Indian J. pure appl. Math., 32(7) : 1041-1050, July 2001

SOME RESULTS ON ADMISSIBLE ALGEBROID SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS*

GAO LINGYUN

Department of Mathematics, Jinan University, Guangzhou 510 632, *P. R. China*

(*Received* 22 *December* 1999; *after revision* 31 *March* 2000; *accepted* 17 *October* 2000)

In this paper we investigate the existence problem of admissible algebroid solutions of two classes of complex differential equations and obtain some results.

Key Words : Algebroid Functions; Admissible Solution; Complex Differential Equations

1. INTRODUCTION

Here we use the standard notation of the Nevanlinna theory of meromorphic functions or algebroid functions.[1]

In 1934, Yosida[2] proved the following :

Theorem A — *If the differential equation with rational coefficients*

$$(w')^n = \frac{\sum\limits_{i=0}^{p} a_i(z) w^i}{\sum\limits_{j=0}^{q} b_j(z)\, w^j},$$

admits at least one transcendental v valued algebroid solution, then

$$q \leq 2n\, v,\ q \leq 2n\,(v-1),$$

holds.

This theorem was extended by several authors[3-10]. In this paper, we shall consider the following two types of differential equation :

$$\Omega\,(z, w) = \frac{P(z, w)}{Q(z, w)} \qquad \text{... (1.1)}$$

*Project supported by NSF of P.R. China

and $$\frac{\Omega(z,w)}{(w-a)^{\lambda}}=\frac{P(z,w)}{Q(z,w)}, \qquad \text{... (1.2)}$$

where $$\Omega(z,w)=\sum_{(i)\in I} a_i(z)\, w^{io}\,(w')^{i_1}\ldots(w^{(n)})^{i_n}\,(n\geq 1),$$

are differential polynomials with meromorphic coefficients $\{a_{(i)}(z)\}$, I being a finite sets of multi-indices $I = (i_0, i_1, ..., i_n)$ for $a_{(i)}\neq 0$ (i_l: non-negative integers), $P(z\,w)=\sum_{i=0}^{p} a_i(z)\,w^i$, $Q(z,w) = \sum_{j=0}^{q} b_j(z)\,w^j$, and the coefficients $\{a_i(z)\}, \{b_j(z)\}$ are meromorphic functions, $a_p\,b_q\neq 0$, a is a nonzero constant.

For a differential monomial $a_{(i)}\,w^{i_0}\,w')^{i_1}\ldots(w^{(n)})^{i_n}$ in $\Omega(z,w)$, $\lambda_{(i)}; u_{(i)}, \Delta_{(i)}, \sigma_{(i)}, l_{(i)}$ are denoted by $\lambda_{(i)}=i_0+i_1+\ldots+i_n$, $u_{(i)}+i_1+2i_2+\ldots+n\,i_n$, $\Delta_{(i)}=i_0+2i_1+\ldots+(n+1)\,i_n$, $\sigma_{(i)} = i_1 + 3i_2+\ldots+(2n-1)\,i_n$, and $l_{(i)}=i_2+2i_3+\ldots+(n-1)\,i_n$, respectively.

For the differential polynomials $\Omega(z,w)$, we adopt the notation :

$$\lambda = \max\,\{\lambda_{(i)}\},\ u = \max\,\{u_{(i)}\},\ \Delta = \max\,\{\Delta_{(i)}\},\ \sigma = \max\,\{\sigma_{(i)}\}\ l = \min\,\{l_{(i)}\}.$$

In addition, put

$$\theta(w,\infty)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,w)}{T(r,w)},\ \xi_x=\limsup_{r\to\infty}\frac{N_x(r,w)}{T(r,w)},$$

$$\xi_b(\infty)=\limsup_{r\to\infty}\frac{N_b(r,w)}{T(r,w)},\ (r\notin I).$$

Gackstatter and Laine[8], Yuzan and Xiao Xiuzhi[9], Toda and Kato[10] have considered the differential equation (1.1). They have shown that if eq. (1.1) admits at least an admissible solution with ν branches, then

(A) : $q\leq 4u\,(\nu-1), p\leq \Delta+4u\,(\nu-1)$;[8]

(B) : $q\leq 2\,\sigma(\nu-1), p\leq q+\lambda+u\,\nu\,(1-\theta(w,\infty))$;[9]

(C) : $\max\,\{p, q+\Delta\}\ \leq \Delta+\sigma\,\xi_x$, and

$p\leq\ \min\,\{q+\lambda+u\,(1-\theta(w,\infty)+\xi_b(\infty)), \Delta+\sigma\,\xi_x\}$,[10]

respectively.

At first, we give two definition as follows :

Definition 1.1 — Put

$$S_1(r, w) = \Sigma\ T(r, a_{(i)}(z)) + \Sigma\ T(r, a_i(z)) + \Sigma\ T(r, b_j(z)).$$

If an algebroid solution $w(z)$ of (1.1) ((1.2)) satisfies

$$\limsup_{r \to \infty} \frac{s_1(r, w)}{T(r, w)} = 0,$$

outside a possible exception set I with a finite linear measure, we call $w(z)$ an admissible solution.

Let $w(z)$ be a ν-value algebroid function and z_0 be a pole of $w(z)$. Then in a neighbourhood of a, we have the following expansion of w :

$$w(z) = (z - z_0)^{\frac{-\tau_i}{\beta_i}}, S((z - z_0)^{\frac{1}{\beta_i}}),$$

where $i = 1, 2, ..., \mu(z_0)\ (\leq \nu)$, $\tau_i \geq 1, \beta_i \geq 1, \Sigma\ \beta_i = \nu$ and $S(t)$ is a regular power of t such that $S(0) \neq 0$.

Definition 1.2[10] : Put

$$n_b(r, w) = \sum_{|z_0| \leq r} \sum_{i=1}^{\nu(z_0)} (\beta_i - 1)$$

$$\nu N_b(r, w) = \int_0^r \frac{n_b(t, w) - nb(r, w))}{t} dt + n_b(0, w) \log r.$$

Obviously, we have

$$N_b(r, w) \leq (\nu - 1)\overline{N}(r, w).$$

2. Main Results

Theorem 2.1 — *If the differential equation (1.1) admits an admissible algebroid solution $w = w(z)$ with ν branches, then*

$$q \leq \sigma\xi_x, p \leq q + \lambda + (\Delta - \lambda)(1 - \theta(w, \infty)) + u\,\xi_b(\infty).$$

Corollary 2.1 — If the differential equation (1.1) admits an admissible algebroid solution $w = w(z)$ with ν branches, then

$$q \leq 2\sigma(\nu - 1), p \leq q + \lambda + u(1 - \theta(w, \infty)) + u\,\xi_b(\infty).$$

Corollary 2.2 — If the differential equation (1.1) admits an admissible algebroid solution $w = w(z)$ with ν branches, then

$$q \leq 2\sigma(\nu - 1), p \leq q + \lambda + u\,\nu(1 - \theta(w, \infty)).$$

Theorem *2.2 — If the differential equation (1.2) admits an admissible algebroid solution $w = w(z)$ with v branches, then*

$$\max\{p, q\} \le \lambda + \sigma \xi_x, \; p \le q + \lambda + (\Delta - \lambda)(1 - \theta(w(\infty)).$$

Theorem *2.3 — If the differential equation (1.2) admits an admissible algebroid solution $w = w(z)$ with v branches, then*

$$\max\{p, q\} \le \lambda + (\Delta - \lambda)(1 - \theta(w, \infty)) + u\, \xi_b(\infty).$$

Theorem *2.4 — If the differential equation (1.2) admits an admissible algebroid solution $w = w(z)$ with v branches, then*

$$p \le \lambda + (\Delta - \lambda)(1 - \theta(w, \infty)) + \sigma \xi_x - l\, \xi_b(\infty), \; q \le \lambda.$$

3. Some Lemmas

Lemma 3.1 — Let $R(z, w) = \dfrac{\sum_{i=0}^{p} a_i(z)\, w^i}{\sum_{j=0}^{q} b_j(z) w^j}$ be an irreducible rational function in $w(z)$ with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$. If $w(z)$ is an algebroid function, then

$$T(r, R(z, w)) = \max\{p, q\}\; T(r, w) + O\left\{\Sigma\; T(r, a_i) + \Sigma\, T(r, b_j)\right\}.$$

Proof : See [4]

Lemma 3.2 — Let $\Omega(z, w), P(z, w), Q(z, w)$ be as in the eq. (1.1), $w(z)$ be an algebroid function and such that $Q(z, w)\, \Omega(z, w) = P(z, w)$. Then

$$m(r, \Omega(z, w))$$

$$\le O\left\{\sum_{(i)} m(r, a_{(i)}) + \sum_{i=0}^{p} m(r, a_i) + \sum_{j=0}^{q}\left[m(r, b_j) + m\left(r, \frac{1}{b_j}\right)\right] + \sum_{l=1}^{n} m\left(r, \frac{w^{(l)}}{w}\right)\right\}$$

for $\quad q \ge p.$

Proof : See [1].

Lemma 3.3 — Let $\Omega(z, w), P(z, w), Q(z, w)$ be as in the eq. (1.1), $w(z)$ be an algebroid function such that $Q(z, w)\dfrac{\Omega(z, w)}{(w-a)^\lambda} = P(z, w)$. Then

$$T\left(r, \frac{\Omega(z, w)}{(w-a)^\lambda}\right) = \lambda T(r, w) + \sigma N_x(r, w)$$

$$+O\left\{\sum_{(i)} T(r, a_{(i)}) + \sum_{i=0}^{p} T(r, a_i) + \sum_{j=0}^{q} T(r, b_j)\right\} + S(r, w), \quad \ldots (3.1)$$

where $S(r, w) = o\,\{\ln \cdot rT(r, w))\}$

PROOF : Since

$$\frac{\Omega(z, w)}{(w-a)^{\lambda}} = \sum_{(i)} a_{(i)} \frac{1}{(w-a)^{\lambda-(i_0+\ldots+i_n)}} \left(\frac{w}{w-a}\right)^{i_0} \left(\frac{w'}{w-a}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w-a}\right)^{i_n},$$

we have $$\left|\frac{\Omega(z, w)}{(w-a)^{\lambda}}\right| \le \sum_{(i)} |a_{(i)}| \left|\frac{1}{(w-a)^{\lambda-(i_0+\ldots+i_n)}}\right| \left|\frac{w}{w-a}\right|^{i_0} \left|\frac{w'}{w-a}\right|^{i_1} \cdots \left|\frac{w^{(n)}}{w-a}\right|^{i_n}$$

But $$\left|\frac{w}{w-a}\right| = \left|\frac{w-a+a}{w-a}\right| = \left|1 + \frac{a}{w-a}\right| \le 1 + |a| \left(\left|\frac{1}{w-1}\right|\right)^{+}$$

$$\le (1 + |a|)\left(\frac{1}{|w-a|}\right)^{+}.$$

Thus $$\left|\frac{\Omega(z, w)}{(w-a)^{\lambda}}\right| \le C \sum_{(i)} |a_{(i)}| \left|\frac{(w-a)'}{w-a}\right|^{i_1} \cdots \left|\frac{(w-a)^{(n)}}{w-a}\right|^{i_n} \left(\left|\frac{1}{w-a}\right|^{+}\right)^{\lambda}.$$

$$m\left(r, \frac{\Omega(z, w)}{(w-a)^{\lambda}}\right) \le \lambda\, n\left(r, \frac{1}{w-a}\right)$$

$$+\,o\left\{\sum_{(i)} m(r, a_{(i)}) + \sum_{l=1}^{n} m\left(r, \frac{(w-a)^{(l)}}{w-a}\right) + 1\right\}. \quad \ldots (3.2)$$

Now we consider $N\left(r, \frac{\Omega(z, w)}{(w-a)^{\lambda}}\right)$.

The poles of $\frac{\Omega(z, w)}{(w-a)^{\lambda}}$ may arise from one of the following cases :

(*i*) The poles of the coefficients $\{a_{(i)}\}$ of $\Omega(z, w)$. It contribute

$$O\left\{\sum_{(i)} N(r, a_{(i)})\right\} \text{ to } n\left(r, \frac{\Omega(z, w)}{(w-a)^{\lambda}}\right).$$

(*ii*) The zeros of $w - a$. It contribute $\lambda N\left(r, \frac{1}{w-a}\right)$ to $N\left(r, \frac{\Omega(z, w)}{(w-a)^{\lambda}}\right)$.

(*iii*) The poles of w. Let z_0 be a pole of the function $w(z)$, we denote by $\tau(a, w)$ its multiple order. We have

$$w(z) = (z - z_0)^{\frac{-\tau(w,\infty)}{s}} w_1(z), \; w_1(z_0) \neq 0, \infty, \; 1 \leq s \leq \nu.$$

$$\tau(P(z,w),\infty) \leq p\,\tau(w,\infty) + \nu \sum_i \tau(\infty, a_i)$$

$$\tau(Q(z,w),\infty) \geq q\,\tau(w,\infty) - q\,\nu \sum_j [\tau(\infty, b_j) + \tau(0, b_j)]$$

$$\tau\left(\frac{\Omega z, w)}{(w-a)^{\lambda}}, \infty\right) = \tau(P(z,w),\infty) - \tau(Q(z,w),\infty)$$

$$\leq (p-q)\,\tau(w,\infty) + \nu \sum_i \tau(\infty, a_i) + q\,\nu \sum_j [\tau(\infty, b_j) + \tau(0, b_j)]$$

$$\leq \nu \sum_i \tau(\infty, a_i) + q\,\nu \sum_j [\tau(\infty, b_j) + \tau(0, b_j)].$$

It contribute

$$O\left\{\sum_{i=0}^{p}\left[N(r, a_i) + N\left(r, \frac{1}{a_i}\right)\right] + \sum_{j=0}^{q}\left[N(r, b_j) + N\left(r, \frac{1}{b_j}\right)\right]\right\} \text{ to } N\left(r, \frac{\Omega(z,w)}{(w-a)^{\lambda}}\right).$$

(*iv*) The poles of $w^{(\alpha)}$ but not the cases (i)-(iii). In fact these points are some branches points of $w(z)$, we suppose that s branches of $w(z)$ take $a \neq \infty$ at z_0. Clearly, we have

$$w^{(\alpha)}(z) = (z - z_0)^{\frac{\tau(w,a) - s\alpha}{s}} w_1(z), \; w_1(z_0) \neq 0, \infty, \; 1 \leq s \leq \nu.$$

Thus $\quad \tau(w^{(\alpha)}(z), \infty) = s\alpha - \tau(w, a) \leq (2\alpha - 1)(s - 1).$

Hence $\quad \tau\left(\frac{\Omega(z,w)}{(w-a)^{\lambda}}, \infty\right) \leq \sigma(\nu - 1).$

It contribute $\quad \sigma N_x(r, w)$ to $N\left(r, \frac{\Omega(z,w)}{(w-a)^{\lambda}}\right)$ at most.

Combining the cases (i)-(iv), we get

$$N\left(r, \frac{\Omega(z,w)}{(w-a)^{\lambda}}\right) \leq \lambda N\left(r, \frac{1}{w-a}\right) + \sigma N_x(r, w) + O\left\{\sum_{(i)} N(r, a_{(i)})\right.$$

$$\left. + \sum_{i=0}^{p}\left[N(r, a_i) + N\left(r, \frac{1}{a_i}\right)\right] + \sum_{j=0}^{q}\left[N(r, b_j) + N\left(r, \frac{1}{b_j}\right)\right]\right\}. \quad \ldots (3.3)$$

By (3.2) and (3.3), we get the inequality (3.1).

Lemma 3.4 — Let $\Omega(z, w)$ be as in the equation (1.1), $w(z)$ be an admissible algebroid solution of the eq. (1.2). Then

$$T\left(r\frac{\Omega(z, w)}{(w-a)^{\lambda}} \right) = \lambda\, T(r, w) + (\Delta - \lambda)\, \overline{N}(r, w) + uN_b(r, w) + S_1(r, w).$$

PROOF : See [6]

4. PROOF OF THEOREM 2.1

As in the proof of Lemma 3.3 we get

$$q \leq \sigma\, \xi_x, (r \notin I).$$

Now we prove the second inequality.

We rewrite (1.2) as :

$$Q(z, w)\, \Omega(z, w) = P(z, w). \qquad \ldots (4.1)$$

By (4.1), we know that the poles of $Q(z, w)\, \Omega(z, w)$ may arise from one of the following cases :

(*i*) The poles of the coefficients of $Q(z, w)\, \Omega(z, w)$.

(*ii*) The poles of $w(z)$.

In case (i), we get $\tau(\infty, Q(z, w)\, \Omega(z, w)) \leq \nu \sum_{(i)} \tau(\infty, a_{(i)}) + \nu \sum_{j} \tau(\infty, b_j)$.

In the case (ii), let z_0 be a pole of $w(z)$, we suppose that s branches of $w(z)$ take ∞ at z_0. Clearly, we have $w^{(\alpha)}(z) = (z - z_0)^{\frac{-[\tau(w, a) + s\,\alpha]}{s}} w_1(z), w_1(z_0) \neq 0, \infty, 1 \leq s \leq \nu$.

Thus, we get

$$\tau(w^{i_0} \ldots (w^{(n)})^{(i_n)}, \infty) \leq \left(\sum_{\alpha=0}^{n} i_\alpha \right) \tau(w^{(\alpha)}, \infty) = \sum_{\alpha=0}^{n} i_\alpha (\tau(w, \infty) + \alpha\, s)$$

$$= \sum_{\alpha=0}^{n} i_\alpha\, \tau(w, \infty) + \sum_{\alpha=1}^{n} \alpha\, i_\alpha (s-1) + \sum_{\alpha=1}^{n} \alpha\, i_\alpha$$

$$= \sum_{\alpha=0}^{n} i_\alpha\, \tau(w, \infty) + \sum_{\alpha=1}^{n} \alpha\, i_\alpha (s-1) + \sum_{\alpha=0}^{n} (\alpha+1)\, i_\alpha - \sum_{\alpha=0}^{n} i_\alpha$$

$$\leq \sum_{\alpha=0}^{n} i_\alpha (\tau(w, \infty) - 1) + \sum_{\alpha=1}^{n} \alpha\, i_\alpha (\nu - 1) + \sum_{\alpha=0}^{n} (\alpha+1)\, i_\alpha.$$

Hence, we have $\tau(\infty, Q(z, w)\, \Omega(z, w)) \leq q\, \tau(w, \infty) + \lambda\, (\tau(w, \infty) - 1) + \Delta$.

Combining the case (i) and (ii), we have

$$N(r, Q(z, w)\, \Omega(z, w)) \leq qN(r, w) + \lambda\, (N(r, w) - \overline{N}(r, w)) + \Delta\, \overline{N}(r, w)$$

$$+ \; uN_b(r, w) + O\left\{ \sum_{(i)} N(r, a_{(i)}) + \sum_{j} N(r, b_j) \right\}. \qquad \ldots (4.2)$$

By Lemma 3.2, we get

$$m(r, Q(z, w)\,\Omega(z, w)) \le m(r, Q) + m(r, \Omega)$$

$$\le (q + \lambda)\, m(r, w) + O\left\{ \sum_{(i)} m(r, a_{(i)} + \sum_{i=0}^{p} m(r, a_i) \right.$$

$$\left. + \sum_{j=0}^{q} \left[m(r, b_j) + m\left(r, \frac{1}{b_j}\right)\right] + \sum_{\alpha=1}^{n} m\left(r, \frac{w^{(\alpha)}}{w}\right)\right]. \qquad \ldots (4.3)$$

By (4.2) and (4.3), we have

$$T(r, Q(z, w)\,\Omega(z, w)) \le (q + \lambda)\, T(r, w) + (\Delta - \lambda)\, \overline{N}(r, w) + uN_b(r, w) + S_1(r, w).$$

According to

$$T(r, Q(z, w)\,\Omega(z, w)) = T(r, P(z, w)) = pT(r, w) + O\left\{ \sum_{i=0}^{p} T(r, a_i) \right\},$$

it follows that $pT(r, w) \le (q + \lambda)\, T(r, w) + (\Delta - \lambda)\, \overline{N}(r, w) + uN_b(r, w) + S_1(r, w)$.

Take $r \to \infty$, outside a possible exception set I with a finite linear measure, we get

$$p \le q + \lambda + (\Delta - \lambda 0)\,(1 - \theta(w, \infty)) + u\, \xi_b(\infty).$$

This completes the proof of Theorem 2.1.

PROOF OF COROLLARY 2.1 : Because $\xi_x \le 2(\nu - 1)$, $\Delta - \lambda \le u$, we have

$$q \le 2\,\sigma(\nu - 1),\; p \le q + \lambda + u\,(1 - \theta(w, \infty)) + u\, \xi_b(\infty).$$

PROOF OF COROLLARY 2.2 :

Since $\xi_x \le 2(\nu) - 1)$, $\Delta - \lambda \le u$ and

$$\xi_b(\infty) \le (\nu - 1)\,(1 - \theta(w, \infty))$$

we have $\quad q \le 2\sigma(\nu - 1),$

$$p \le q + \lambda + u(1 - \theta(w, \infty)) + u(\nu - 1)\,(1 - \theta(w, \infty))$$

$$= p + \lambda + u\nu(1 - \theta(w, \infty)).$$

5. Proof of Theorem 2.2

According to Lemma 3.3, we get $T\left(r, \frac{\Omega(z, w)}{(w-a)^{\lambda}}\right) = \lambda T(r, w) + \sigma N_x(r, w) + S_1(r, w)$.

But
$$T\left(r, \frac{P(z, w)}{Q(z, w)}\right) = \max\{p, q\}\ T(r, w) + O\left\{\sum_{i=0}^{p} T(r, a_i) + \sum_{j=0}^{q} T(r, b_j)\right\}.$$

Thus we have $\max\{p, q\} \le \lambda + \sigma \xi_x, (r \notin I)$.

Proceeding similarly as in q the proof of the second inequality of Theorem 2.1, we have

$$p \le q + (\Delta - \lambda)(1 - \theta(q, \infty)) + u\, \xi_b(\infty) \cdot (r \notin I)$$

6. Proof of Theorem 2.3

According to Lemma 3.4, we get

$$T\left(r, \frac{\Omega(z, w)}{(w-a)^{\lambda}}\right) = \lambda T(r, w) + (\Delta - \lambda)\overline{N}(r, w) + uN_b(r, w) + S_1(r, w).$$

But
$$T\left(r, \frac{P(z, w)}{Q(z, w)}\right) = \max\{p, q\}\ T(r, w) + O\left\{\sum_{i=0}^{p} T(r, a_i) + \sum_{j=0}^{q} t(r, b_j)\right\}.$$

Thus we have $\max\{p, q\}\ T(r, w) \le \lambda T(r, w)\ (\Delta - \lambda)\overline{N}(r, w) + u N_b(r, w) + S_1(r, w)$.

Take $r \to \infty$, outside a possible exception set I with a finite linear measure, then

$$\max\{p, q\} \le \lambda + (\Delta - \lambda)(1 - \theta(w, \infty)) + u\, \xi_b(\infty).$$

7. Proof of Theorem 2.4

Clearly, we have

$$m(r, \Omega) \le \lambda m(r, w) + \sum_{(i)} (r, a_{(i)} + \sum_{l=0}^{n} m\left(r, \frac{w^{(\alpha)}}{w}\right).$$

Now we consider $N(r, \Omega)$.

By Lemma 5 of the paper [6], we have

$$n(r, w^{(\alpha)}) \le n(r, w) + \alpha \bar{n}(r, w) + (2\alpha - 1)\, n_x(r, w) - (\alpha - 1)\, n_b(r, w).$$

Hence,

$$n(r, a_{(i)} w^{i_0} \ldots (w^{(n)})^{(i_n)} \le n(r, a_{(i)}) + \left(\sum_{\alpha=0}^{n} i_\alpha\right) n(r, w) + \left(\sum_{\alpha=1}^{n} \alpha\, i_\alpha\right) \bar{n}(r, w)$$

$$+\left(\sum_{\alpha=1}^{n}(2\alpha-1)\,i_\alpha\right)n_x(r,w)-\left(\sum_{\alpha=1}^{n}(\alpha-1)\,i_\alpha\right)n_b(r,w)$$

$$= n\,r,a_{(i)})+\left(\sum_{\alpha=0}^{n} i_\alpha\right)[\,n(r,w)-\overline{n}(r,w)]+\left(\sum_{\alpha=1}^{n}(\alpha+1)\,i_\alpha\right)\overline{n}(r,w)$$

$$+\left(\sum_{\alpha=1}^{n}(2\alpha-1)\,i_\alpha\right)n_x(r,w)-\left(\sum_{\alpha=1}^{n}(\alpha-1)\,i_\alpha\right)n_b(r,w).$$

It follows that

$$N(r,\Omega)\le\sum_{(i)}N(r,a_{(i)})+\lambda\,[N(r,w)-\overline{N}(r,w)]+\Delta\overline{N}(r,w)+\sigma N_x(r,w)-lN_b(r,w)$$

i.e.,

$$N(r,\Omega)\le\sum_{(i)}N(r,a_{(i)})+\lambda\,N(r,w)+(\Delta-\lambda)\,\overline{N}(r,w)+\sigma N_x(r,w)-lN_b(r,w),$$

which implies that

$$T(r,\Omega)\le\lambda\,T(r,w)+(\Delta-\lambda)\,\overline{N}(r,w)+\sigma N_x(r,w)-lN_b(r,w)+S_1(r,w).$$

In addition,

$$T(r,(w-a)^{\lambda})\le\lambda\,T(r,w)+O(1).$$

As in the proof of Theorem 2.1 of the paper [11], we get

$$p\le\lambda+(\Delta-\lambda)\,(1-\theta(w,\infty))+\sigma\,\xi_x-l\,\xi_b(\infty),\ q\le\lambda.\ \ (r\notin I)$$

This completes the proof of Theorem 2.4.

ACKNOWLEDGEMENT

The author thanks the referee for his helpful suggestions.

REFERENCES

1. He Yuzan and Xiao Xiuzhi, *Algebroid Functions and Ordinary Differential Equations*, Beijing Science Press, 1988.
2. K. Yosida, *Japan J. Math.*, **10** (1934) 199-208.
3. A. Z. Mokhon'ko, *Sib. Math. Z.*; **23** (1982) 80-88.
4. He Yuzan, *Acta Math. Sinica*, **24** (1981) 464-71.
5. Xiao Xiuzhi and He Yuzan, *Sci. China (Series A)*, **10** (1983) 1034-43.
6. Chen Tewei, *Chinese Quarterly J. Math.*, **6** 4 (1991) 45-51.
7. Gao Shian, *Math. Sp. Issue, J. South China Norma Univ.*, (1984) 74-81.
8. F. Gackstatter and I. Laine, *Ann. Polon. Math.*, **38** (1980) 259-87.
9. He Yuzan and Xiao Xiuzhi, *Contemporary Math.*, **25** (1983) 51-61.
10. N. Toda and M. Kato, *Proc. Japan Acad. Ser. A*, **61** (1985) 325-28.
11. Gao Lingyun, *Ann. Math. Ser. A*, **20** 2 (1999), 221-28.