theorem is then utilized to obtain optimality conditions and Lagrangian duality results for multiobjective fractional programming problems involving n-set functions.

Key Words : Quasiconvex n-set functions; Theorem of Alternative; Multiobjective Fractional Programming; Optimality Conditions; Duality relations

1. INTRODUCTION

Theorems of alternative have become an important tool in developing various results in mathematical programming. These theorems for convex, nonconvex and a subclass of quasiconvex functions were investigated by several authors[1-10] in the past.

Chou *et al.*[11] obtained the theorem of alternative for convex set functions. This theorem was generalized to cone-convex set functions by Hsia *et al.*[12] and was then utilized to develop Lagrangian multiplier theorems and duality results for multiobjective optimization problems with set functions.

The theory of optimizing set functions was introduced in the literature by Morris[13]. Subsequently many authors[14-20] have made significant contributions in developing the optimality conditions and duality results for various classes of optimization problems involving set functions.

In this paper, we introduce a class of r-quasiconvex n-set functions as a subclass of quasiconvex n-set functions and prove a new theorem of alternative for this class of functions using Sion's minimax theorem. This alternative theorem is then utilized to derive optimality conditions and Lagrangian duality results for a class of multiobjective fractional programming problems involving n-set functions.

2. PRELIMINARIES

Let $I\!R^p$ denote the p-dimensional Euclidean space of reals partially ordered by a pointed closed convex cone P i.e. for y_1, y_2 $I\!R^p$,

$$y_1 \leq_P y_2 \Leftrightarrow y_2 - y_1 \in P$$

$$y_1 \leq_P y_2 \Leftrightarrow y_2 - y_1 \in P\backslash\{0\}$$

$$y_1 <_P y_2 \Leftrightarrow y_2 - y_1 \in \text{int } P$$

The set of all P-minimal and P-maximal points of a subset Y of $\mathbb{R}^p$ are defined as
$\text{Min}_P\ Y = \{y \in Y \mid \text{there exist no } \hat{y} \in Y \text{ such that } \hat{y} \leq_P y\}$,
$\text{Max}_P\ Y = \{y \in Y \mid \text{there exist no } \hat{y} \in Y \text{ such that } y \leq_P \hat{y}\}$.
The polar cone P^* of P is defined by

$$P^* = \{y^* \in \mathbb{R}^p \mid \langle y^*, y \rangle \geqq 0,\ \forall\ y \in P\}.$$

Lemma 2.1 (Jeyakumar *et al.*[6]) — If int $P \neq \Phi$ then

(a) $0 \neq y \in \text{int } P \Leftrightarrow \langle y^*, y \rangle > 0,\ \forall\ y^* \in P^*/\{0\}$.

(b) for $0 \neq y \in \text{int } P$, there exists a compact convex subset B of P^*, called the base of P, defined as

$$B = \{y^* \in P^* \mid \langle y^*, y \rangle = 1\}$$

such that $$P^* = \bigcup_{\alpha \geqq 0} \alpha B.$$

We now state the **Sion's minimax theorem**[22] which will be used as a basic tool in deriving the main result of this contribution i.e. the theorem of alternative.

Theorem 2.1 — *Let A and B be convex subsets of some real topological vector space with B compact, and let $\gamma: A \times B \rightarrow \mathbb{R}$. If $\gamma(a, \cdot)$ is upper semicontinuous, quasiconcave on* B *for all $a \in A$, and if $\gamma(\cdot, b)$ is lower semicontinuous, quasiconvex on A for all $b \in B$, then*

$$\inf_{a \in A} \max_{b \in B} \gamma(a, b) = \max_{b \in B} \inf_{a \in A} \gamma(a, b).$$

Let $(X, \mathcal{A}, \mu)$ be a finite atomless measure space, $\mathcal{A}$ is a σ-algebra of subsets of a given set X, $\mathcal{A}^n$ is n-fold product of $\mathcal{A}$, 'd' is a pseudo-metric on $\mathcal{A}^n$ defined by

$$d(R, S) = \left[\sum_{i=1}^{n} \mu^2 (R_i \Delta S_i)\right]^{1/2}$$

$R = (R_1, \ldots, R_n), S = (S_1, \ldots, S_n) \in \mathcal{A}^n$ and Δ denotes the symmetric difference of sets. For $S \in \mathcal{A}$, let $I_S \in L_\infty (X, \mathcal{A}, \mu)$ denote the characteristic function. Also, let $L_1 (X, \mathcal{A}, \mu)$ be separable.

Remark 2.1 (Chou *et al.*[11], Rudin[23]) : For any $\Gamma \subset \mathcal{A}$, $\overline{\Gamma}$ denote the w^*-closure of the set $\{I_S : S \in \Gamma\}$ in $L_\infty (X, \mathcal{A}, \mu)$. Then $\overline{\mathcal{A}} = \{f \in L_\infty (X, \mathcal{A}, \mu) : 0 \leqq f \leqq 1\}$ is w^*-compact. Further, since

$L_1(X, \mathcal{A}, \mu)$ is assumed to be separable, therefore, $\overline{\mathcal{A}}$ is metrizable. Hence $\overline{\mathcal{A}}^n$ is also metrizable and thus sequentially compact.

The following definitions will be used in the sequel.

Definition 2.1 (Hsia *et al.*[12]) — An n-set function $\psi : I\!A \subseteq \mathcal{A}^n \to I\!R$ is said to be w^*-continuous on $I\!A$ if for each $f \in \overline{I\!A}$ and for any sequence $\{S^k\}$ in $I\!A$ with $I_S^k \to f$, the sequence $\{\psi(S^k)\}$ converges to the same limit.

Definition 2.2 (Lin[20]) — An n-set function $\psi : \mathcal{A}^n \to I\!R$ is said to be quasiconvex on a convex subfamily $I\!A$ of $\mathcal{A}^n$ if for each $S \in I\!A, R \in I\!A$ and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in $\mathcal{A}$ associated with (S_i, R_i, λ) such that $I_{V_i^k(\lambda)} \xrightarrow{w^*} (1-\lambda) I_{S_i} + \lambda I_{R_i}$ for each $i = 1, 2, \ldots, n$, $V^k(\lambda) \in I\!A$ for each $k \in I\!N$ and

$$\overline{\lim_{k \to \infty}} \; \psi(V^k(\lambda)) \leq \max\{\psi(S), \psi(R)\}.$$

Definition 2.3 — A vector function $\psi : \mathcal{A}^n \to I\!R^p$ is said to be p-quasiconvex on a convex subfamily $I\!A$ of $\mathcal{A}^n$ if for each $S \in I\!A$, $R \in I\!A$ and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in $\mathcal{A}$ associated with (S_i, R_i, λ) such that $I_{V_i^k(\lambda)} \xrightarrow{w^*} (1-\lambda) I_{S_i} + \lambda_{R_i}$ for each $i = 1, \ldots, n$, $V^k(\lambda) \in I\!A$ for each $k \in I\!N$ and

$$\left.\begin{array}{l} \psi(S) \leqq_P \xi \\ \psi(R) \leqq_P \xi \end{array}\right] \Rightarrow \overline{\lim_{k \to \infty}} \; \psi(V^k(\lambda)) \leqq_p \xi.$$

Definition 2.4 — An n-set function $\psi : \mathcal{A}^n \to I\!R^p$ is said to be r-quasiconvex on a convex subfamily $I\!A$ of $\mathcal{A}^n$ if for each $y^* \in P^*$, the real-valued n-set function $S \mapsto \langle y^*, \psi(S) \rangle$ is quasiconvex and w^*-continuous on $I\!A$.

The authors are motivated to carry out the present study because there exist n-set functions that are not convex but are r-quasiconvex, as can be seen by the following example.

Example 2.1 — Let $P = I\!R_+^2$, and let $\psi = I\!A \subseteq \mathcal{A} \to I\!R^2$ be defined as

$$\psi(S) = (\psi_1(S), \psi_2(S)) = \left(f_1\left(\int_S g\, d\mu \right), f_2\left(\int_S g\, d\mu \right) \right)$$

where
$$\psi_1(S) = f_1\left(\int_S g\, d\mu \right) = \min\left\{ \int_S g\, d\mu, 2 \right\},$$

$$\psi_2(S) = f_2\left(\int_S g\, d\mu \right) = \max\left\{ -\int_S g\, d\mu, -1 \right\},$$

$g \in L_1(x, \mathcal{A}, \mu)$ and $S \in I\!A$.

The functions f_1 and f_2 are quasiconvex on $I\!R$, but f_1 is not a convex function. Hence, the set functions ψ_1 and ψ_2 are quasiconvex on $I\!A$, but ψ_1 is not a convex set function (see, Lin[21]). Thus the function ψ is P-quasiconvex on $I\!A$, but not convex. Moreover, ψ is r-quasiconvex on $I\!A$, as for $y = (y_1, y_2) \in P^* = I\!R^2_+$, we have

$$\langle y, \psi(S) \rangle = \begin{cases} (y_1 - y_2) \int_S g\,d\mu, & \text{if } \int_S g\,d\mu \leqq 1, \\ y_1 \int_S g\,d\mu - y_2, & \text{if } 1 < \int_S g\,d\mu \leqq 2, \\ 2y_1 - y_2, & \text{if } \int_S g\,d\mu > 2. \end{cases}$$

which is quasiconvex and w^*-continuous on $I\!A$.

3. Weakened Quasiconvexity and Theorem of Alternative

A separation theorem was used by Lin[19] to establish an alternative theorem for convex n-set functions. It may be pointed out that this theorem of Lin does not hold for quasiconvex n-set functions. In this section, using Sion's minimax theorem, a new theorem of alternative is proved that holds only for a class of r-quasiconvex n-set functions.

Remark 3.1 : It is a well known fact that a nonnegative linear combination of quasiconvex functions is, in general, not quasiconvex. However, the converse holds under certain mild conditions as proved in the following Lemma.

Lemma 3.1 — Let $\psi : I\!A \to I\!R^p$ be r-quasiconvex fnction on $I\!A$. Then ψ is P-quasiconvex on $I\!A$.

Proof : Let $\lambda \in [0, 1]$ and $S, R \in I\!A$ with

$$\psi(S) \leqq_P \xi, \ \psi(R) \leqq_P \xi.$$

Therefore, for each $y^* \in P^*$, we have

$$\langle y^*, \psi(S) \rangle \leqq \langle y^*, \xi \rangle, \langle y^*, \psi(R) \rangle \leqq \langle y^*, \xi \rangle.$$

Since the function $S \mapsto \langle y^*, \psi(S) \rangle$ is quasiconvex on $I\!A$ there exists a Morris sequence $\{V_i^k(\lambda)\}$ associated with (S_i, R_i, λ) in $\mathcal{A}$ such that $V^k(\lambda) \in I\!A$ for each $k \in I\!N$ and

$$\lim_{k \to \infty} \langle y^*, \psi(V^k(\lambda)) \rangle \leqq \langle y^*, \xi \rangle \quad \text{... (3.1)}$$

Also, since $S \mapsto \langle y^*, \psi(S) \rangle$ is w^*-continuous, we have

$$\overline{\lim_{k \to \infty}} \langle y^*, \psi(V^k(\lambda)) \rangle = \lim_{k \to \infty} \langle y^*, \psi(V^k(\lambda)) \rangle \quad \text{... (3.2)}$$

It follows from (3.1) and (3.2) that

$$\lim_{k \to \infty} \langle y^*, \psi(V^k(\lambda)) \rangle \leqq \langle y^*, \xi \rangle, \ \forall \ y^* \in P^*$$

which implies

$$\lim_{k \to \infty} \psi(V^k(\lambda)) \leqq_P \xi.$$

Hence, ψ is P-quasiconvex on $I\!A$.

Let, CLQ = $\{\psi : I\!A \to I\!R^p \mid \psi$ is P-quasiconvex on $I\!A\}$, and let

CLRQ = $\{\psi : I\!A \to I\!R^p \mid \psi$ is r-quasiconvex on $I\!A\}$.

It follows from the above Lemma that CLRQ is subclass of CLQ.

Also, in the above lemma, let $P = I\!R^p_+$. Then we get that if, for every $y \in I\!R^p_+$, $\sum_{i=1}^{p} y_i \psi_i(S)$ is quasiconvex and w^*-continuous on $I\!A$ then $\psi = (\psi_1, \ldots, \psi_p)$ is quasiconvex on $I\!A$.

We now prove the theorem of alternative for a class of r-quasiconvex n-set functions.

Theorem 3.1 — *Let $I\!A$ be a convex subfamily of $\mathcal{A}^n$, and let P be a pointed closed convex cone in $I\!R^p$ with int $P \neq \Phi$. If $\psi : I\!A \to I\!R^p$ is r-quasiconvex on $I\!A$ then exactly one of the following systems is solvable :*

(i) $\exists S \in I\!A$ such that $\psi(S) <_P 0$

(ii) $\exists y^* \in P^* \{0\}$ such that $\langle y^*, \psi(S) \rangle \geqq 0, \ \forall \ S \in I\!A$.

PROOF : Since int $p \neq \Phi$, it follows from Lemma 2.1 (b) that for some $0 \neq y \in$ int P, there exists a compact convex subset B of P^* given by

$$B = \{y^* \in P^* \mid \langle y^*, y \rangle = 1\}$$

such that $$P^* = \bigcup_{\alpha \geqq 0} \alpha B.$$

Define a function $\gamma : I\!A \times B \to I\!R$ as

$$\gamma(S, b) = \langle b, \psi(S) \rangle, \text{ for } S \in I\!A, b \in B.$$

By hypothesis, $\gamma(\cdot, b)$ is quasiconvex and w^*-continuous on $I\!A$ for each $b \in B$. Also, $\gamma(S, \cdot)$ is linear, hence, quasiconcave and continuous on B for each $S \in I\!A$. It follows from Theorem 2.1 that

$$\inf_{S \in I\!A} \max_{b \in B} \gamma(S, b) = \max_{b \in B} \inf_{S \in I\!A} \gamma(S, b) \quad \text{... (3.3)}$$

Now, if the system (I) has no solution, then

$$\Leftrightarrow \forall\ S \in IA\ \exists\ \hat{y}^* \in P^* \setminus\{0\} \text{ such that } \langle \hat{y}^*, \psi(S) \rangle \geqq 0$$

$$\Leftrightarrow \forall\ S \in IA\ \exists\ y^* \in B \text{ such that } \gamma(S, y^*) \geqq 0$$

$$\Leftrightarrow \inf_{S \in IA} \max_{b \in B} \gamma(S, b) \geqq 0$$

$$\Leftrightarrow \max_{b \in B} \inf_{S \in IA} \gamma(S, b) \geqq 0 \text{ (by using (3.3))}$$

$$\Leftrightarrow \exists\ y^* \in B \text{ such that that } \gamma(S, y^*) \geqq 0,\ \forall\ S \in IA$$

$$\Leftrightarrow y^* \in P^* \setminus\{0\} \text{ such that } \langle y^*, \psi(S) \rangle \geqq 0,\ \forall\ S \in IA$$

$\Leftrightarrow$ the system (II) has a solution.

4. Optimality Conditions

In this section, we first develop the Lagrange multiplier theorem for a class of scalar-valued fractional programming problems involving n-set functions. This result is then extended to the multiobjective case.

Consider the following single objective fractional programming problem :

(SFP) Min $(F_0(S)/G_0(S))$

subject to $H(S) \leqq_Q 0$,

$S \in IA$,

where Q is a pointed closed convex cone in IR^m with int $Q \neq \Phi$, $F_0, G_0 : IA \to IR, H : A \to IR^m$ are n-set functions defined on a convex subfamily IA of $\mathcal{A}^n$ and for each $S \in IA$, $G_0(S) > 0$.

The problem (SFP) is said to satisfy Generalized Slater's Constraint Qualification if there exists some $S_0 \in IA$ such that $H(S_0) <_Q 0$.

***Theorem* 4.1** — *Let S^* be an optimal solution of (SFP) with $\xi^* = F_0(S^*)/G_0(S^*)$. If, $(F_0 - \xi^* G_0, H)$ is τ-quasiconvex on IA, and generalized Slater's constraint qualification holds then there exists $\mu_0^* \in Q^*$ such that*

$$F_0(S) - \xi^* (G_0(S) + \langle \mu_0^*, H(S) \rangle \geqq 0,\ \forall\ S \in IA,$$

$$\langle \mu_0^*, H(S^*) \rangle = 0.$$

PROOF : Since S^* is an optimal solution of (SFP) with optimal objective value $\xi^* = F_0(S^*)/G_0(S^*)$, hence the system

$$\left.\begin{aligned} F_0(S) - \xi^* G_0(S) < 0 \\ H(S) <_Q 0 \end{aligned}\right] \quad \text{... (4.1)}$$

has no solution $S \in IA$.

Define a function $\psi : IA \times IR^m$ as

$$\psi(S) = (F_0(S) - \xi^* G_0(S), H(S)), \text{ for } S \in IA$$

and let $p = IR_+ \times Q$.

Then P is a pointed closed convex cone in $IR \times IR^m$ with int $P \neq \Phi$.

Also, $P^* = IR_+ \times Q^*$.

From (4.1), we get that the system

$$\psi(S) <_P 0$$

has no solution $S \in IA$.

By hypothesis ψ is r-quasiconvex on IA, hence from Theorem 3.1, we get the existence of Lagrange multipliers $\lambda_0^* \in IR_+$ and $\mu_0^* \in Q^*$ such that

$$\langle \lambda_0^*, F_0(S) - \xi^* G_0(S) \rangle + \langle \mu_0^*, H(S) \rangle \geqq 0, \ \forall\ S \in IA \quad \text{... (4.2)}$$

$$(\lambda_0^*, \mu_0^*) \neq 0 \quad \text{... (4.3)}$$

Now, if $\lambda_0^* = 0$, then from (4.2) and (4.3) we have that for $\mu_0^* \in Q^* \backslash \{0\}$,

$$\langle \mu_0^*, H(S) \rangle \geqq 0, \ \forall\ S \in IA$$

which contradicts the generalized Slater's constraint qualification. So, $\lambda_0^* \neq 0$. Without loss of generality, we can take $\lambda_0^* = 1$. Hence (4.2) can be rewritten as

$$F_0(S) - \xi^* G_0(S) + \langle \mu_0^*, H(S) \rangle \geqq 0, \ \forall\ S \in IA \quad \text{... (4.4)}$$

Set $S = S^*$ in (4.4), we get

$$\langle \mu_0^*, H(S^*) \rangle \geqq 0 \quad \text{... (4.5)}$$

Also, $H(S^*) \leqq_Q 0$ and $\mu_0^* \in Q^*$ implies

$$\langle \mu_0^*, H(S^*) \rangle \leqq 0 \qquad \dots (4.6)$$

(4.5) and (4.6) together yields

$$\langle \mu_0^*, H(S^*) \rangle = 0.$$

We now consider the following multiobjective fractional programming problem

(VFP) Min $F(S)/G(S) = (F_1(S)/G_1(S), \dots, F_p(S)/G_p(S))$

subject to $H(S) \leqq_Q 0$,

$S \in I\!A$,

where Q is a pointed closed convex cone in $I\!R^m$ with int $Q \neq \Phi$, $F, G : I\!A \to I\!R^p$, $H : I\!A \to I\!R^m$ are vector-valued n-set functions defined on a convex subfamily $I\!A$ of $\mathcal{A}^n$ and for each $S \in I\!A$, $G(S) > 0$. Here it not necessary to assume that $F(S) \geqq 0$, for each $S \in I\!A$. Moreover, minimization is taken in terms of finding efficient solutions of (VFP).

Let $E = \{S \in I\!A \mid H(S) \leqq_Q 0\}$ denote the set of all feasible solutions of (VFP), and let $\mathcal{L}^+$ be the set of all $p \times m$ real matrices M such that $M(Q) \subset I\!R^p_+$.

Definition 4.1 — A vector-valued Lagrangian function, $L : I\!A \times \mathcal{L}^+ \to I\!R^p$, for the problem (VFP) is defined by

$$L(S, M) = (F(S) + MH(S))/G(S), \text{ for } S \in I\!A, M \in \mathcal{L}^+.$$

We associate the following sequence of programs $(SFP)_r^*$, $r = 1, \dots, p$, each with single objective function, with the problem (VFP) :

$(SFP)_r^*$ Min $F_r(S)/G_r(S)$

subject to $(F_i(S)/G_i(S)) \leqq (F_i(S^*)/G_i(S^*))$,

$i = 1, \dots, p, \ i \neq r$

$H(S) \leqq_Q 0$,

$S \in I\!A$.

Lemma 4.1 (Bhatia and Kumar[14]) — $S^* \in I\!A$ is an efficient solution of (VFP) if and only if it is an optimal solution of $(SFP)_r^*$, for each r, $r = 1, \dots, p$.

The following theorem is an extension of Theorem 4.1 to the multiobjective case.

Theorem 4.2 — *Let $S^* \in I\!A$ be an efficient solution of (VFP) with efficient objective value $v^* = F(S^*)/G(S^*)$. If $(F - v^* G, H)$ is r-quasiconvex on $I\!A$, and generalized Slater's constraint qualification is satisfied by* $(\text{SFP})_r^*$, *for each r, r = 1, ..., p, then there exists $M^* \in \mathcal{L}^+$ such that*

(i) $F(S^*)/G(S^*) \in \text{Min } \{L(S, M^*) \mid S \in E\}$

(ii) $M^* \; H \; (S^*) = 0.$

PROOF : Since $S^* \in I\!A$ is an efficient solutionof (VFP), it follows from Lemma 4.1, that S^* is an optimal solution of $(\text{SFP})_r^*$ for each r, $r = 1, ..., p$. Using Theorem 4.1, we have that there exist $\lambda^*_{0_r} = (\lambda^*_{0_{r1}}, \ldots, \lambda^*_{0_{rp}}) \in I\!R^p_+$ with $\lambda^*_{0_{rr}} = 1$ and $\mu^*_{0_r} \in Q^*$ such that, for each $r = 1, ..., p$.

$$\lambda^*_{0_{rr}} (F_r(S) - v_r^* G_r(S)) + \sum_{\substack{i=1 \\ i \neq r}}^{p} \lambda^*_{0_{ri}} (F_i(S) - v_i^* G_i(S)) + \langle \mu^*_{0_r}, H(S) \rangle \geqq 0 \quad \forall \; S \in I\!A,$$

$$\langle \mu^*_{0_r}, H(S^*) \rangle = 0$$

which can be rewritten as

$$\langle \lambda^*_{0_r}, F(S) - v^* (G(S) \rangle + \langle \mu^*_{0_r}, H(S) \rangle \geqq 0, \;\; \forall \;\; S \in I\!A \qquad \text{... (4.7)}$$

$$\langle \mu^*_{0_r}, H(S^*) \rangle = 0 \qquad \text{... (4.8)}$$

Summing over r in (4.7) and (4.8) and setting $\lambda^*_0 = \sum_{r=1}^{p} \lambda^*_{0_r} > 0, \mu^*_0 = \sum_{r=1}^{p} \mu^*_{0_r} \in Q^*$, we obtain

$$\langle \lambda^*_0, F(S) - v^* G(S) \rangle + \langle \mu^*_0, H(S) \rangle \geqq 0, \;\; \forall \;\; S \in I\!A \qquad \text{... (4.9)}$$

$$\langle \mu^*_0, H(S^*) \rangle = 0 \qquad \text{... (4.10)}$$

Choose a vector $e \in \text{int } I\!R^p_+$ such that $\langle \lambda^*_0, e \rangle = 1$ and for $q \in I\!R^m$, define $M^* \in I\!R^{p \times m}$ by $M^* q = \langle \mu^*_0, q \;\; e \rangle$. Then,

$$\lambda^*_0 M^* = \mu^*_0, M^* (Q) \subset I\!R^p_+, M^* H(S^*) = 0.$$

If for this M^*,

$$F(S^*) \; / \; G(S^*) \notin \text{Min } \{L(S, \; M^*) \mid S \in E\}$$

Then there exists some $S \in E$ such that

$$\frac{F(S^*)}{G(S^*)} - \frac{F(S)}{G(S)} - \frac{M^* H(S)}{G(S)} \in I\!R^p_+ \setminus \{0\}$$

which, in view of the fact that $G(S) > 0$, yields

$$F(S) + M^* H(S) \leq v^* G(S)$$

Since $\lambda_0^* > 0$, the above inequalities imply that

$$\langle \lambda_0^*, F(S) - v^* G(S) \rangle + \langle \lambda_0^* M^*, H(S) \rangle < 0$$

which can be rewritten as

$$\langle \lambda_0^*, F(S) - v^* G(S) \rangle + \langle \mu_0^* H(S) \rangle < 0.$$

But this contradicts (4.9). Therefore,

$$F(S^*)/G(S^*) \in \text{ Min } \{L(S, M^*) \mid S \in E\}.$$

5. Lagrange Duality

In this section, we establish Lagrangian duality for (VFP) under generalized Slater's constraint qualification.

Define a dual function $J(M)$ as

$$J(M) = \underset{S}{\text{Min}} \ \{L(S, M) \mid S \in E\}, \text{ for } M \in \mathcal{L}^+.$$

The Lagrange dual problem (DFP) associated with (VFP) is given by

(DFP) Max $J(M)$

subject to $M \in \mathcal{L}^+$.

Definition 5.1 — A matrix $M^* \in \mathcal{L}^+$ is said to be an efficient solution of (DFP) with $\tau^* \in J(M^*)$ as an efficient objective value of (DFP) if for every $M \in \mathcal{L}^+$,

$$(J(M) - \tau^*) \cap (I\!R^p_+ \setminus \{0\}) = \Phi.$$

Theorem 5.1 (*Weak Duality*) — *Let S^0 be feasible for (VFP) and M^0 be feasible for (DFP). Then, for each $\tau \in J(M^0)$, the following cannot hold :*

$$F(S^0)/G(S^0) \leq \tau.$$

PROOF : Let, if possible, for some $\tau^0 \in J(M^0)$, we have

$$F(S^o)/G(S^o) \leq \tau^o \quad \text{... (5.1)}$$

As $M^o \in \mathcal{L}^+$ and $S^o \in E$, we have

$$M^o H(S^o) \leqq 0$$

which, together with the fact that $G(S^o) > 0$, yields

$$M^o H(S^o)/G(S^o) \leqq 0 \quad \text{... (5.2)}$$

(5.1) alongwith (5.2) implies

$$L(S^o, M^o) \leq \tau^o$$

which contradicts the fact that

$$\tau^o \in J(M^o) = \text{Min } \{L(S, M^o) \mid S \in E\}$$

Hence the result.

Theorem 5.2 (*Strong Duality*) — *Let S^* be an efficient solution of (VFP) with efficient objective value $v^* = F(S^*)/G(S^*)$ and let, $(F - v^* G, H)$ is r-quasiconvex on IA. Further, let the generalized Slater's constraint qualification hold for each* $(\text{SFP})_r^*$, *$r = 1, ..., p$. Then there exists $M^* \in \mathcal{L}^+$ such that* M^* *is an efficient solution of (DFP) with efficient objective value v^*.*

PROOF : Since S^* is an efficient solution of (VFP) and all the conditions of Theorem 4.2 are satisfied, hence there exists $M^* \in \mathcal{L}^+$ such that

$$v^* = (F(S^*)/G(S^*)) \in \text{Min } \{L(S, M^*) \mid S \in E\} = J(M^*)$$

which implies that M^* is feasible for (DFP). Efficiency of M^* for (DFP) follows from the Weak Duality Theorem.

6. CONCLUSIONS

Sion's minimax theorem (Sion[22]) is used to establish the theorem of alternative for a subclass of quasiconvex n-set functions, called r-quasiconvex functions. This theorem of alternative is then applied to derive Lagrange multiplier theorems for the cases of single objective and multiobjective fractional programming problems with n-set functions. Lagrangian dual is formulated and duality results are obtained for multiobjective fractional programming problems involving n-set functions.

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