Indian J. pure appl. Math., 32(7) : 1059-1064, July 2001

GRACEFULNESS OF ARBITRARY SUPERSUBDIVISIONS OF GRAPHS

G. SETHURAMAN

School of Mathematics, Anna University, Chennai 600 025, India
(E-mail : Sethu@annauniv.edu)

AND

P. SELVARAJU

Department of Mathematics, Jerusalem College of Engineering, Chennai 601 302, India
(E-mail : psekvar@yahoo.com)

(*Received* 15 *July* 1998; *after revision* 1 *May* 2000; *accepted* 1 *September* 2000)

In this paper we prove that arbitrary supersubdivisions of paths are graceful and that every cycle has a graceful supersubdivision. Further, we discuss related open problems.

Key Words : Graph Labelling; Graceful Labelling; Supersubdivision of Graphs; Arbitrary Supersubdivision of Graphs.

1. INTRODUCTION

Let G be a graph with m edges. A *graceful* labelling of G is an injection from the set of its vertices to the set $\{0, 1, 2, 3..., m\}$ such that the values of the edges are all integers from 1 to m, the value of an edge being the absolute value of the difference between the integers attributed to its end vertices.

Many mathematicians have constructed larger graceful graphs from standard graphs by using various operations. Join and product operations are used extensively among graphs such as paths, cycles, stars, complete graphs, complete bipartite graphs, complement of complete graphs and graceful trees etc., to get larger graceful graphs[1,2,6,9,12&14]. On the otherhand, when many copies of certain standard graphs such as complete graphs, complete bipartite graphs, cycles etc., are adjoined at one common vertex, the resultant graphs have been proved to be graceful[3-5, 13, 15 & 16]. Similarly, when many copies of certain graphs, such as the complete graph K_4, edge deleted subgraphs of the complete graph K_4, cycles C_n with $n - 3$ consecutive chords etc.,[8, 17 & 18], are adjoined at one common edge the resultant graphs have been proved to be graceful. For an exhaustive survey of these topics refer the excellent survey paper[11]. Most recently, Burzio and Ferrarese[7] have introduced an interesting method of construction of larger graceful trees from two given graceful trees by means of adjoining each vertex of one graceful tree with the other graceful tree. In this paper we introduce a new method of construction called supersubdivisions of a graph and show that arbitrary supersubdivisions of paths are graceful and that there exist a supersubdivision of cycles that are graceful. Finally, we discuss related open problems.

In the complete bipartite graph $K_{2,m}$, we call the part consisting of two vertices *the 2-vertices part* of $K_{2,m}$ and the part consisting of m vertices *the m-vertices part of* $K_{2,m}$.

Let G be a graph with n vertices and t edges. A graph H is said to be a *supersubdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \le i \le t$ in such a way that the ends of e_i are merged with the two vertices of the 2-vertices part of K_{2,m_i} after removing the edge e_i from G. A supersubdivision H of a graph G is said to be an *arbitrary supersubdivision* of a graph of G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily).

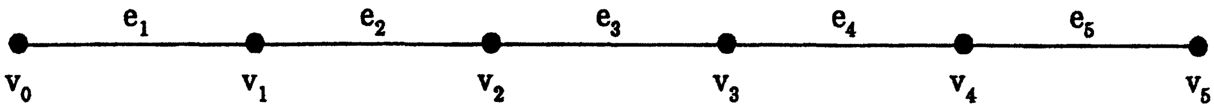

FIG. 1(a). Path P_6

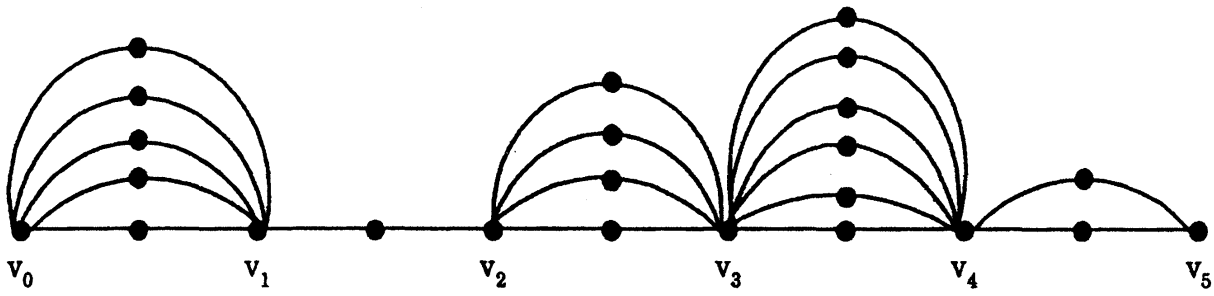

FIG. 1(b). An arbitrary supersubdivision of P_6

2. ARBITRARY SUPERSUBDIVISIONS OF PATHS ARE GRACEFUL

In this section we prove that arbitrary supersubdivisions of paths are graceful and then we show that there exist a supersubdivision of any cycle C_n that is graceful.

***Theorem* 1** — *Arbitrary supersubdivisions of any path are graceful.*

PROOF : Let P_n be a path with successive vertices $v_0, v_1, \ldots, v_{n-1}$ and let e_i denote the edge $v_{i-1} v_i$ of P_n, for $1 \le i \le n-1$. Let H be an arbitrary supersubdivision of a path P_n. That is, for $1 \le i \le n-1$, each edge e_i of P_n is replaced by a complete bipartite graph K_{2,m_i}, where m_i is any positive integer. Observe that H has

$$M = 2\,(m_1 + m_2 + \ldots + m_{n-1}) \text{ edges.}$$

Let $N_0 = M$, and let $N_{i-1} = M - 2\,(m_1 + m_2 + \ldots + m_{i-1}) + (i-1)$, for $2 \le i \le n-1$. Now for each i, $1 \le i \le n-1$, in the increasing order, (starting with 1) we shall give a graceful labelling to the vertices of K_{2,m_i} of a supersubdivision graph H of P_n corresponding to the edge e_i of P_n as shown in Fig. 2.

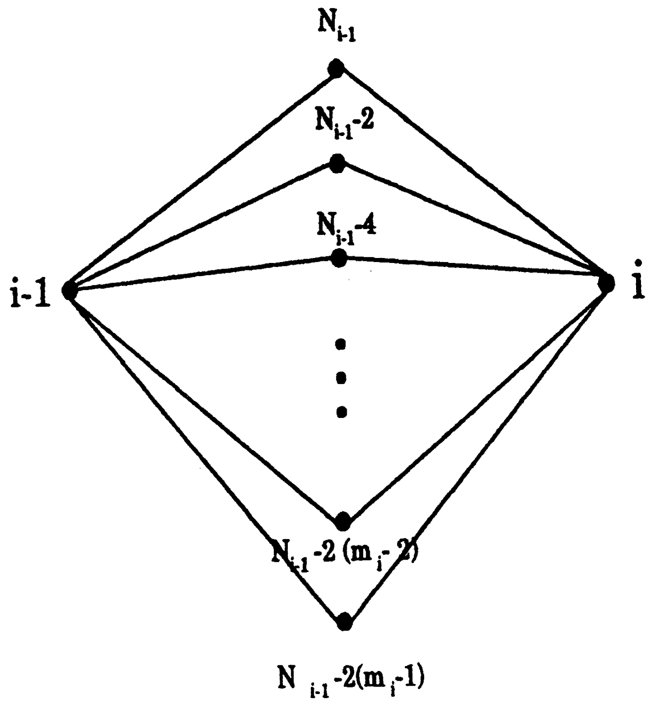

FIG. 2. Graceful labelling of K_{2,m_i} of H

Note that the two vertices of the 2-vertices part of K_{2,m_i} get the labels $i - 1$ and i and the label $N_{i-1}, N_{i-1}, g - N_{i-1} - 4, \ldots, N_{i-1} - 2(m_i - 1)$ are assigned to the m_i vertices of m_i-vertices part of K_{2,m_i}. It is clear from the above labelling that the $m_i + 2$ vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels. Therefore, the vertices of each K_{2,m_i}, $1 \leq i \leq n - 1$, in the supersubdivision H of P_n have distinct labels and hence the edges of each K_{2,m_i}, $1 \leq i \leq n - 1$, in the supersubdivision graph H of P_n also have distinct labels. Hence, H is graceful.

Illustrative examples for the labelling of Theorem 1.

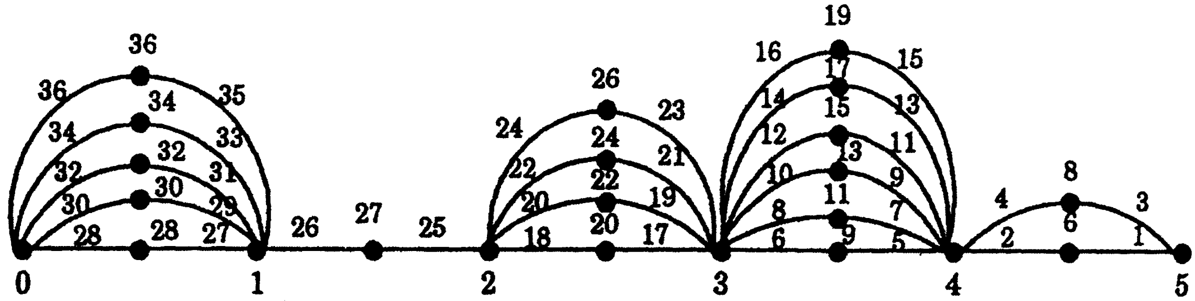

FIG. 3(a). Graceful labelling of the supersubdivision of P_6 that is shown in Fig. 1

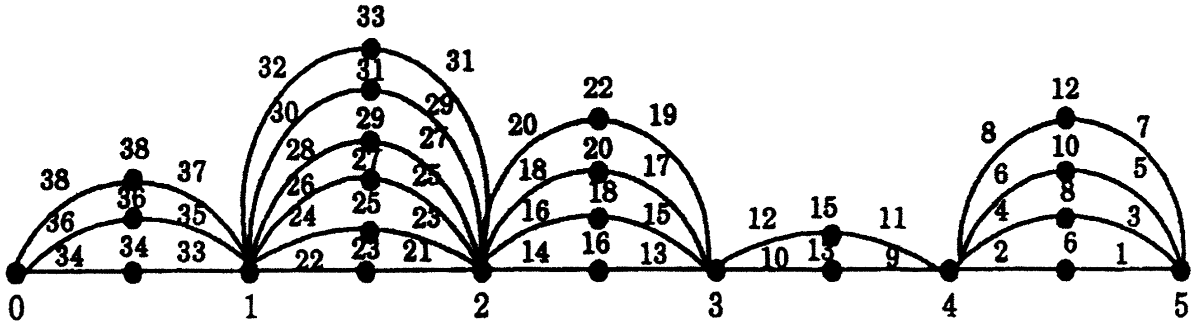

FIG. 3(b). Graceful labelling of a supersubdivision of P_6 diferent from that is shown in Fig. 1

Observe that in the first example, the edge 34 of P_6 is replaced by $K_{2,6}$ in the construction of the supersubdivision, while in the second example, the edge 34 of P_6 is replaced by $K_{2,2}$ for the construction of the supersubdivision.

2.1. A Graceful Supersubdivision of Cycles

In this section we prove that there exist a supersubdivision of C_n which is graceful.

***Theorem* 2** — *From any $n \geq 3$, there exists a supersubdivision of C_n that is graceful.*

PROOF : Let C_n be a cycle with consecutive vertices $v_0, v_1, \ldots, v_{n-1}$. Let G be a supersubdivision of cycle C_n obtained by replacing each edge e_i of C_n by a complete bipartite graph K_{2,m_i} where $m_i \in Z^+$, for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that G has $M = 2(m_1 + m_2 + \ldots + m_n)$ edges.

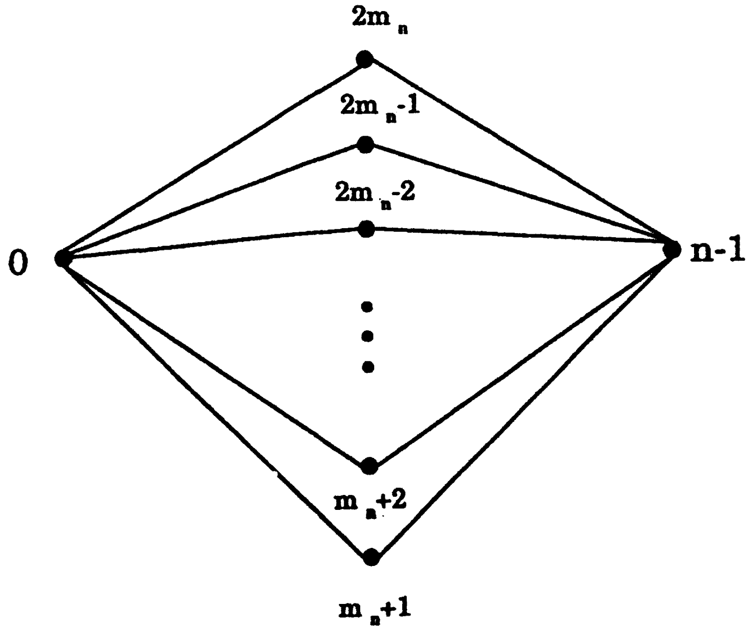

FIG. 4. Graceful labelling of K_{2,m_n} in G

Let $N_{i-1} = M - 2(m_1 + m_2 + \ldots + m_{i-1}) + i - 1$, for $2 \le i \le n-1$. When $i = 1$, $N_{i-1} = N_0 = M$. Now, for each i, $1 \le i \le n-1$, the vertices of K_{2, m_i} of G (obtained by replacing the corresponding edge e_i of C_n, $1 \le i \le n-1$) are labelled as shown in Fig. 2

For the vertices K_{2, m_n} of G (obtained by replacing the corresponding edge e_n of C_n) give labels as shown in Fig. 4.

It follows that the vertices of K_{2, m_i}, for $1 \le i \le n$, have distinct labels and hence the edges of K_{2, m_i} , for $1 \le i \le n$, also have distinct labels.

Therefore, all the vertices and all the edges of G have distinct labels.

Hence, G is graceful.

Illustration of labelling of Theorem 2

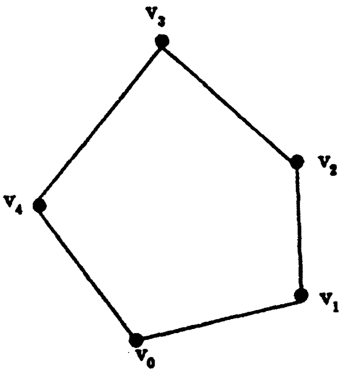

FIG. 5(a). Cycle C_5

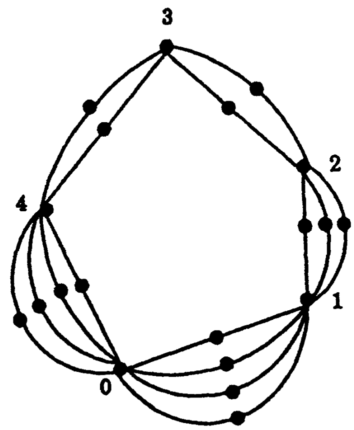

FIG. 5(b). Supersubdivision of C_5

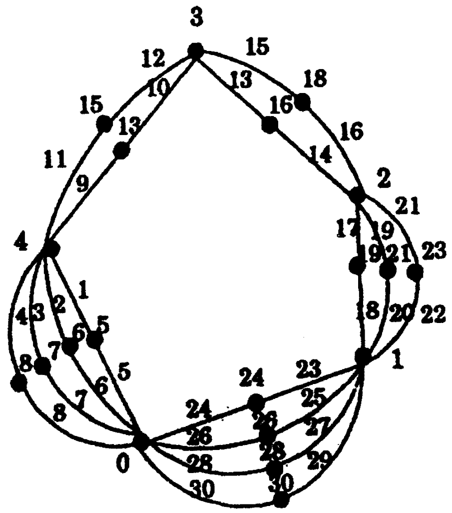

FIG. 5(c). Graceful labelled graph of SS (C_5)

Note that the edge $v_0 v_4$ is replaced by $K_{2,4}$ for the construction of supersubdivision of C_5.

Remark

"Supersubdivision of a graph" can be used as a powerful operation to get larger graphs from a given graph. In obtaining arbitrary supersubdivisions of a given graph G, each edge of G can be replaced by $K_{2,m}$, for any value of m, independently. Thus, each edge of G gives raise to an infinite family of supersubdivision of G.

3. DISCUSSION

In this paper, we have shown that arbitrary supersubdivisions of paths are graceful and for every cycle C_n, $n \geq 3$, there exist supersubdivisions of C_n which are graceful. We suspect that for many connected graphs there exist supersubdivisions which are graceful but only paths have graceful arbitrary supersubdivisions. So, we conclude this paper with the following two open problems :

Problem 1 Are there any graphs different from paths whose arbitrary supersubdivisions are graceful?

Problem 2 Is it true that every connected graph has at least one supersubdivision which is graceful?

ACKNOWLEDGEMENT

The authors are grateful to the referees for their valuable comments and suggestions.

REFERENCES

1. B. D. Acharya and M. K. Gill. *Indian J. Math.*, **23** (1981) 81-94.
2. R. Balakrishnan and R. Kumar, *Utilitas Math.*, **46** (1994) 97-102.
3. J. C. Bermond, A. E. Brouwer and A. Germa, *Colloq. Intern. ducentre National Recherche Scientifique, Paris* (1978) 35-38.
4. J. C. Bermond, A. Kotzig and J. Turgeon, In: *Combinatorics*, *Eds*: A. Hajnal and V. T. Sós), *Colloq, Math. Soc. János Bolyai*, **18** 2 vols, North-Holland, Amsterdam, (1978) 135-49.
5. R. Bodendiek, H. Schumacher and H. Wegner, *Mitt. Math. Gesellsh. Hamburg*, **10** (1975) 241-48.
6. C. Bu, *J. Harbin Ship building Engng. Inst.*, **15** (1994) 91-93.
7. M. Burzio and G. Ferrarese, *Discrete Math.*, **181** (1998) 275-81.
8. C. Delorme, *J. Graph Theory*, **4** (1980) 247-50.
9. R. Frucht and J. A. Gallian, *Ars Combin.* **26** (1988) 69-82.
10. J. A. Gallian, *Discrete Appl. Math.*, **49** (1994) 213-29.
11. J. A. Gallian, *Electron. J. Combinatorics*, **5** (1998) #DS6, http://www.combinatorics.org.
12. T. Grace, *J. Graph Theory*, **7** (1983) 195-201.
13. J. Huang and S. Skiena, *Ars Combin.*, **38** (1994) 225-42.
14. D. Jungreis and M. Reid, *Ars Combin.*, **34** (1992) 167-82.
15. K. Kathiresan, *Indian J. pure appl. Math.*, **23** (1992) 21-23.
16. K. M. Koh, D. G. Rogers, P. Y. Lee and C. W. Toh, *Nanta Math.*, **12** (1979) 133-36.
17. G. Sethuraman and R. Dhavamani, *Discret. Math.* **218** (2000) 283-87.
18. G. Sethuraman and S. P. M. Kishore, *Indian J. pure appl. Math.*, **30**(8) (1999) 801-8
19. S. C. Shee, *Discrete math.*, **28** (1991) 73-80.