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DIFFRACTION OF WATER WAVES BY A NEARLY VERTICAL WALL WITH A GAP

SUDESHNA BANERJEA AND C. C. KAR

Department of Mathematics, Jadavpur University, Calcutta 700 032, India

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The present paper is concerned with the study of scattering of water waves by a nearly vertical wall with a gap. Using a simplified perturbational analysis together with application of Green's integral theorem, first order correction to reflection and transmission coefficients are determined. Considering an explicit shape of the barrier some numerical calculations are performed.

Key Words : Diffraction; Water Waves; Nearly Vertical Wall; Green's Integrals Theorem; Perturbation Analysis; Reflection and Transmission Coefficients

1. INTRODUCTION

Water wave scattering by a nearly vertical barrier was first considered by Shaw[1] wherein the barrier was partially immersed in deep water. He used an integral equation formulation based on Green's integral theorem and employed a perturbational analysis to solve the integral equation upto first order. The first order correction to reflection and transmission coefficients were then obtained by him using some laborious calculations. Later, Mandal and Chakrabarty[2] and Mandal and Kundu[3] applied a simplified perturbation analysis to the governing differential equation to reduce it to two independent boundary value problems. One of the boundary value problems correspond to the problem of diffraction of water waves by vertical barrier, the solution of which is very well known in the literature. The first order correction to the reflection and transmission coefficients were then obtained in a simple manner from the second boundary value problem by a judicious application of Green's integral theorem.

In the present paper we have considered the problem of diffraction of water waves by a nearly vertical wall with a gap present in deep water. Employing the perturbation technique as in [2] and [3], the problem is reduced to two independent boundary value problems. One of the boundary value problems correspond to the problem of scattering of water waves by a vertical wall with a gap whose solution is well known in the literature (cf [4], [5]). Following [2] and [3], the first order corrections to the reflection and transmission coefficients were then obtained from the second boundary value problem by a judicious application of Green's integral theorem. Considering an explicit shape of the barrier, reflection and transmission coefficient are computed numerically, and compared with the results of corresponding problem involving vertical wall.

2. Statement and Formulation of the Problem

We consider a nearly vertical wall with a gap present in deep water, described by $x = \varepsilon (cy)$, $y \in B$ where $\varepsilon \ll 1$, $c(y)$ is a continuous function of y with and $B = (0, a) + (b, \infty)$. An incoming wave of frequency σ is incident upon the wall, it is partially reflected by the wall and partially transmitted through the gap in the wall. Under the assumption of linearised theory, the two dimensional motion is described by velocity potential $Re\ \{\phi (x, y)\ exp\ (-i\,\sigma t)\}$ where ϕ satisfies the following boundary value problem.

$$\nabla^2 \phi = 0 \text{ in } y \geq 0, \qquad \text{... (2.1)}$$

and

$$K\phi + \phi_y = 0 \text{ on } y = 0 \qquad \text{... (2.2)}$$

where $K = \frac{\sigma^2}{g}$, g being the acceleration due to gravity,

$$\phi_\eta = 0 \text{ on } x = \varepsilon\, c(y), y \in B \qquad \text{... (2.3)}$$

where η being the outward drawn normal,

$$r^{1/2}\, \nabla \phi \text{ is bounded as } r \to 0, \qquad \text{... (2.4)}$$

where r denotes the distance of any field point from sharp edges of the barrier,

$$\phi \sim \begin{cases} exp\,(-Ky + iKx) + R\, exp\,(-Ky - iKx) & as\ x \to -\infty, \\ T\, exp\,(-Ky + iKx) & as\ x \to \infty, \end{cases} \qquad \text{... (2.5)}$$

where R and T are the reflection and transmission coefficients respectively.

Assuming ε to be very small and neglecting $o(\varepsilon^2)$ terms, the boundary condition (2.3) on the nearly vertical barrier can be expressed approximately in the form (cf [2], [3])

$$\frac{\partial \phi}{\partial x}(\pm 0, y) = \varepsilon \frac{d}{dy}\left\{ c(y) \frac{\partial \phi}{\partial y}(\pm 0, y) \right\}, y \in B. \qquad \text{... (2.6)}$$

This form of boundary condition suggests (cf [2]) that we may assume the following perturbation expansion for the unknown $\phi(x, y), R$ and T in terms of ε as

$$\left.\begin{aligned} \phi(x, y) &= \phi_0 (x, y) + \varepsilon\, \phi_1 (x, y) + o(\varepsilon^2), \\ R &= R_0 + \varepsilon R_1 + o(\varepsilon^2), \\ T &= T_0 + \varepsilon T_1 + o(\varepsilon^2). \end{aligned}\right] \qquad \text{... (2.7)}$$

Next we shall proceed to find out R_0, R_1, T_0 and T_1. We substitute (2.7) in (2.1), (2.2), (2.4) to (2.6) and then equating the coefficient of identical powers of ε from both sides of the results thus derived, we find that ϕ_0 and ϕ_1 satisfy the following two independent boundary value problems.

BVP I

The function $\phi_0 (x, y)$ satisfies

$$\nabla^2 \phi_0 = 0 \text{ in } y \geq 0, \qquad \text{... (i)}$$

$$K\phi_0 + \phi_{0y} = 0 \text{ on } y = 0, \qquad \text{... (ii)}$$

$$\phi_{0x} = 0 \text{ on } x = 0 \;\; y \in B = (0, a) + (b, \infty), \qquad \text{... (iii)}$$

$$r^{1/2}\,\nabla\phi_0 \text{ is bounded as } r \to 0 \qquad \text{... (iv)}$$

with $r = \left\{ x^2 - (y - C)^2 \right\}^{\frac{1}{2}}$, where c is either a or b,

$$\nabla\phi_0 \text{ is bounded as } y \to \infty, \qquad \text{... (v)}$$

$$\phi_0 \sim \begin{cases} exp\,(-Ky + iKx) + R_0\, exp\,(-Ky - iKx) & as\; x \to -\infty, \\ T_0\, exp\,(-Ky + iKx) & as\; x \to \infty, \end{cases} \qquad \text{... (vi)}$$

BVP II

$\phi_1(x, y)$ satisfies

$$\nabla^2 \phi_1 = 0 \text{ in } y \geq 0, \qquad \text{... (vii)}$$

$$K\phi_1 + \phi_{1y} = 0 \text{ on } y = 0, \qquad \text{... (viii)}$$

$$\phi_{1x}(\pm 0, y) = \frac{d}{dy}\cdot\{c(y)\,\phi_{0y}(\pm 0, y)\},\;\; y \in B = (0, a)\; U\, (b, \infty), \qquad \text{... (ix)}$$

$$\nabla\phi_1 \text{ is bounded as } y \to \infty, \qquad \text{... (x)}$$

$$r^{1/2}\,\nabla\phi_1 \text{ is bounded as } r \to 0 \qquad \text{... (xi)}$$

with $r = \left\{ x^2 - (y - C)^2 \right\}^{\frac{1}{2}}$, where c is either a or b,

$$\phi_1 \sim \left.\begin{matrix} R_1 \exp\,(-Ky - iKx) & as\; x \to -\infty, \\ T_1 \exp\,(-Ky + iKx) & as\; x \to \infty. \end{matrix}\right] \qquad \text{... (xii)}$$

BVP I corresponds to the problem of scattering of water waves by a vertical wall with a gap. The solution of this problem is well known (cf [4], [5]) in the literature and is given by

$$= \phi_0(x, y) = \begin{cases} \exp\,(-Ky + iKx) + R_0 \exp\,(-Ky - iKx) - \int\limits_0^\infty A(k)\, M(k, y) \exp\,(kx)\, dk, & x < 0, \\ T_0 \exp\,(-Ky + iKx) + \int\limits_0^\infty A(k) M(k, y) \exp\,(-kx)\, dk, & x > 0, \end{cases} \qquad \text{... (2.8)}$$

where $M(k, y) = k \cos ky - K \sin ky$,

$$R_0 = 1 - T_0 = A_1 I,$$

$$A_1 = \frac{i}{(J + Ii)},$$

$$J = \frac{\exp(-Ka)}{K} + \delta \alpha_2(K) - \frac{2\alpha_2(K, F_1)}{\pi},$$

$$I = \delta\{\alpha_1(K) - \alpha_3(K)\} - \frac{2}{\pi}\{\alpha_1(K, F_1) - \alpha_3(K, F_1)\},$$

$$\delta = \frac{\left\{K^{-1}\exp(Ka) + \frac{2}{\pi}\alpha_2(-K, F_1)\right\}}{\alpha_2(-K)},$$

and $$\alpha_i(K) \equiv \alpha_i(K, 1),\ \alpha_i(K, F_1) = \int_{t_i} \frac{uF_1(a, b, u)}{R_0(u)} \exp(-Ku)\, du$$

with $$R_0(u) = |u^2 - a^2|^{1/2} |u^2 - b^2|^{1/2},$$

$$t_i = \begin{cases} (-a, a), & i = 1 \\ (a, b), & i = 2 \\ (b, \infty) & i = 3 \end{cases}$$

$$F_1(a, b, u) = \int_0^a \frac{R_0(v)}{v^2 - u^2}\, dv,$$

$$-A(k) = \frac{2}{\pi} \frac{A_1}{k(k^2 + K^2)} \left\{ -\sin ka + k \int_a^b \frac{uS(t)}{R_0(u)} \cos ku\, du \right\} \quad \ldots (2.9)$$

and $$S(u) = \left\{ \delta - \frac{2}{\pi} F_1(a, b, u) \right\}.$$

The BVP II is a special type of boundary value problem which involves two different boundary conditions (*cf* (*ix*)) on the two sides of the barrier $x = 0$; $y \in B$. We shall now proceed to find out first order corrections to the reflection and transmission coefficients R_1 and T_1 respectively without solving BVP II.

To find R_1 we apply Green's integral theorem to the functions ϕ_0 and ϕ_1 in the region bounded by the lines

$$x = X, 0 \le y \le Y; y = 0, 0 < x \le X; x = 0^+, 0 \le y \le a;$$

$$x = 0^-, 0 \le y \le a; y = 0, -X \le x < 0; x = -X, 0 \le y \le Y;$$

$$y = Y, -X \le x < 0; x = 0^-, b \le y \le Y; x = 0^+, b \le y \le Y;$$

$$y = Y, 0 < x \le X$$

Making $X, Y \to \infty$ we obtain

$$iR_1 = \int_0^a \{\phi_0(0^+, y)\, \phi_{1x}(0^+, y) - \phi_0(0^-, y)\, \phi_{1x}(0^-, y)\}\, dy$$

$$+ \int_b^\infty \{\phi_0(0^+ y)\, \phi_{1x}(0^+, y) - \phi_0(0^-, y)\, \phi_{1x}(0^-, y)\}\, dy.$$

Using (*ix*) of BVP II together with A1 and A9 (see appendix) we have

$$iR_1 = 4\left[-K^2 \int_0^a c(y) \exp(-2Ky) \int_a^y \frac{A_1\, t\, S(t) \exp(Kt)}{R(t)}\, dt\, dy \right.$$

$$+ K \int_0^a \frac{A_1\, c(y)\, y\, S(y) \exp(-Ky)}{R(y)}\, dy$$

$$+ K^2 \int_b^\infty c(y) \exp(-2Ky) \int_b^y \frac{A_1\, t\, S(t) \exp(Kt)}{R(t)}\, dt\, dy$$

$$\left. - K \int_b^\infty \frac{A_1\, c(y)\, S(y) \exp(-Ky)}{R(y)}\, dy \right] \quad \text{... (2.10)}$$

This result gives rise to R_1 = 4P in the notation of Shaw[1]. To find T_1, we apply Green's integral theorem to $\chi_0(x, y) = \phi_0(-x, y)$ and $\phi_1(x, y)$ in the region defined by described *a priori*, we have

$$iT_1 = -\int_0^a \{\phi_0(0^+, y)\, \phi_{1x}(0^-, y) - \phi_0(0^-, y)\, \phi_{1x}(0^+, y)\}\, dy$$

$$- \int_b^\infty \{\phi_0(0^+, y)\, \phi_{1x}(0^-, y) - \phi_0(0^-, y)\, \phi_{1x}(0^+, y)\}\, dy.$$

Using (*ix*) and assuming $c(y) \to 0$ as $y \to 0$ or ∞,

$$iT_1 = -\{\phi_0(0^+, a)\, \phi_{0y}(0^-, a) - \phi_0(0^-, a)\, \phi_{0y}(0^+, a)\}\, c(a)$$

$$+ \{\phi_0(0^+, b)\, \phi_{0y}(0^-, b) - \phi_0(0^-, b)\, \phi_{0y}(0^+, b)\}\, c(b)$$

using (A2) we have $iT_1 = 0$ ... (2.11)

Similar conclusion was also obtained by [1], [2] and [3].

3. DISCUSSION

Now we consider an explicit shape of the barrier viz, $x = \varepsilon c(y)$, where $c(y) = \left(1 - \frac{a}{b}\right) y e^{-\lambda y}$, a/b = 0.01, $a\lambda$ = 5.0, 10.0 and presented $|R_0|$ and $|R| = |R_0 + \varepsilon R_1|$ in tabular form for ε = 0.03 and 0.003 and for various values of the wave number Ka.

TABLE I

$c(y) = \left(1 - \frac{a}{b}\right) y \exp(-\lambda y), \ \frac{a}{b} = 0.01$

		$\varepsilon = 0.03$		$\varepsilon = 0.003$	
		$a\lambda = 5$	$a\lambda = 10$	$a\lambda = 5$	$a\lambda = 10$
Ka	$\lvert R_0 \rvert$	$\lvert R \rvert$	$\lvert R \rvert$	$\lvert R \rvert$	$\lvert R \rvert$
0.1	0.939819	0.939819	0.939819	0.939819	0.939819
0.5	0.992884	0.992892	0.992885	0.992884	0.992884
1.0	0.996742	0.996782	0.996749	0.996743	0.996743
1.5	0.998277	0.998363	0.998285	0.998278	0.998277
2.0	0.999196	0.999310	0.999200	0.999197	0.999196

From Table I it is observed that for a particular ε and $a\lambda$, as the wave number increases, $|R_0|$ and $|R|$ increases. Also $|R|$ differs from $|R_0|$ as Ka increases for a particular ε and $a\lambda$. For ε = 0.03 and for large Ka, this difference occurs at fourth or fifth decimal places while for ε = 0.003 this difference is not appreciable. This shows that the nearly vertical barrier in this case has some effect on the reflection coefficient for ε = 0.03. However, as ε decreases, the effect is not much significant.

APPENDIX

To obtain R_1 and T_1 we require to calculate $\phi_0(+0, y) \pm \phi_0(-0, y)$ on the barrier and in the gap. From [4] we have

$$\phi_0(+0, y) - \phi_0(-0, y) = \begin{cases} A_1 \exp(-Ky) \int_a^y \frac{2t\, S(t) \exp(Kt)}{R_0(t)} dt, & 0 < y < a, \\ -A_1 \exp(-Ky) \int_b^y \frac{2t S(t) \exp(Kt)}{R_0(t)} dt, & b < y < \infty, \\ 0, & a < y < b \end{cases} \quad \text{(A1)}$$

From (2.8)

$$K\phi_0(\pm 0, y) + \phi_{0y}(\pm 0, y) = \mp \int_0^\infty A(k)(k^2 + K^2) \sin ky\, dk.$$

Noting (2.9), on integration this gives

$$\phi_0(\pm 0, y) = \begin{cases} c_1^{\pm} \exp(-Ky) \pm \exp(-Ky) \int_a^y \frac{A_1\, t\, S(t) \exp K(t)}{R_0(t)} dt, & 0 < y < a, \\ c_2^{\pm} \exp(-Ky) \pm \exp(-Ky) \int_b^y \frac{A_1\, t\, S(t) \exp(Kt)}{R_0(t)} dt, & b < y < \infty, \\ c_3^{\pm} \exp(-Ky), & a < y < b \end{cases} \quad \text{... (A2)}$$

In view of (A1)

$$c_3{}^{+} = c_3{}^{-} = c_3 \text{ (say)} \quad \text{... (A3)}$$

Using (2.8) in (A1) for $a < y < b$, we have

$$R_0 \exp(-Ky) = \int_0^\infty A(k)\, M(k, y)\, dk, \; a < y < b. \quad \text{... (A4)}$$

Again from (2.8)

$$\phi_0(+0, y) = T_0 \exp(-Ky) + \int_0^\infty A(k)\, M(k, y)\, dk \quad \text{... (A5)}$$

Using (A4) in (A5) and noting that $T_0 + R_0 = 1$,

$$\phi_0(+0, y) = \exp(-Ky), \; a < y < b. \quad \text{... (A6)}$$

Comparing (A6) with (A2) we have

$$c_3 = 1$$

Computing $\phi_0(+0, y) - \phi_0(-0, y)$ directly from (A2) and comparing this with (A1) we have

$$\left. \begin{array}{l} c_1{}^{+} = c_1{}^{-} \\ c_2{}^{+} = c_2{}^{-} \end{array} \right] \quad \text{... (A8)}$$

Again from (2.8)

$$\phi_0(+0, y) + \phi_0(-0, y) = 2 \exp(-Ky), \; 0 < y < \infty. \quad \text{... (A9)}$$

Using (A2), (A9) and (A8) we have

$$c_1{}^{+} = c_2{}^{+} = 1. \quad \text{... (A10)}$$

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