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INTERVAL OSCILLATION CRITERIA FOR NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS*

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New oscillation criteria are established for the nonlinear differential equation $(r(t)\,y'(t))' + q(t)f(y(t)) = 0$ that are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. Our results are sharper than some previous results. In particular, several examples that dwell upon the sharp conditions of our results are also included.

Key Words : Interval Criteria; Oscillation; Second Order; Nonlinear Differential Equations

1. INTRODUCTION

We consider the oscillation behaviour of solutions of the second-order nonlinear differential equation

$$(r(t)y'(t))' + q(t)f(y(t)) = 0 \qquad \text{... (1.1)}$$

on the half-line $[t_0, \infty)$. In eq. (1.1) we assume that where $q \geq 0$ for $t \geq t_0$, $r(t) > 0$ is an eventually positive function, and f is a continuous real-valued function on the real line R and satisfies $uf(u) > 0$ and $f(u)/u \geq \mu > 0$ for every $u \neq 0$.

We recall that a function $y : [t_0, t_1) \to (-\infty, \infty)$, $t_1 > t_0$ is called a solution of eq. (1.1) if $y(t)$ satisfies eq. (1.1) for all $t \in [t_0, t_1)$. In the sequel it will be always assumed that solutions of eq. (1.1) exist for any $t_0 \geq 0$. A solution $x(t)$ of eq. (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

The oscillation problem for eq. (1.1) and for the less general equations such as the linear differential equation

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$$(r(t)\, y'(t))' + q(t)\, y(t) = 0, \quad \ldots (1.2)$$

and the nonlinear differential equation

$$y''(t) + q(t) f(y(t)) = 0 \quad \ldots (1.3)$$

has been discussed by numerous authors and by different metods (see, for example, [1-12]). For the case where $r(t) = 1$, one of the most important condition that guarantee that every solution of eq. (1.2) is oscillatory is as follows :

$$\limsup_{t \to \infty} \frac{1}{t^{\lambda}} \int_{t_0}^{t} (t-s)^{\lambda} q(s) ds = \infty \text{ for some } \lambda > 1 \text{ (Kamenev[5]).}$$

Most oscillation results involve the interval of q and hence require the information of q on the entire half-line $[t_0, \infty)$.

However, from the Sturm Separation Theorem, we see that oscillation is only an interval property, i.e., if there exists a sequence of sunintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \infty$, such that for each i there exists a solution of eq. (1.2) that has at least two zeros in $[a_i, b_i]$, then every solution of eq. (1.2) is oscillatory.

Ei-Sayed[3] established an interval criterion for oscillation of a forced second-order equation, but the result is not very sharp, because a comparison with equations of constant coefficient is used as in the proof.

In 1997, Huang[4] presented the following interval criteria for oscillation and nonoscillation of the second order linear differential equation

$$y''(t) + q(t) y(t) = 0, \quad \ldots (1.4)$$

where $q(t) \geq 0$ for $t \in [t_0, \infty)$.

Theorem A — *(i) If there exists $t_0 > 0$ such that for every $n \in N$,*

$$\int_{2^n t_0}^{2^{n+1} t_0} q(s) ds \leq \frac{\alpha_0}{2^{n+1} t_0}, \quad \ldots (1.5)$$

then every solution of eq. (1.4) is nonoscillatory, where $\alpha_0 = 3 - 2\sqrt{2}$.

(ii) If there exists $t_0 > 0$ and $\alpha > \alpha_0$ such that for every $n \in N$,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(s) ds \geq \frac{\alpha}{2^{n+1} t_0}, \quad \ldots (1.6)$$

then every solution of eq. (1.4) is oscillatory, where $\alpha_0 = 3 - 2\sqrt{2}$.

As an application, Huang[4] obtained the following corollary.

Corollary A — (*i*) If

$$\lim_{t \to \infty} t \int_t^{2t} q(s)ds = \alpha < \frac{\alpha_0}{2}, \qquad \text{... (1.7)}$$

then every solution of eq. (1.4) is nonoscillatory.

(*ii*) If

$$\lim_{t \to \infty} t \int_t^{2t} q(s)ds = \alpha > \alpha_0, \qquad \text{... (1.8)}$$

then every solution of eq. (1.4) is oscillatory, where $\alpha_0 = 3 - 2\sqrt{2}$.

We note that, the above result seems surprisingly interesting because the interval $(\alpha_0/2^{n+1} t_0, \alpha_0/2^n t_0)$ is not convered by the conditions (1.5) and (1.6). In particular, if $q(t) = \gamma/t^2$, where $\gamma > 0$ is constant, then

$$\lim_{t \to \infty} t \int_t^{2t} \frac{\gamma}{s^2} ds = \frac{\gamma}{2} < \frac{\alpha_0}{2} \to \gamma < 3 - 2\sqrt{2} < 1/4.$$

and

$$\lim_{t \to \infty} t \int_t^{2t} \frac{\gamma}{s^2} ds = \frac{\gamma}{2} = \alpha > \alpha_0 \to \gamma > 6 - 4\sqrt{2} > 1/4.$$

That implies that Huang's result remains open on $(3 - 2\sqrt{2}, 6 - 4\sqrt{2})$. That is to say, Huang's oscillation criterion is not sharp. In fact, that Euler equation

$$y''(t) + \frac{\gamma}{t^2} y(t) = 0$$

is oscillatory if $\gamma > 1/4$, and nonoscillatory if $\gamma \leq 1/4$ [7].

We remark that, Kong[6] employed the technique in the work of Philos[11] and obtained several interval oscillation results for sécond order linear equation (1.2). However, they cannot be applied to the nonlinear differential equation (1.1).

Motivated by the idea of Li[9], in this paper we obtain, by using a generalized Riccati techniquė due to Li[9], several new interval criteria for oscillation, that is, criteria given by the behaviour of eq. (1.1) (or of r, q and f) only on a sequence of subintervals of $[t_0, \infty)$. Our results involve the Kamenev's type condition and improve and extend the results of Huang[4], Kamenev[5] and Philos[11]. Finally, several examples that dwell upon the sharp conditions of our results are also included.

In what follows we always assume that

$$\frac{f(y)}{y} \geq \mu > 0, y \neq 0. \qquad \text{... (1.9)}$$

2. Oscillation Results

In the sequel we say that a function $H = H(t, s)$ belongs to a function class X, denoted by $H \in X$, if $H \in C(D, R_+)$, where $D = \{(t, s) : -\infty < s \le t < \infty\}$, which satisfies

$$H(t, t) = 0,\ H(t, s) > 0, \text{ for } t > s, \qquad \dots (2.1)$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t, s)\, H(t, s)^{1/2} \text{ and } \frac{\partial H}{\partial s} = -h_2(t, s)\, H(t, s)^{1/2}, \qquad \dots (2.2)$$

where $\quad h_1, h_2 \in L_{loc}(D, R)$.

The following lemmas will be useful for establishing oscillation criteria for eq. (1.1).

Lemma 2.1 — Suppose that (1.9) holds and that y is a solution of eq. (1.1) such that $y(t) > 0$ on $[c, b)$. For any $\upsilon \in C^1[t_0, \infty)$, let

$$u(t) = \upsilon(t)\, r(t) \left\{ \frac{y'(t)}{y(t)} + g(t) \right\} \qquad \dots (2.3)$$

on $[c, b)$. Then for any $H \in X$,

$$\int_c^b H(b, s)\, \phi(s)\, ds \le H(b, c)\, u(c) + \frac{1}{4} \int_c^b \upsilon(s)\, r(s)\, h_2^2(b, s)\, ds, \qquad \dots (2.4)$$

where $\quad \upsilon(t) = \exp\left(-2 \int^t g(s) ds \right)$ and

$$\phi(t) = \upsilon(t)\ \{\mu\, q(t) + r(t) g^2(t) - [r(t)\, g(t)]'\}.$$

Proof : From (1.1), (1.9) and (2.3) we have for $s \in [c, b)$

$$u'(t) = \upsilon(t) \left\{ -q(t) \frac{f(y(t))}{y(t)} - r(t)\, g^2(t) + [r(t)\, g(t)]' \right\} - \frac{1}{r(t)\, \upsilon(t)} u^2(t)$$

$$\le -\frac{u^2(t)}{\upsilon(t)\, r(t)} - \phi(t).$$

That is,

$$\phi(t) \le -u'(t) - \frac{u^2(t)}{\upsilon(t)\, r(t)}. \qquad \dots (2.5)$$

Multiplying (2.5) by $H(t, s)$, integrating it with respect to s from c to t for $t \in [c, b)$, and using (2.1) and (2.2) we get that

$$\int_c^t H(t,s)\,\phi(s)ds \le -\int_c^t H(t,s)u'(s)ds - \int_c^t H(t,s)\frac{u^2(s)}{v(s)\,r(s)}ds$$

$$= H(t,c)u(c) - \int_c^t \left\{h_2(t,s)\sqrt{H(t,s)}\,u(s) + H(t,s)\frac{u^2(s)}{v(s)\,r(s)}\right\}ds$$

$$= H(t,c)u(c) - \int_c^t \left\{\sqrt{\frac{H(t,s)}{r(s)\,v(s)}}\,u(s) + \frac{1}{2}\sqrt{r(s)\,v(s)}\,h_2(t,s)\right\}^2 ds$$

$$+\frac{1}{4}\int_c^t r(s)\,v(s)\,h_2^2(t,s)\,ds$$

$$\le H(t,c)u(c) + \frac{1}{4}\int_c^t r(s)\,v(s)\,h_2(t,s)\,ds.$$

Letting $t \to b^-$ in the above, we obtain (2.4). The proof is complete.

Lemmå 2.2 — Suppose that (1.9) holds and that y is a solution of eq. (1.1) such that $y(t) > 0$ on $(a, c]$. For any $g \in C^1[t_0, \infty)$ let $u(t)$ be defined by (2.3) on $(a, c]$. Then for any $H \in X$,

$$\int_a^c H(s,a)\,\phi(s)\,ds \le -H(c,a)\,u(c) + \frac{1}{4}\int_a^c r(s)\,v(s)\,h_1^2(s,a)\,ds, \qquad \text{... (2.6)}$$

where $v(t) = \exp\left(-2\int^t g(s)ds\right)$ and

$$\phi(t) = v(t)\ \{\mu\,q(t) + r(t)\,g^2(t) - [r(t)\,g(t)]'\}.$$

PROOF : Similar to the proof of Lemma 2.1, we multiply (2.5) by $H(s, t)$, integrate it with respect to s from c for $t \in (a, c]$, and use (2.1) and (2.2), then we get that

$$\int_t^c H(s,t)\,\phi(s)\,ds \le -\int_t^c H(s,t)u'(s)ds - \int_t^c H(s,t)\frac{u^2(s)}{v(s)\,r(s)}ds$$

$$= -H(c,t)\,u(c) + \int_t^c \left\{h_1(s,t)\sqrt{H(s,t)}\,u(s)b - H(s,t)\frac{u^2(s)}{v(s)\,r(s)}\right\}ds$$

$$= -H(c,t)\,u(c) - \int_t^c \frac{1}{r(s)\,v(s)}\{[\sqrt{H(s,t)}\,u(s)]^2$$

$$- h_1(s,t)\sqrt{H(s,t)}\,u(s)\,r(s)\,v(s) + \frac{1}{4}r^2(s)\,v^2(s)\,h_1^2(s,t)\}\,ds$$

$$+ \frac{1}{4}\int_t^c r(s)\,v(s)\,h_1^2(s,t)\,ds$$

$$= -H(c,t)\,u(c) - \int_t^c \frac{1}{r(s)\,v(s)}\left[\sqrt{H(s,t)}\,u(s) - \frac{1}{2}r(s)\,v(s)\,h_1(s,t)\right]^2 ds$$

$$+ \frac{1}{4}\int_t^c r(s)\,v(s)\,h_1^2(s,t)\,ds$$

$$\leq -H(c,t)\,u(c) + \frac{1}{4}\int_t^c r(s)\,v(s)\,h_1^2(s,t)\,ds.$$

Letting $t \to a^-$ in the above, we obtain (2.6). The proof is complete.

The following theorem is an immediate result from Lemmas 2.1 and 2.2.

***Theorem* 2.1** — *Assume that (1.9) holds and that for some $c \in (a,b)$ and for some $H \in X$, $g \in C^1[t_0, \infty)$,*

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\,\phi(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)\,\phi(s)ds$$

$$> \frac{1}{4}\left(\frac{1}{H(c,a)}\int_a^c r(s)\,v(s)\,h_1^2(s,a)\,ds + \frac{1}{H(b,c)}\int_c^b r(s)\,v(s)\,h_2^2(b,s)\,ds\right) \quad \ldots (2.7)$$

where $\quad v(t) = \exp\left(-2\int^t g(s)ds\right)$ and

$$\phi(t) = v(t)\;\{\mu\,q(t) + r(t)\,g^2(t) - [r(t)\,g(t)]'\}.$$

Then every solution of eq. (1.1) has at least one zero in (a, b).

PROOF : Suppose the contrary. Then without loss of generality we may assume that there is a solution $y(t)$ of eq. (1.1) such that $y(t) > 0$ for $t \in (a,b)$. From Lemmas 2.1 and 2.2 we see that both (2.4) and (2.6) hold. By dividing (2.4) and (2.6) by $H(b, c)$ and $H(c, a)$, respectively, and then adding them, we have that

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\,\phi(s)\,ds + \frac{1}{H(b,c)}\int_c^b H(b,s)\,\phi(s)\,ds$$

$$\leq \frac{1}{4}\left(\frac{1}{H(c,a)} \int_a^c r(s)\, v(s)\, h_1^2(s,a)\, ds + \frac{1}{H(b,c)} \int_c^b r(s)\, v(s)\, h_2^2(b,s)\, ds \right)$$

which contradicts the assumption (2.7) and completes the proof.

***Theorem* 2.2** — *Assume that (1.9) holds. If, for each $T \geq t_0$, there exist $H \in X$, $g \in C^1[t_0, \infty)$ and $a, b, c \in R$ such that $T \leq a < c < b$ and (2.7) holds, then every solution of eq. (1.1) is oscillatory.*

PROOF : Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumption, for each $i \in N$, there exist $a_i, b_i, c_i \in R$ such that $T_i \leq a_i < c_i < b_i$ and (2.7) holds, where a, b, c are replaced by a_i, b_i, c_i respectively. From Theorem 2.1, every solution $y(t)$ has at least one zero, $t_i \in (a_i, b_i)$. Noting that $t_i > a_i \geq T_i$, $i \in N$, we see that every solution has arbitrary large zeros. Thus, every solution of eq. (1.1) is oscillatory. The proof is complete.

***Theorem* 2.3** — *Assume that (1.9) holds. If*

$$\limsup_{t \to \infty} \int_l^t \left[H(s,l)\, \phi(s) - \frac{1}{4} r(s)\, v(s)\, h_1^2(s,l) \right] ds > 0, \qquad \text{... (2.8)}$$

and

$$\limsup_{t \to \infty} \int_l^t \left[H(t,s)\, \phi(s) - \frac{1}{4} r(s)\, v(s)\, h_2^2(t,s) \right] ds > 0, \qquad \text{... (2.9)}$$

for some $H \in X$, $g \in C^1[t_0, \infty)$ and for each $l \geq t_0$, where

$$v(t) = \exp\left(-2 \int^t g(s)ds \right)$$

and

$$\phi(t) = v(t)\ \{\mu\, q(t) + r(t)\, g^2(t) - [r(t)\, g(t)]'\},$$

then every solution of eq. (1.1) is oscillatory.

PROOF : For any $T \geq t_0$, let $a = T$. In (2.8) we choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c \left[H(s,a)\, \phi(s) - \frac{1}{4} r(s)\, v(s)\, h_1^2(s,a) \right] ds > 0. \qquad \text{... (2.10)}$$

In (2.9) we choose $l = c$. Then there exists $b > c$ such that

$$\int_a^b \left[H(b,s)\, \phi(s) - \frac{1}{4} r(s)\, v(s)\, h_2^2(b,s) \right] ds > 0. \qquad \text{... (2.11)}$$

Combining (2.10) and (2.11) we obtain (2.7). The conclusion thus comes from Theorem 2.2. The proof is complete.

For the case where $H := H(t-s) \in X$, we have that $h_1(t-s) = h_2(t-s)$ and denote them by $h(t-s)$. The subclass of X containing such $H(t-s)$ is denoted by X_0. Applying Theorem 2.2 to X_0, we obtain

***Theorem* 2.4** — *Assume that (1.9) holds. If for each $T \geq t_0$, there exist $H \in X_0$, $g \in [t_0 \infty)$ and $a, c \in R$ such that $T \leq a < c$ and*

$$\int_a^c H(s-a)\,[\phi(s) + \phi(2c-s)]\,ds$$

$$> \frac{1}{4} \int_a^c [r(s)\,v(s) + r(2c-s)\,v(2c-s)]\,h^2(s-a)\,ds,$$

where $$v(t) = \exp\left(-2\int^t g(s)ds\right) \text{ and}$$

$$\phi(t) = v(t)\{\mu\, q(t) + r(t) g^2(t) - [r(t)\,g(t)]'\},$$

then every solution of eq. (1.1) is oscillatory.

PROOF : Let $b = 2c - a$. Then $H(b-c) = H(c-a) = H((b-a)/2)$, and for any $w \in L[a, b]$, we have

$$\int_c^b w(s)\,ds = \int_a^c w(2c-s)\,ds.$$

Hence, $$\int_c^b H(b-s)\,\phi(s)ds = \int_a^c H(s-a)\,\phi(2c-s)\,ds$$

and $$\int_c^b r(s)\,v(s)\,h^2(b-s)\,ds = \int_a^c r(2c-s)\,v(2c-s)\,h^2(s-a)\,ds.$$

Thus that (2.12) holds implies that (2.7) holds for $H \in X_0$, $g \in [t_0, \infty)$ and therefore every solution of eq. (1.1) is oscillatory by Theorem 2.2. The proof is complete.

From above oscillation criteria, we can obtain different sufficient conditions for oscillation of all solutions of eq. (1.1) by different choices of $H(t, s)$.

Let $h(t, s) = (t-s)^{\lambda}$, $t \geq s \geq t_0$,

where $\lambda > 1$ is a constant.

Corollary 2.1 — Assume that (1.9) holds. Then every solution of eq. (1.1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > 1$, the following two inequalities hold :

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t \left[(s-l)^\lambda \phi(s) - \frac{1}{4} r(s)\, \upsilon(s)\, (s-\upsilon)^{\lambda-2} \right] ds > 0, \qquad \text{... (2.13)}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_l^t \left[(t-s)^\lambda \phi(s) - \frac{1}{4} r(s)\, \upsilon(s)\, (t-s)^{\lambda-2} \right] ds > 0, \qquad \text{... (2.14)}$$

where

$$\upsilon(t) = \exp\left(-2 \int^t g(s)\, ds \right) \text{ and}$$

$$\phi(t) = \upsilon(t) \ \{\mu\, q(t) + r(t) g^2(t) - [r(t)g(t)]'\}.$$

The proof is similar to that of Theorem 2.3, we omit it here.

Define

$$R(t) = \int_l^t \frac{1}{r(s)} ds, \ t \geq l \geq t_0, \qquad \text{... (2.15)}$$

and let

$$H(t,s) = [R(t) - R(s)]^\lambda, \ t \geq t_0, \qquad \text{... (2.16)}$$

where $\lambda > 1$ is a constant.

By Theorem 2.3, we have the following oscillation criterion.

Theorem 2.5 — *Assume that (1.9) holds and that* $\lim_{t\to\infty} R(t) = \infty$. *Then every solution of eq. (1.1) is oscillatory provided that for each* $l \geq t_0$ *and for some* $\lambda > 1$, *the following two inequalities hold :*

$$\limsup_{t\to\infty} \frac{\mu}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda q(s) ds > \frac{\lambda^2}{4(\lambda-1)} \qquad \text{... (2.17)}$$

and

$$\limsup_{t\to\infty} \frac{\mu}{R^{\lambda-1}(t)} \int_l^t [R(t) - R(s)]^\lambda q(s) ds > \frac{\lambda^2}{4(\lambda-1)}. \qquad \text{... (2.18)}$$

PROOF : Pick up $g(t) \equiv 0$, then $\upsilon(t) = 1, \phi(t) = \mu\, q(t)$,

$$h_1(t,s) = \lambda\, [R(t) = R(s)]^{(\lambda-2)/2} \frac{1}{r(t)},$$

and

$$h_2(t,s) = \lambda\, [R(t) = R(s)]^{(\lambda-2)/2} \frac{1}{r(s)}.$$

Noting that

$$\int_l^t r(s)\,h_1^2(s,l)\,ds = \int_l^t r(s)\,\lambda^2\,[R(s)-R(l)]^{\lambda-2}\frac{1}{r^2(s)}\,ds = \frac{\lambda^2}{\lambda-1}[R(t)-R(l)]^{\lambda-1},$$

and
$$\int_l^t r(s)\,h_2^2(t,s)\,ds = \int_l^t r(s)\,\lambda^2\,[R(t)-R(s)]^{\lambda-2}\frac{1}{r^2(s)}\,ds = \frac{\lambda^2}{\lambda-1}[R(t)-R(l)]^{\lambda-1}.$$

In view of $\lim_{t\to\infty} R(t)=\infty$, we have

$$\lim_{t\to\infty}\frac{1}{4(R^{\lambda-1}(t)}\int_l^t r(s)\,h_1^2(s,l)\,ds = \frac{\lambda^2}{4(\lambda-1)}, \qquad \text{... (2.20)}$$

and
$$\lim_{t\to\infty}\frac{1}{4R^{\lambda-1}(t)}\int_l^t r(s)\,h_2^2(t,s)\,ds = \frac{\lambda^2}{4(\lambda-1)}. \qquad \text{... (2.21)}$$

From (2.17) and (2.20) we have that

$$\limsup_{t\to\infty}\frac{\mu}{R^{\lambda-1}(t)}\int_l^t\left\{[R(s)-R(l)]^{\lambda}q(s)-\frac{1}{4}r(s)\,h_1^2(s,l)\right\}ds$$

$$= \limsup_{t\to\infty}\frac{\mu}{R^{\lambda-1}(t)}\int_l^t[R(s)-R(l)]^{\lambda}q(s)\,ds - \lim_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t\frac{1}{4}r(s)\,h_1^2(s,l)\,ds$$

$$= \limsup_{t\to\infty}\frac{\mu}{R^{\lambda-1}(t)}\int_l^t[R(s)-R(l)]^{\lambda}q(s)ds - \frac{\lambda^2}{4(\lambda-1)} > 0,$$

i.e., (2.8) holds. Similarly, (2.18) implies that (2.9) holds. By Theorem 2.3, every solution of eq. (1.1) is oscillatory. The proof is complete.

3. EXAMPLES

In this section we will show the applications of our oscillation criteria by two examples. We will see that the equations in the examples are oscillatory based on the results in Section 2, though the oscillation cannot be demonstrated by the results of Huang[4] and most other known criteria.

Example .1 — Consider the nonlinear differential equation

$$\left(\frac{1}{2t}y'(t)\right)' + \frac{2\gamma t}{(t^2-1)^2}y(t)\left(1+\frac{1}{1+y^2(t)}\right) = 0,\ t\ge 1. \qquad \text{... (3.1)}$$

Then
$$R(t) = \int_1^t \frac{1}{r(s)}\,ds = \int_1^t 2sds = (t^2-1),\ R'(t) = 2t,$$

$$\lim_{t\to\infty} R(t) = \infty, \frac{f(y)}{y} = 1 + \frac{1}{1+y^2} \geq 1, \mu > 0,$$

and for $\lambda > 1$,

$$\lim_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda q(s)ds$$

$$= \lim_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^\lambda \frac{2\gamma s}{(s^2-1)^2} ds \qquad \text{... (3.2)}$$

$$= \lim_{t\to\infty} \frac{[R(t) - R(l)]^\lambda}{(\lambda - 1) R^{\lambda-1}(t) R'(t)} \frac{2\gamma t}{(t^2-1)^2}$$

$$= \frac{\gamma}{(\lambda-1)}.$$

Next, we will prove that

$$\int_l^t [R(t) - R(s)]^\lambda \frac{2\gamma s}{(s^2-1)^2} ds \geq \int_l^t [R(s) - R(l)]^\lambda \frac{2\gamma s}{(s^2-1)^2} ds. \qquad \text{... (3.3)}$$

Let $$F(t) = \int_l^t \{[R(t) - R(s)]^\lambda - [R(s) - R(l)]^\lambda\} \frac{2\gamma s}{(s^2-1)^2} ds.$$

Then $F(l) = 0$, and for $t \geq l$,

$$F'(t) = \int_l^t \lambda [R(t) - R(s)]^{\lambda-1} R'(t) \frac{2\gamma s}{(s^2-1)^2} ds - [R(t) - R(l)]^\lambda \frac{2\gamma t}{(t^2-1)^2}$$

$$\geq \int_l^t \lambda [R(t) - R(s)]^{\lambda-1} R'(s) \frac{2\gamma s}{(s^2-1)^2} ds - [R(t) - R(l)]^\lambda \frac{2\gamma t}{(t^2-1)^2}$$

$$\geq \frac{2\gamma t}{(t^2-1)^2} \int_l^t \lambda [R(t) - R(s)]^{\lambda-1} R'(s)\, ds - [R(t) - R(l)]^\lambda \frac{2\gamma t}{(t^2-1)^2}$$

$$= -\frac{2\gamma t}{(t^2-1)^2} \int_l^t \lambda [R(t) - R(s)]^{\lambda-1} d[R(t) - R(s)]$$

$$- [R(t) - R(l)]^\lambda \frac{2\gamma t}{(t^2-1)^2}$$

$$= \frac{2\gamma t}{(t^2-1)^2} [R(t) - R(l)]^\lambda - [R(t) - R(l)]^\lambda \frac{2\gamma t}{(t^2-1)^2}$$

$$= 0.$$

Hence $F(t) \geq F(l) = 0$ for $t \geq l$, i.e., (3.3) holds. By (3.2) and (3.3), for any $\gamma > 1/4$ there exists $\lambda > 1$ such that $\gamma/(\lambda - 1) > \lambda^2/4(\lambda - 1)$. This means that (2.17) and (2.18) hold for the same λ. Applying Theorem 2.5, we find that (3.1) is oscillatory for $\gamma > 1/4$.

Example 2 — Consider the Euler equation

$$y''(t) + \frac{\gamma}{t^2} y(t) = 0, \qquad \text{... (3.4)}$$

where $\gamma > 0$ is a constant. Then $q(t) = \gamma/t^2$, $\mu = 1$ and $R(t) = t$. Similar to the proof of Example 1, we have that every solution of eq. (3.4) is oscillatory if $\gamma > 1/4$. However, Theorem A of Huang only apply to the condition : $\gamma > 2(3 - 2\sqrt{2})$. This implies that our results are sharp.

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