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Δ^m-STATISTICAL CONVERGENCE

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In this paper, we define a general sequence space $\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$, $(m \in N)$, where X is any sequence space. We establish some inclusion relations, topological results and we characterize the continuous duals of $\Delta^m(X)$. Furthermore we introduce of Δ^m-statistical convergence and given inclusion relation between $\Delta^m(w_p)$-convergence and Δ^m-statistically convergence. The results are more general than those of Kizmaz [3], Fridy [6], Conor [7], Basarir [8], Et-Colak [9] and Salat [10].

1. INTRODUCTION

Let l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in N = \{1, 2, ...\}$, the set of positive integers.

Kizmax [3] defined the sequence spaces

$$l_\infty(\Delta) = \{x = (x_k) : \Delta x \in l_\infty\}$$

$$c(\Delta) = \{x = (x_k) : \Delta x \in c\}$$

$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\}$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaces with norm

$$\|x\|_1 = |x_1| + \|\Delta x\|_\infty.$$

Then Colak [11] defined the sequence space $\Delta_v(X) = \{x = (x_k) : \Delta_v(x_k) \in X\}$, where $(\Delta_v(x_k)) = (v_k x_k - v_{k+1} x_{k+1})$ and X is any sequence space, and investigated some topological properties of this space.

Recently Et and Colak [9] generalized the above sequence spaces to the following sequence spaces.

$$l_\infty(\Delta^m) = \{x=(x_k) : \Delta^m x \in l_\infty\}$$

$$c(\Delta^m) = \{x=(x_k) : \Delta^m x \in c\}$$

$$c_0(\Delta^m) = \{x=(x_k) : \Delta^m x \in c_0\}$$

where $m \in N$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$,

and
$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$$

These are Banach spaces with norm

$$\| x \|_\Delta = \sum_{i=1}^{m} | x_i | + \| \Delta^m x \|_\infty.$$

It is trivial that $c_0(\Delta^m) \subset c_0(\Delta^{m+1})$, $c(\Delta^m) \subset c(\Delta^{m,+1})$, $l_\infty(\Delta^m) \subset l_\infty(\Delta^{m+1})$, and $c_0(\Delta^m) \subset c\ (\Delta^m) \subset l_\infty\ (\Delta^m)$ are satisfied and strict [9]. For convenience we denote these spaces

$\Delta^m(l_\infty)$, $\Delta^m(c)$, and $\Delta^m(c_0)$ and call the constituent sequences Δ^m-bounded, Δ^m-convergent and Δ^m-null sequences, respectively.

2. Some Properties of $\Delta^m(X)$

We define the general sequence space $\Delta^m(X)$ as follows

$$\Delta^m(X) = \{x=(x_k) : (\Delta^m x_k) \in X\}$$

where, $m \in N$ and X is any sequence space. Now we give some relations between $\Delta^m(X)$ and X, and we discuss some topological properties of $\Delta^m(X)$.

Theorem 2.1 — *If X is a linear space, then $\Delta^m(X)$ is also a linear space.*

Proof is trivial

Theorem 2.2 — *If X is a Banach space normed by $\| \ \|$, then $\Delta^m(X)$ is also a Banach space normed by*

$$\| x \|_\Delta = \sum_{i=1}^{m} | x_i | + \| \Delta^m x \|. \qquad \text{... (2.1)}$$

Proof : It is a routine verification that $\Delta^m(X)$ is a normed space normed by (2.1). Now we show that $\Delta^m(X)$ is complete. Let (x^s) be a Cauchy sequence in $\Delta^m(X)$, where

$x^s = (x_1^s, x_2^s, \ldots) \in \Delta^m(X)$. Then

$$\| x^s - x^t \|_\Delta \to 0 \text{ as } s, t \to \infty$$

that is,
$$\| x_k^s - x_k^t \|_\Delta \to 0 \text{ as } s, t \to \infty.$$

Hence,
$$\sum_{i=1}^{m} | x_i^s - x_i^t | + \| \Delta^m (x_k^s - x_k^t) \| \to 0 \text{ as } s, t \to \infty.$$

Therefore $(x_i^1, x_i^2, \ldots)$, $(i \le m)$, and $(\Delta^m(x_k^1), \Delta^m(x_k^2), \ldots)$ are Cauchy sequences in **C**, the set of complex numbers, and X, respectively. Since **C** and X are complete, they are convergent. Suppose that $x_i^s \to x_i$, $(i \le m)$, in **C** and $(\Delta^m(x^s)) \to (y_k)$ in X, $s \to \infty$. Let $y_k = \Delta^m x_k$ so that

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m},$$

$$y_{1-m} = y_{2-m} = \ldots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. (we suppose that $\binom{-1}{-1} = 1$, in some literature it assume that $\binom{r}{k} = 0$ for $k < 0$). Then $(\Delta^m(x^s)) = ((\Delta^m(x_k^1), \Delta^m(x_k^2), \ldots)$ converges to $(\Delta^m(x_k))$ in X. Hence

$$\| x^s - x \|_\Delta \to 0 \text{ as } s \to \infty.$$

Therefore $\Delta^m(X)$ is a Banach space.

Lemma 2.3 — If $X \subset Y$, then $\Delta^m(X) \subset \Delta^m(Y)$.

Proof is trivial.

Theorem 2.4 — *Let X be a Banach space and A, a closed subset of X. Then $\Delta^m(A)$ is also closed in $\Delta^m(X)$.*

PROOF : Since $A \subset X$, then $\Delta^m(A) \subset \Delta^m(X)$ by Lemma 2.3. Now we show that $\overline{\Delta^m(A)} = \Delta^m(\overline{A})$, where $\overline{\Delta^m(A)}$, the closure of $\Delta^m(A)$ and $\overline{A}$, the closure of A|. *Let $x \in \overline{\Delta^m(A)}$. Then there exists a sequence (x^n) in $\Delta^m(A)$ such that*

$$\| x^n - x \|_\Delta \to 0 \text{ as } n \to \infty$$

in $\Delta^m(A)$, (see [1], pp 30). Hence

$$\| (x_k^n) - (x_k) \|_\Delta \to 0 \text{ as } n \to \infty$$

in $\Delta^m (A)$ so that

$$\sum_{i=1}^{m} | x_i^n - x_i | + \| \Delta^m (x_k^n) - \Delta^m (x_k) \| \to 0 \text{ as } n \to \infty.$$

in A. Thus $\Delta^m x \in \bar{A}$. This implies that $x \in \Delta^m (\bar{A})$. Conversely, if $x \in \overline{\Delta^m (A)}$. Then $x \in \overline{\Delta^m (A)}$. Since A is closed $\overline{\Delta^m (A)} = \Delta^m (A)$. Hence $\Delta^m (A)$ is a closed subset of $\Delta^m (X)$.

Theorem 2.5 — *Let X be a seprable space, then $\Delta^m (X)$ is also a separable space.*

PROOF : The proof is similar to that of Theorem 2.4.

Corollary 2.6 — $\Delta^m (c)$, $\Delta^m (c_0)$ and $\Delta^m (l_p)$, $(1 \le p < \infty)$, are separable spaces.

Theorem 2.7 — *In general, $\Delta^m (X)$ need not be sequence algebra.*

Proof is trivial.

Theorem 2.8 — If X is a BK-space normed by $\| \ \|$, then $\Delta^m (X)$ is a also BK-space normed by (2.1).

PROOF : Since X is Banach space $\Delta^m (X)$ is Banach space by Theorem 2.2. Now suppose that

$$\| x_k^n - x_k \|_\Delta \to 0 \text{ for each } k \in N \text{ as } n \to \infty.$$

Then we can write for $k \le m$,

$$| x_k^n - x_k | \to 0, \text{ as } n \to \infty$$

and for each $k \in N$

$$\| \Delta^m (x_k^n - x_k \| \to 0 \text{ as } n \to \infty.$$

On the other hand we have the inequality

$$| x_{k+m}^n - x_{k+m}) | \le \| \Delta^m (x_k^n - x_k) \| + \binom{m}{0} | x_k^n - x_k | + \ldots +$$
$$\binom{m}{m-1} | x_{k+m-1}{}^n - x_{k+m-1}) |$$

for each $k \in N$. Hence we obtain

$$| x_k^n - x_k | \to 0, \text{ as } n \to \infty$$

for each $k \in N$. Consequently $\Delta^m (X)$ is BK-space.

Now suppose that $X \subseteq l_\infty$ and X is Banack space normed by $\| \ \|$.

Let us define the operator

$$D : \Delta^m (X) \to \Delta^m (X)$$

defined by $Dx = (0, 0, ..., x_{m+1}, x_{m+2}, \ldots)$, where $x = (x_1, x_2, x_3, \ldots)$. It is trivial that D is a bounded linear operator on $\Delta^m (X)$. Furthermore the set

$$D [\Delta^m (X)] = D \Delta^m (X) = \{x = (x_k) : x \in \Delta^m (X), x_1 = x_2 = \ldots = x_m = 0\}$$

is a subspace of $\Delta^m (X)$ and $\| x \|_\Delta = \| \Delta^m x \|$ in $D \Delta^m (X)$. $D \Delta^m (X)$ and X are equivalent as topological space since

$$\Delta^m : D \Delta^m (X) \to X, \text{ defined by } \Delta^m x = y = (\Delta^m x_k) \tag{2.2}$$

is a linear homeomorphism [4].

Let X' and $[D \Delta^m (X)]'$ denote the continuous duals of X and $D \Delta^m (X)$, respectively. It can be shown that

$$T : [D \Delta^m (X)]' \to X', f_\Delta \to f_\Delta \circ (\Delta m)^{-1} = f$$

is a linear isometry. So $[D \Delta^m (X)]'$ is equivalent to X' [4].

3. New Sequence Space

We define

$$\Delta^m = (w_p) \left\{ x = (x_k) : \frac{1}{n} \sum_{k=1}^{n} | \Delta^m x_k - L |^p \to 0, n \to \infty, p > 0 \text{ for some } L \right\}$$

In the case $x \in \Delta^m (w_p)$, we write $x_k \to L (\Delta^m (w_p))$ and we write $\lim_n$ for $\lim_{n \to \infty}$.

Theorem 3.1 — *The sequence space $\Delta^m (w_p)$ is a Banach space for $1 \le p < \infty$ normed by*

$$\| x \|_{\Delta 1} = \sum_{i=1}^{m} | x_i | + \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} | \Delta^m x_k |^p \right)^{1/p}$$

and a complete p-normed space for, $0 < p < 1$, p-normed by

$$\| x \|_{\Delta 1} = \sum_{i=1}^{m} | x_i |^p + \sup_n \frac{1}{n} \sum_{k=1}^{m} | \Delta^m x_k |^p$$

PROOF : Proof follows from Theorem 2.2

We now define norm in $\Delta^m (w^p)$ which are different from those mentioned in Theorem 3.1. For $x \in \Delta^m (w_p)$ define

$$\| x \|_{\Delta 2} = \sum_{i=1}^{m} | x_i | + \sup_r \left(2^{-r} \sum_r | \Delta^m x_k |^p \right)^{1/p}, (p \geq 1),$$

$$\| x \|_{\Delta 2} = \sum_{i=1}^{m} | x_i |^p + \sup_r 2^{-r} \sum_r | \Delta^m x_k |^p, \ 0 < p < 1$$

where $r \geq 0$ and $\sum_r$ denotes sum over $k \in [2^r, 2^{r+1})$. It is trivial that the norms $\| x \|_{\Delta 1}$ and $\| x \|_{\Delta 2}$ are equivalent. It can be shown that $\Delta^{m-1}(w_p) \subset \Delta^m(w_p)$ and the inclusion is strict since $x = (k^m)$, for example, belongs to $\Delta^m(w_p)$, does not belong to $\Delta^{m-1}(w_p)$ for $0 < p < \infty$.

Corollary 3.2 — Let $p, q \in \boldsymbol{R}, 0 \leq p < q < \infty$. Then $\Delta^m(w_q) \subset \Delta^m(w_p)$.

Corollary 3.3 — $w_p \subset \Delta^m(w_p)$.

Corollary 3.4 — w_p and $D\,\Delta^m(w_p)$ are equivalent as topological space.

4. Δ^m-STATISTICAL CONVERGENCE

The definition of statistical convergence was introduced by Fast [2] in a short note. Schoenberg [5] studied statistical convergence as a summability method and listed some of the elementary properties of statistical convergence. Recently statistical convergence has been studied by Salat [10], Fridy [6], and Connor [7].

If $x = (x_k)$ is a sequence that satisfies some property p for all k except a set of natural density zero, then we say that x_k satisfies p for "almost all k" and we abbreviate this by "a.a.k".

In this section we give the definition of Δ^m-statistical convergence which gives the statistical convergence in the case $m = 0$ and inclusion theorems between the set of all Δ^m-statistical convergent sequence and other some sequence spaces. The set of all statistical convergent sequence is denoted by S.

Definition 4.1 — Let w be the linear space of all complex sequences and $x = (x_k) \in w$. The sequence x is said to be Δ^m-statistically convergent if there is a complex number L such that

$$\lim_n n^{-1} |\{k \leq n : | \Delta^m x_k - L | \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$, in which case we say that x is Δ^m-statistically convergent to L. In this case we write $x_k \to L(\Delta^m(S))$, where the vertical bars indicate the number of elements in the enclosed set. The set of Δ^m-statistically convergent sequences will be denoted by $\Delta^m(S)$. In the case $L = 0$, we shall write $\Delta^m(S_0)$.

***Theorem* 4.2** — *Let $p \in \boldsymbol{R}, 0 < p < \infty$. If $x_k \to L(\Delta^m(w_p))$, then $x_k \to L(\Delta^m(S))$. If $x \in \Delta^m(l_\infty)$ and $x_k \to L(\Delta^m(S))$ then $x_k \to L(\Delta^m(w_p))$.*

PROOF : Observe that for any $x = (x_k) \in w$, and $\varepsilon > 0$, we have that

$$\sum_{k=1}^{n} |\Delta^m x_k - L|^p \geq |\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}| \varepsilon^p$$

It follows that if $x_k \to L(\Delta^m(w_p))$, then $x_k \to L(\Delta^m(S))$.

Now suppose that $x \in \Delta^m(l_\infty)$ and $x_k \to L(\Delta^m(S))$ and set $M = \|\Delta^m x\|_\infty + |L|$. Let $\varepsilon > 0$ be given and select N_ε such that

$$n^{-1} \left| \left\{ k \leq n : \Delta^m x_k - L| \geq (\varepsilon/2)^{1/p} \right\} \right| \leq \frac{\varepsilon}{2M^p}$$

for all $n > N_\varepsilon$ and set $L_n = \{k \leq n : |\Delta^m x_k - L| \geq (\varepsilon/2)^{1/p}\}$

Now for $n > N_\varepsilon$ we have that

$$\frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_k - L|^p = \frac{1}{n} \left(\sum_{k \in L_n} |\Delta^m x_k - L|^p + \sum_{k \notin L_n} |\Delta^m x_k - L|^p \right)$$

$$< \frac{1}{n} \left(\frac{n\varepsilon}{2M^p} \right) M^p + \frac{1}{n} n \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $x_k \to l(\Delta^m(w_p))$.

Corollary 4.3 — $S \cap l_\infty \subset \Delta^m(S) \cap \Delta^m(l_\infty)$.

Corollary 4.4 — $\Delta^m(S) \cap \Delta^m(l_\infty) = \Delta^m(w_p)$.

Definition 4.5 — Let $x = (x_k) \in w$. The sequence x is said to be Δ^m-statistically Cauchy if there exists a number $N(= N(\varepsilon))$ such that

$$\lim_n n^{-1} |\{k \leq n : |\Delta^m x_k - \Delta^m x_N| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$.

Theorem 4.6 — *If x is a Δ^m-statistically convergent sequence, then x is a Δ^m-statistically Cauchy sequence.*

PROOF : Suppose that $x_k \to L(\Delta^m(S))$ and $\varepsilon > 0$. Then $|\Delta^m x_k - L| < \varepsilon/2$ for almost all k, and if N is chosen so that $|\Delta^m x_N - L| < \varepsilon/2$, then we have

$$|\Delta^m x_k - \Delta^m x_N| < |\Delta^m x_k - L| + |\Delta^m x_N - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for almost all } k.$$

Hence x is Δ^m-statistically Cauchy sequence.

Theorem 4.7 — *If x is a sequence for which there is a Δ^m-statistically convergent sequence y such that $\Delta^m x_k = \Delta^m y_k$ for almost all k then x is Δ^m-statistically convergent sequence.*

PROOF : Assume that $\Delta^m x_k = \Delta^m y_k$ for almost all k and $y_k \to L(\Delta^m(S))$, suppose $\varepsilon > 0$. Then for each n,

$$\{k \le n : |\Delta^m x_k - L| \ge \varepsilon\} \subseteq \{k \le n : \Delta^m x_k \ne \Delta^m y_k\} \cup \{k \le n : |\Delta^m x_k - L| \le \varepsilon\}$$

since $y_k \to l(\Delta^m(S))$, the latter set contains a fixed number of integers, say $g = g(\varepsilon)$. Therefore

$$\lim_n n^{-1} |\{k \le n : |\Delta^m x_k - L| \ge \varepsilon\}| \le \lim_n n^{-1} |\{k \le n : |\Delta^m x_k \ne \Delta^m y_k|\}|$$

$$+ \lim_n \frac{g}{n} = 0.$$

because $\Delta^m x_k = \Delta^m y_k$ for almost all k. Hence $x_k \to L(\Delta^m(s))$.

Now we will prove the following theorems help of Salat's [10] results.

Firstly, it follows from Lemma 1.2 [10] and Theorem 2.1 that the set of all Δ^m-bounded statistically convergent sequences of reel numbers is a linear subspace of all the linear normed space of all Δ^m-bounded sequences of reel numbers.

Now let $\Delta^m(m_0)$ and $\Delta^m(m)$ be the set of all Δ^m-bounded statistically convergent sequences of reel numbers and the set of all Δ^m-bounded sequences of reel numbers, respectively.

Theorem 4.8 — *The set of $\Delta^m(m_0)$ is a closed linear space of the linear normed space $\Delta^m(m)$.*

Proof follows from Theorem 2.1 [10] and Theorem 2.4.

Theorem 4.9 — *The set $\Delta^m(m_0)$ is a nowhere dense set in $\Delta^m(m)$.*

PROOF : According to Salat [10] that every closed linear subspace of an arbitrary linear normed space E, different from E is a nowhere dense set in E. Hence on account of Theorem 4.8 it suffices to prove that $\Delta^m(m_0) \ne \Delta^m(m)$. But this is evident, since the sequence $x = \{(-1)^k\} \in \Delta^m(m)$, but does not belong to $\Delta^m(m_0)$. (If $x = \{(-1)^k\}$, then $\Delta^m(x_k) = \{(-1)^k 2^m\}$).

Corollary 4.10 — m_0 and $D\Delta^m(m_0)$ are equivalent as topological spaces.

Finally, we prove that the following theorem.

Theorem 4.11 —

i) $\Delta^m(c) \subset \Delta^m(S)$ and the inclusion is strict,

ii) $\Delta^m(S)$ and $\Delta^m(l_\infty)$, overlap but neither one contains the other,

iii) $\Delta^m(S)$ and l_∞, overlap but neither one contains the other,

iv) S and $\Delta^m(S)$, overlap but neither one contains the other,

v) S and $\Delta^m(c)$, overlap but neither one contains the other,

vi) S and $\Delta^m(c_0)$, overlap but neither one contains the other,

vii) S and $\Delta^m(l_\infty)$, overlap but neither one contains the other.

PROOF : i) Since $c \subset S$, then $\Delta^m(c) \subset \Delta^m(S)$. If we choose

$$\Delta^m x_k = \begin{cases} \sqrt{k}\ , & k = n^2 \\ 0\ , & k \neq n^2 \end{cases} \quad , n = 1, 2, \ldots \qquad \ldots (4.1)$$

Then we obtain $\Delta^m x \in S$ but $\Delta^m x \in c$

Since $c \subset \Delta^m(S), c \subset \Delta^m(c), c \subset \Delta^m(l_\infty), c \subset S, c \subset l_\infty$, and $c \cap \Delta^m(c_0) \neq \Theta$ then $\Delta^m(S)$ and $\Delta^m(l_\infty), \Delta^m(S)$ and l_∞, S and $\Delta^m(S), S$ and $\Delta^m(c), S$ and $\Delta^m(c_0)$, S and $\Delta^m(l_\infty)$ is overlap. We show that an example, for each one that they don't include each other.

ii) If we define $\Delta^m X = (\Delta^m x_k)$ by (4.1), then $\Delta^m x \in S$ but $\Delta^m x \notin l_\infty$. Now we choose x = (1, 0, 1, 0, ...), then $\Delta^m x_k = (-1)^{k+1} 2^{m-1}$, and $x \in \Delta^m(l_\infty)$ but $x \notin \Delta^m(S)$.

iii) The proof is the same as (ii).

iv) Define $x_k = 1$ if k is a square and $x_k = 0$ otherwise. Then $x \in S$ but $x \notin \Delta^m(S)$. Conversely if we take $x = (k)$, then $x \in \Delta^m(S)$ but $x \notin S$.

The proof of (v) and (vi) are the same as (iv).

vii) Define $x_k = \sqrt{k}$ if k is a square and $x_k = 0$ otherwise. Then $x \in S$ but $x \notin \Delta^m(l_\infty)$. Conversely if we take $x = (k)$, then $x \in \Delta^m(l_\infty)$ but $x \notin S$.

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