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MULTIPLE POSITIVE SOLUTIONS OF SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS AT NONRESONANCE*

WEI ZHONGLI

Department of Fundamental Courses, Shandong Institute of Architectural and Engineering, Jinan, Shandong, 250014, *People's Republic of China*

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This paper investigates the existence of multiple positive solutions of singular Dirichlet boundary value problems at nonresonance. A and sufficient conditions for the existence of C [0, 1] multiple positive solutions as well as C^1 [0, 1] multiple positive solutions is given by means of the fixed point theorems on cones.

Key Words : Singular Dirichlet Boundary Value Problem; Positive Solution; Fixed Point Theorem; Cone; Nonresonance

1. INTRODUCTION

The theory of singular boundary value problems has become an important area of investigation in recent years (see [1-6] and the references therein). Consider the singular Dirichlet boundary value problems of second order ordinary differential equation

$$\left.\begin{array}{l} -x'' + \rho\, p(t)x = f(t, x), \quad t \in (0, 1), \\ x(0) = x(1) = 0, \end{array}\right] \qquad \text{... (1.1)}$$

where $\rho > 0$ is such that

$$\left.\begin{array}{l} -x'' + \rho\, p(t)x = 0, \quad t \in (0, 1), \\ x(0) = x(1) = 0 \end{array}\right\} \qquad \text{... (1.2)}$$

has only the trivial solution. For convenience, we list the following hypothesis.

(H_1) $p(t) \in C(0, 1), p(t) \geq 0, t \in (0, 1)$, and

$$\int_0^1 t(1-t)\, p(t)dt < \infty; \qquad \text{... (1.3)}$$

and $$\lim_{t \to 0^+} t^2 p(t) = 0 \text{ if } \int_0^1 (1-t)\, p(t)dt = \infty; \qquad \text{... (1.4)}$$

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and $$\lim_{t \to 1^-} (1-t)^2 p(t) = 0 \text{ if } \int_0^1 tp(t)\, dt = \infty; \quad \text{... (1.5)}$$

(H_2) $p(t) \in C(0, 1), p(t) \geq 0, t \in (0, 1)$,

and $$\int_0^1 p(t)\, dt < \infty; \quad \text{... (1.6)}$$

(H_3) $f(t, x) = f_1(t, x) + f_2(t, x), f_1(t, x), f_2(t, x) \in C((0, 1) \times (0, +\infty), [0, +\infty)), f(t, 1) \not\equiv 0$ for $t \in (0, 1)$, and there exist constants $\lambda_1, \mu_1 \ (0 < \lambda_1 \leq \mu_1 < 1)$, $\lambda_2, \mu_2 \ (1 < \lambda_2 \leq \mu_2 < \infty)$, $N, M \ (0 < N \leq 1 \leq M)$ such that for $t \in (0, 1)$ and $x \in (0, +\infty)$,

$$c^{\mu_1} f_1(t, x) \leq f_1(t, cx) \leq c^{\lambda_1} f_1(t, x), \text{ if } 0 \leq c \leq N; \quad \text{... (1.7)}$$

and $$c^{\lambda_1} f_1(t, x) \leq f_1(t, cx) \leq c^{\mu_1} f_1(t, x), \text{ if } c \geq M. \quad \text{... (1.8)}$$

$$c^{\mu_2} f_2(t, x) \leq f_2(t, cx) \leq c^{\lambda_2} f_2(t, x), \text{ if } 0 \leq c \leq N; \quad \text{... (1.9)}$$

and $$c^{\lambda_2} f_2(t, x) \leq f_2(t, cx) \leq c^{\mu_2} f_2(t, x), \text{ if } c \geq M. \quad \text{... (1.10)}$$

Typical functions that satisfy the above sublinear (1.7) and (1.8)) and superlinear ((1.9) and (1.10)) hypothesis are those taking the form $f_1(t, x) = \sum_{k=1}^{n} p_k(t) x^{\eta_k}$; here $p_k(t) \in C(0, 1), p_k(t) > 0$ on (0, 1), $0 < \eta_k < 1$, $k = 1, 2, ..., n$; and $f_2(t, x) = \sum_{j=1}^{m} q_j(t) x^{\rho_j}$; here $q_j(t) \in C(0, 1), q_j(t) > 0$ on (0, 1), $\rho_j > 1, j = 1, 2, \dots m$, respectively.

By singularity we mean that the functions p, f in (1.1) are allowed to be unbounded at the end points $t = 0$ and $t = 1$. A function $x(t) \in C[0, 1] \cap C^2(0, 1)$ is called a C [0, 1] (positive) solution of (1.1) if it satisfies (1.1) ($x(t) > 0$, for $t \in (0, 1)$). A C [0, 1] (positive) solution of (1.1) is called a C^1 [0, 1] (positive) solution if $x'(0^+)$ and $x'(1^-)$ both exist ($x(t) > 0$ for $t \in (0, 1)$).

In the special cases i) — $p(t) = 0, f_2(t, x) \equiv 0, f(t, x) = f_1(t, x) = p_1(t) x^{\eta_1}, 0 < \eta_1 < 1$, where $p_1(t) \in C(0, 1), p_1(t) > 0$ on (0, 1), and ii) : $f_2(t, x) \equiv 0, f(t, x) = f_1(t, x) = \sum_{k=1}^{n} p_k(t) x^{\eta_k}$, where $p_k(t) \in C(0, 1), p_k(t) > 0$ on (0, 1), $0 < \eta_k < 1, k = 1, 2, ..., n$, the existence and uniqueness of positive solutions of (1.1) have been studied completely by Zhang in [4] and by Wei in [5] with the method of lower and upper solutions and with Schauder fixed point theorems, respectively. A sufficient condition for the existence of C [0, 1] solutions of the singular problem (1.1) was given by O'Regan D. in [6] with a continuous Theorem.

Now, in this paper, we shall give a sufficient condition for the existence of C[0, 1] multiple positive solutions as well as C^1 [0, 1] multiple positive solutions of the singular problem (1.1) by using the fixed point theorems of cone expansion and compression of norm type, which is different from that of [3-6].

2. Several Lemmas

Lemma 2.1 — (see Lemma 2.1 of [5]) Suppose (H_1) holds.

(i) Then

$$\left.\begin{array}{l} -x'' + \rho\, p(t)x = 0, \quad t \in (0, 1), \\ x(0) = 0, x'(0) = 1 \end{array}\right] \qquad \text{... (2.1)}$$

has a unique positive increasing solution $e_4(t) = tw_1(t) \in C\,[0, 1] \cap C^1\,[0, 1)$, where $w_1 \in C\,[0, 1]$ is a unique solution of the following integral equation

$$w_1(t) = 1 + \frac{\rho}{t} \int_0^t \int_0^s \tau p(\tau)\, w_1(\tau)\, d\tau\, ds. \qquad \text{... (2.2)}$$

(*ii*) Then

$$\left.\begin{array}{l} -x'' + \rho\, p(t)x = 0, \quad t \in (0, 1), \\ x(1) = 0, x'(1) = -1 \end{array}\right\} \qquad \text{... (2.3)}$$

has a unique positive decreasing solution $e_2(t) = (1 - t)\, w_2(t) \in C[0, 1] \cap C^1\,(0, 1]$, where $w_2 \in C\,[0, 1]$ is a unique solution of the following integral equation

$$w_2(t) = 1 + \frac{\rho}{1 - t} \int_t^1 \int_s^1 (1 - \tau)\, p(\tau)\, w_2(\tau)\, d\tau\, ds. \qquad \text{... (2.4)}$$

Remark 1 : If $p(t) = 0$, then $e_1(t) = t, e_2(t) = 1 - t, w_1(t) = w_2(t) = 1$.

Let

$$\omega(t) = \begin{vmatrix} e_2(t) & e_2'(t) \\ e_1(t) & e_1'(t) \end{vmatrix} \qquad \text{... (2.5)}$$

be the Wronskian of $e_2(t)$ and $e_1(t)$ at t. Then $\omega(t) = \omega > 0, t \in (0, 1)$ is a constant. And let

$$G(t, s) = \left.\begin{array}{ll} \frac{1}{\omega} e_1(t)\, e_2(s), & 0\,, t \le s, \\ \frac{1}{\omega} e_1(s)\, e_2(t), & s \le t < 1, \end{array}\right] \qquad \text{... (2.6)}$$

where $e_1(t)$ and $e_2(t)$ are given by Lemma 2.1, ω is given by (2.5).

By means of Lemma 2.1, we can obtain the following Lemma 2.2.

Lemma 2.2 — Suppose that (H_1) holds. Then for all $\alpha, \beta\,(0<\alpha<\beta<1)$, we have

$$G(t, s) \geq \tau,\ (t, s) \in J_0 \times J_0, \quad \text{... (2.7)}$$

and

$$G(t, s) \geq \varepsilon_0\, G(\xi, s),\ t \in J_0,\ (\xi, s) \in J \times J. \quad \text{... (2.8)}$$

Here $J = [0,\ 1],\ J_0 = [\alpha, \beta]$,

$$\tau = \frac{1}{\omega} e_1(\alpha)\, e_2(\beta),\quad \varepsilon_0 = \frac{e_1(\alpha)}{e_1(1)} \cdot \frac{e_2(\beta)}{e_2(0)}. \quad \text{... (2.9)}$$

Lemma 2.3 (see Theorem 2.3.4 of [7]) — Let E be a Banach space, $P \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded and open subsets of E with $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ and $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either

$$\| Ax \| \leq \| x \|,\ \forall\, x \in P \cap \partial\Omega_1,\ \| Ax \| \geq \| x \|,\ \forall\, x \in P \cap \partial\Omega_2,$$

or

$$\| Ax \| \geq \| x \|,\ \forall\, x \in P \cap \partial\Omega_1,\ \| Ax \| \leq \| x \|,\ \forall\, x \in P \cap \partial\Omega_2.$$

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Positive Solution in the Superlinear Case

In this section, we shall give a sufficient condition for the existence of $C[0,\ 1]$ positive solutions as well as $C^1[0, 1]$ positive solutions of the singular problem (1.1) which $f(t,\ x)$ satisfies the superlinear condition ((1.9) and (1.10)) by using the fixed point theorems of cone expansion of norm type

Theorem 3.1 — *Suppose (H_1) holds, and $f_1(t, x) \equiv 0, f(t, x) = f_2(t, x)$ satisfies (H_3).*

$$0 < \int_0^1 t(1-t) f_2(t,\ 1)\, dt < \infty, \quad \text{... (3.1)}$$

$$\lim_{t \to 0^+} t \int_t^1 (1-s) f_2(s,\ 1) ds = 0 \text{ if } \int_0^1 (1-s) f(s,\ 1)\, ds = \infty, \quad \text{... (3.2)}$$

and

$$\lim_{t \to 1^-} t \int_t^1 (1-t)\, s f_2(s,\ 1) ds = 0 \text{ if } \int_0^1 s f(s,\ 1)\, ds = \infty, \quad \text{... (3.3)}$$

then problem (1.1) has at least one C[0, 1] positive solution.

***Theorem* 3.2** — *Suppose (H_2) holds, and $f_1(t,x) \equiv 0, f(t,x) = f_2(t,x)$ satisfies (H_3).*

$$0 < \int_0^1 f_2(t, 1))\, dt < \infty, \qquad \text{... (3.4)}$$

then problem (1.1) has at least one $C^1[0, 1]$ positive solution.

PROOF OF THEOREM 3.1 : Let $J = [0, 1]$, $X = C[0, 1]$, $\|x\| = \sup_{t \in J} |x(t)|$. Then X is a Banach space with $\|\cdot\|$.

For each fixed $x(t) \in C[0,1], x(t) \geq 0$, let $c_1 > 0$ be a positive constant such that $c_1 \|x\| \leq N$, $1/c_1 \geq M$. From (1.9) and (1.10), we have

$$f(t, x(t)) \leq f(t, y(t)) \text{ for } x(t) \leq y(t), t \in J, \qquad \text{... (3.5)}$$

$$f(t, x(t)) \leq (1/c_1)^{\mu_2} f(t, c_1 x(t)) \leq (c_1)^{\lambda_2 - \mu_2} \|x\|^{\lambda_2} f(t, 1) \leq c_1^{-\mu_2} f(t, 1). \qquad \text{... (3.6)}$$

By Lemma 2.1, (2.6), (3.1) and (3.6)

$$\int_0^1 G(t, s) f(s, x(s))\, ds \leq \frac{\|w_1\| \cdot \|w_2\|}{\omega} c_1^{-\mu_2} \int_0^1 s(1-s) f(s, 1)\, ds < +\infty, \; t \in J, \qquad \text{... (3.7)}$$

where $G(t, s)$ is given by (2.6), $e_1(t), e_2(t), w_1(t), w_2(t)$ are given by Lemma 2.1.

By means of (3.7), if (3.1)-(3.3) hold, then the problem (1.1) is equivalent to the following integral equation

$$x(t) = Ax(t) = \int_0^1 G(t, s,) f(s, x(s))\, ds, \qquad \text{... (3.8)}$$

i.e., $A : X \to X$ has a fixed point.

Since $f(t, 1) \not\equiv 0$, there exist $0 < \alpha < \beta < 1$ such that

$$\min_{t \in J_0} f(t, 1) = \tau_0 > 0, \qquad \text{... (3.9)}$$

here $\qquad J_0 [\alpha, \beta]$.

Let $P = \left\{ x \mid x \in C[0, 1], x(t) \geq 0 \min_{t \in J_0} x(t) \geq \varepsilon_0 \|x\| \right\}$, where ε_0 is given by (2.9). Then P is a cone of $C[0, 1]$. For $x \in P$, by virtue of Lemma 2.2, we have

$$Ax(t) \geq \varepsilon_0 \int_0^1 G(z, s) f(s, x(s))\, ds = \varepsilon_0 Ax(z), \; t \in J_0, z \in J.$$

So, $\qquad \min_{t \in J_0} Ax(t) \geq \varepsilon_0 \|Ax\|$, i.e.: $Ax \in P$, and therefore

$$A(P) \subset P. \qquad \text{... (3.10)}$$

In the following, we show that $A : P \to P$ is completely continuous.

Let $B \subset P$ be bounded. Then $\forall\, x \in B$, there exists posive consant L such that $\| x \| \le L$. Let $c_2 > 0$ be a constant such that $c_2 L \le N$, $1/c_2 \ge M$. From (1.9) and (1.10), we have

$$f(t, x(t)) \le c_2^{-\mu_2} f(t, 1), \ \ \forall\, x \in B, t \in (0, 1).$$

Hence,
$$| Ax(t) | \le c_2^{-\mu_2} L_1 \int_0^1 s(1-s) f(s, 1)\, ds < \infty,$$

where
$$L_1 = c_2^{-\mu_2} \| w_1 \| \cdot \| w_2 \| / \omega. \quad \text{... (3.11)}$$

i.e., $A(B)$ is uniformly bounded.

From (3.1)-(3.3) and $w_1(t), w_2(t) \in C[0, 1]$, $\forall\, \varepsilon > 0, \exists\, \delta = \delta(\varepsilon)$, when $| t_1 - t_2 | < \delta$ (without loss of generality, we assume $0 \le t_1 < t_2 \le 1$), we have

$$| e_1(t_2) - e_1(t_1) | = | t_2 (w_1(t_2) - w_1(t_1)) + (t_2 - t_1) w_1(t_1) |$$

$$\le t_2 | w_1(t_2) - w_1(t_1) | + (t_2 - t_1) \| w_1 \|, \quad \text{... (3.12)}$$

$$| e_2(t_2) - e_2(t_1) | = | [(1 - t_2) - (1 - t_2) - (1 - t_1)] w_2(t_2) + (1 - t_1) [w_2(t_2) - w_2(t_1)] |$$

$$\le (t_2 - t_1) \| w_2 \| + (1 - t_1) | w_2(t_2) - w_2(t_1) |, \quad \text{... (3.13)}$$

$$\int_{t_1}^{t_2} s(1-s) f(s \cdot 1)\, ds < \frac{\varepsilon}{8 L_1}, \quad \text{... (3.14)}$$

$$(t_2 - t_1) \cdot \int_{t_2 - t_1}^{1} (1-s) f(s, 1)\, ds \,, \frac{\varepsilon}{8 L_1}, \quad \text{... (3.15)}$$

$$(t_2 - t_1) \int_0^{1-(t_2-t_1)} s f(s, 1)\, ds < \frac{\varepsilon}{8 L_1}, \quad \text{... (3.16)}$$

$$| w_1(t_2) - w_1(t_1) | < \frac{\varepsilon \| w_1 \|}{8 L_1 \int_0^1 s(1-s) f(s, 1)\, ds}, \quad \text{... (3.17)}$$

$$| w_2(t_2) - w_2(t_1) | < \frac{\varepsilon \| w_2 \|}{8 L_1 \int_0^1 s(1-s) f(s, 1)\, ds}, \quad \text{... (3.18)}$$

where L_1 is given by (3.11). For any $x \in B$, from (3.12)-(3.18), we have

$$|Ax(t_2) - Ax(t_1)| = \Bigg| \frac{1}{\omega} \int_0^{t_2} e_2(t_2)\, e_1(s) f(s, x(s))\, ds + \frac{1}{\omega} \int_{t_2}^{1} e_1(t_2)\, s e_2(s) f(s, x(s))\, ds$$

$$- \frac{1}{\omega} \int_0^{t_1} e_2(t_1)\, e_1(s) f(s, x(s))\, ds - \frac{1}{\omega} \int_{t_1}^{1} e_1(t_1) e_2(s) f(s, x(s)) ds \Bigg|$$

$$= \Bigg| \frac{(e_2(t_2) - e_2(t_1))}{\omega} \int_0^{t_1} e_1(s) f(s, x(s))\, ds + \frac{1}{\omega} \int_{t_1}^{t_2} e_2(t_2)\, e_1(s) f(s, x(s))\, ds$$

$$+ \frac{e_1(t_2) - e_1(t_1)}{\omega} \int_{t_2}^{1} e_2(s) f(s, x(s)) ds - \frac{1}{\omega} \int_{t_1}^{t_2} e_1(t_1) e_2(s) f(s, x(s))\, ds \Bigg|$$

$$\le L_1(t_2 - t_1) \int_0^{1-(t_2-t_1)} s f(s, 1) ds + L_1(t_2 - t_1) \int_{t_2-t_1}^{1} (1-s) f(s, 1)\, ds$$

$$+ \frac{c_1^{-\mu_2} \| w_1 \|}{\omega} | w_2(t_2) - w_2(t_1) | \int_0^1 s(1-s) f(s, 1) ds$$

$$+ \frac{c_1^{-\mu_2} \| w_2 \|}{\omega} | w_1(t_2) - w_1(t_1) | \int_0^1 s(1-s) f(s, 1) ds$$

$$+ 2L_1 \int_{t_1}^{t_2} s(1-s) f(s, 1) ds < \varepsilon.$$

This means $A(B)$ is a relatively compact set.

Let $x_n, x_0 \in P, x_n \to x_0\ (n \to \infty)$. Then $L_2 = \sup\{\| x_n \|, n = 0, 1, 2, ...\} < \infty$. Let $c_3 > 0$ be a constant such that $c_3 L_2 \le N$, $1/c_3 \ge M$. From (1.9) and (1.10), we have

$$f(t, x_n(t)) \le c_3^{-\mu_2} f(t, 1),\ n = 0, 1, 2, ...,\ t \in (0, 1), \qquad \text{... (3.19)}$$

$$| Ax_n(t) - Ax_0(t) | \le \frac{\| w_1 \| \cdot \| w_2 \|}{\omega}$$

$$\int_0^1 s(1-s)\,|f(s, x_n(s)) - f(s, x_0(s))|\,ds, \; t \in J. \qquad \text{... (3.20)}$$

(3.19), (3.20), (3.1), condition (H_3) and Lebesgue dominated convergence theorem imply :

$$\| Ax_n - Ax_0 \| \to 0, \; (n \to \infty).$$

i.e., $A : P \to P$ is continuous. Therefore, $A : P \to P$ is completely continuous.

Let $c_4 > 0$ be a constant such that $c_4 \varepsilon_0 \geq M$, $1/c_4 \leq N$. For $x(t) \in P, \| x \| > 1$, we have $c_4 x(t) \geq M$ for $t \in J_0$. From (1.9) and (1.10), we have

$$f(t, x(t)) \geq c_4^{\lambda_2 - \mu_2} x^{\lambda_2}(t)\, f(t, 1) \text{ for } x \in P, \| x \| > 1, t \in J_0. \qquad \text{... (3.21)}$$

By virtue of (2.7), (3.9), (3.21), we obtain

$$Ax(t) \geq \int_\alpha^\beta G(t, s) f(s, x(s))\, ds$$

$$\geq \tau c_4^{\lambda_2 - \mu_2} \tau_0 (\beta - \alpha) \varepsilon_0^{\lambda_2} \| x \|^{\lambda_2} \text{ for } x \in P, t \in J_0, \| x \| > 1.$$

Since $\lambda_2 > 1$, if we choose

$$R = \max \left\{ 2, \left[\tau c_4^{\lambda_2 - \mu_2} \tau_0 (\beta - \alpha) \varepsilon_0^{\lambda_2} \right]^{-\frac{1}{\lambda_2 - 1}} > 1 \right\},$$

then $$\| Ax \| \geq \| x \|, \; \forall \, x \in P, \| x \| = R. \qquad \text{... (3.22)}$$

On the other hand, let $c_5 > 0$ be a constant such that $c_5 \geq M$, $1/c_5 \leq N$. From (1.9), (1.10), we have

$$f(t, x(t)) \leq c_5^{\mu_2 - \lambda_2} x^{\lambda_2}(t) f(t, 1) \text{ for } x(t) \in P, \| x \| < 1, t \in (0, 1). \qquad \text{... (3.23)}$$

$$| Ax(t) | \leq \frac{\| w_1 \| \cdot \| w_2 \| c_5^{\mu_2 - \lambda_2}}{\omega} \int_0^1 s(1-s) f(s, 1)\, ds \, \| x \|^{\lambda_2}$$

for $\quad x(t) \in P, \| x \| < 1, t \in J.$

Therefore, choosing

$$r = \min\left\{\frac{1}{2}, \left[(c_5^{\mu_2-\lambda_2}\|w_1\|\cdot\|w_2\|/\omega)\int_0^1 s(1-s)f(s,1)\,ds\right]^{-\frac{1}{\lambda_2-1}}\right\} < 1,$$

we get $$\|Ax\| \le \|x\| \quad \forall\, x \in P, \|x\| = r. \qquad \text{... (3.24)}$$

Now, observing (3.10), (3.22), (3.24) and using Lemma 2.3, we conclude that A has a fixed point $x^*(t) \in P$ such that

$$r < \|x^*\| < R. \qquad \text{... (3.25)}$$

In the following, we show that $x^*(t)$ is a positive solution of (1.1).

Obviously, $x^*(t) \ge \varepsilon_0\|x^*\| > 0$, for $t \in J_0$. If $t \in (0, 1)\setminus J_0$, then there are two cases 1 : $t \in (0, \alpha)$, 2 : $t \in (\beta, 1)$.

For case 2 — $t \in (\beta, 1)$, let $c_6 > 0$ be a constant such that $c_6 R \le N$, $1/c_6 \ge M$. From (1.9), (1.10), we get

$$f(t, x^*(t)) \ge c_6^{\mu_2-\lambda_2}(x^*(t))^{\mu_2} f(t, 1), t \in (0, 1). \qquad \text{... (3.26)}$$

Therefore,

$$x^*(t) = Ax^*(t) = \frac{1}{\omega}\int_0^t e_2(t)e_1(s)f(s, x^*(s))\,ds + \frac{1}{\omega}\int_t^1 e_2(s)e_1(t)f(s, x^*(s))\,ds$$

$$\ge \frac{c_6^{\mu_2-\lambda_2}}{\omega}\int_\alpha^\beta e_2(t)e_1(s)(x^*(s))^{\mu_2} f(s, 1)ds$$

$$\ge c_6^{\mu_2-\lambda_2}\tau_0\,\varepsilon_0^{\mu_2} r^{\mu_2} e_1(\alpha)\,e_2(t)/\omega > 0, t \in (\beta, 1)$$

i.e., $x^*(t) > 0, t \in (\beta, 1)$. Similarly, we can show $x^*(t) > 0$, $t \in (0, \alpha)$. Hence, $x^*(t)$ is a $C[0, 1]$ positive solution of (1.1). The proof of Theorem 3.1 is complete.

PROOF OF THEOREM 3.2 : (H_2) and (3.4) imply (H_1) and (3.1)-(3.3). By the proof of Theorem 3.1, there is a $C[0, 1]$ positive solution $x^*(t)$ of (1.1) satisfying (3.25). In the following, we shall show that $x^*(t) \in C^1[0, 1] \cap C^2(0, 1)$.

Let $c_7 > 0$ be a constant such that $c_7\|x^*\| \le N$, $1/c_7 \ge M$. From (1.9) and (1.10) we have :

$$\|x^{*''}(t)| \le \rho p(t)\|x^*\| + f(t, x^*(t)) \le \rho p(t)\|x^*\| + c_7^{-\mu_2} f(t, 1), t \in (0, 1). \qquad \text{... (3.27)}$$

By means of (H_2), (3.4) and (3.27), we get that $x^{*''}(t)$ is absolutely integrable on [0, 1]. This implies $x^*(t) \in C^1[0, 1]$, so $x^*(t)$ is a $C^1[0, 1]$ positive solution of the problem (1.1). The proof is complete.

§4. Multiple Positive Solutions

In this section, we shall give a sufficient condition for the existence of $C[0, 1]$ multiple positive solutions as well as C^1 [0, 1] multiple positive solutions of the singular problem (1.1) which $f(t, x)$ satisfies condition (H_3) by using the fixed point theorems of cone expansion and cone compression of norm type.

***Theorem* 4.1** — *Suppose (H_1) and (H_3) hold, and* $f_1(t, 1) > 0, f_2(t, 1) > 0, t \in (0, 1)$.

$$0 < \int_0^1 t(1-t)f(t,-1)\,dt < \frac{\omega}{\|w_1\| \cdot \|w_2\|}, \quad \text{... (4.1)}$$

$$\lim_{t \to 0^+} t \int_t^1 (1-s)f(s, 1))ds = 0 \text{ if } \int_0^1 (1-s)f(s, 1)\,ds = \infty, \quad \text{... (4.2)}$$

and

$$\lim_{t \to 1^-} (1-t) \int_0^t sf(s, 1))\,ds = 0 \text{ if } \int_0^1 sf(s, 1)\,ds = \infty, \quad \text{... (4.3)}$$

then the singular problem (1.1) has at least two C[0, 1] positive solution $x_1(t)$ and $x_2(t)$ such that

$$0 < r < \|x_1\| < 1 < \|x_2\| < R, \quad \text{... (4.4)}$$

where r and R are constant.

***Theorem* 4.2** — *Suppose (H_2) and (H_3) hold, and* $f_1(t, 1) > 0, f_2(t, 1) > 0, t \in (0, 1)$.

$$0 < \int_0^1 f(t, 1)\,dt < \frac{\omega}{\|w_1\| \cdot \|w_2\|}, \quad \text{... (4.5)}$$

then the singular problem (1.1) at least two C^1 [0, 1] positive solution $x_1(t)$ and $x_2(t)$ such that (4.4).

Proof of Theorem 4.1 : For each fixed $x(t) \in C[0, 1], x(t) \geq 0$, let $c_1 > 0$ be a constant such that $c_1 \|x\| \leq N$, $1/c_1 \geq M$. From (1.7), (1.8), (1.9) and (1.10), we obtain

$$f(t, x(t)) \leq f(t, y(t)) \text{ for } x(t) \leq y(t), t \in (0, 1), \quad \text{... (4.6)}$$

$$\begin{aligned} f(t, x(t)) &\leq (1/c_1)^{\mu_1} f_1(t, c_1 x(t)) + (1/c_1)^{\mu_2} f_2(t, c_1 x(t)) \\ &\leq (c_1)^{\lambda_1 - \mu_1} \|x\|^{\lambda_1} f_1(t, 1) + (c_1)^{\lambda_2 - \mu_2} \|x\|^{\lambda_2} f_2(t, 1) \quad \text{... (4.7)} \\ &\leq c_1^{-\mu_1} f_1(t, 1) + c_1^{-\mu_2} f_2(t, 1), t \in (0, 1). \end{aligned}$$

By means of (4.7) : if (4.1)-(4.3) hold, then the problem (1.1) is equivalent to the following integral equation

$$x(t) = Ax(t) = \int_0^1 G(t, , s) f(s, x(s))\, ds, \qquad \text{... (4.8)}$$

i.e., $A : X \to X$ has a fixed point, where $G(t, s)$ is given by (2.6).

Since $f_1(t, 1) > 0, f_2(t, 1) > 0, t \in (0, 1)$, there exist $0 < \alpha < \beta < 1$ such that

$$\min_{t \in J_0} f_1(t, 1) = \tau_1 > 0, \ \min_{t \in J_0} f_2(t, 1) = \tau_2 > 0, \qquad \text{... (4.9)}$$

where $\quad J_0 = [\alpha, \beta]$.

Let $P = \left\{ x \mid x \in C[0, 1], x(t) \geq 0, \ \min_{t \in J_0} x(t) \geq \varepsilon_0 \| x \| \right\}$, where ε_0 is given by (2.9). Then P is cone of C [0, 1]. In view of the proof of Theorem 3.1, we get that $A : P \to P$ is completely continuous.

For $x(t) \in P, \| x \| = 1$, then $0 \leq x(t) \leq \| x \| = 1$. From (4.1) and (4.6), we obtain

$$| Ax(t) | \leq \frac{1}{\omega} \int_0^1 e_1(s)\, e_2(s) f(s, 1)\, ds \leq \frac{\| w_1 \| \cdot \| w_2 \|}{\omega} \int_0^1 s(1-s) f(s, 1)\, ds$$

$$\leq \| x \|, \text{ for } x \in P, \| x \| = 1, t \in J. \qquad \text{... (4.10)}$$

Now, we prove $Ax \neq x, \ \forall\, x \in P, \| x \| = 1$. ... (4.11)

In fact, if there is $x^* \in P, \| x^* \| = 1 = x^*(t_0), t_0 \in (0, 1)$ such that $Ax^* = x^*$, then from (4.10),

$$1 = x^*(t_0) = \| Ax^* \| = | Ax^*(t_0) | \leq \frac{\| w_1 \| \cdot \| w_2 \|}{\omega} \int_0^1 s(1-s) f(s, 1) ds < 1,$$

which is contradiction.

From (2.7) and (2.8), we have

$$Ax(t) \geq \int_\alpha^\beta G(t, s) f(s, x(s))\, ds \geq \tau \left[\int_\alpha^\beta f_1(s, x(s))\, ds + \int_\alpha^\beta f_2(s, x(s)) ds \right],$$

$$\forall\, x \in P, t \in J_0. \qquad \text{... (4.12)}$$

Let $c_2 > 0$ be a constant such that $c_2 \varepsilon_0 \geq M, 1/c_2 \leq N$. For $x(t) \in P, \| x \| > 1$, we have $c_2 x(t) \geq M, t \in J_0$. From (1.9) and (1.10), we have

$$f_2(t, x(t)) \geq c_2^{\lambda_2 - \mu_2} x^{\lambda_2}(t) f_2(t, 1), \ x \in P, \| x \| > 1, t \in J_0. \qquad \text{... (4.13)}$$

Let $c_3 > 0$ be a constant such that $c_3 \leq N, 1/c_3 \geq M$. For $x(t) \in P, \| x \| < 1$, we have $c_3 x(t) \leq N$ for $t \in J$. From (1.7) and (1.8), we have

$$f_1(t, x(t)) \geq c_3^{\mu_1 - \lambda_1} x^{\mu_1}(t) f_1(t, 1), x \in P, \|x\| < 1, t \in (0, 1). \quad \ldots (4.14)$$

Choosing $r = \min \left\{ \frac{1}{2}, \left[\tau \tau_1 c_3^{\mu_1 - \lambda_1} \varepsilon_0^{\mu_1} (\beta - \alpha) \right]^{\frac{1}{1-\mu_1}} \right\} < 1,$

from (4.12) and (4.14), we have

$$\|Ax\| \geq \tau \int_\alpha^\beta f_1(s, x(s))\, ds \geq \tau \tau_1 c_3^{\mu_1 - \lambda_1} \varepsilon_0^{\mu_1} (\beta - \alpha) \|x\|^{\mu_1} \quad \ldots (4.15)$$

$$\geq r = \|x\|, \ \forall \ x \in P, \|x\| = r.$$

Choosing

$$R = \min \left\{ 2, \left[\tau \tau_2 c_2^{\lambda_2 - \mu_2} \varepsilon_0^{\lambda_2} (\beta - \alpha) \right]^{\frac{1}{1-\lambda_2}} \right\} > 1,$$

from (4.12) and (4.13), we obtain

$$\|Ax\| \geq \tau \int_\alpha^\beta f_2(s, x(s)) ds \geq \tau \tau_2 c_2^{\lambda_2 - \mu_2} \varepsilon_0^{\lambda_2} (\beta - \alpha) \|x\|^{\lambda_2}$$

$$\geq R = \|x\|, \ \forall \ x \in P, \|x\| = R. \quad \ldots (4.16)$$

(4.10), (4.11), (4.15), (4.16) and Lemma 2.3 imply, A has at least two fixed points $x_1(t), x_2(t) \in P$ such that

$$r < \|x_1\| < 1 < \|x_2\| < R/. \quad \ldots (4.17)$$

By proof of Theorem 3.1, we have $x_1(t), x_2(t) \in C[0, 1] \cap C^2(0, 1)$ are positive solutions of the singular (1.1). The proof is complete.

PROOF OF THEOREM 4.2 : is from the proof of Theorem 4.1 and Theorem 3.2.

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