Indian J. pure appl. Math., 32(7) : 1077-1080, July 2001

LOCATION OF ZEROS OF A POLYNOMIAL

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(*Received* 24 *March* 2000; *accepted* 1 *September* 2000)

The estimate of maximum number of zeros of a polynomial with complex co-efficients in a prescribed region has been obtained with restriction on the coefficients.

Key Words : Polynomial; Complex Number; Number of Zeros; Prescribed Region

INTRODUCTION AND RESULTS

Let $p(z) = \sum_{i=0}^{a} a_i z'$ be a polynomial in complex variable z. A well-known result due to Enstrom-Kakeya states that if $a_n \geq a_{n-1} \geq \ldots\ldots \geq a_0 > 0$ then all the zeros of $p(z)$ lie on $|z| \leq 1$.

Concerning the number of zeros of this polynomial in the region $|z| \leq \frac{1}{3}$, a result of Mohammad[4] states that

Theorem A — *Let* $p(z) = \sum_{i=0}^{a} a_i z^i$ *be a polynomial and* a_i*'s are positive real numbers such that* $a_n \geq a_{n-1} \geq \ldots\ldots \geq a_0 > 0$ *then number of zeros in* $|z| \leq \frac{1}{3}$ *does not exceed* $1 + \frac{1}{\log 2} \log\left(\frac{a_n}{a_0}\right)$

The result is analogous to the following theorem of Titcmarsh[1].

Theorem B — *Let* $f(z)$ *be regular,* $f(0) \neq 0, |f(z)| \leq M$ *in* $|z| \leq 1$*, then the number of zeros in* $|z| \leq \frac{1}{3}$*, does not exceed*

$$\frac{1}{\log 2} \log\left(\frac{M}{f(0)}\right).$$

In this article we consider the co-efficients a_is to be complex such that $\arg a_i \leq \alpha < \frac{\pi}{2}$ for each i and $|a_n| \geq |a_{n-1}| \geq \ldots\ldots \geq |a_0| > 0$, we obtain the estimate of number of zeros in the disc $|z| \leq \frac{1}{3}$. We prove the following theorem :

Theorem — *Let* $p(z) = \sum_{i=0}^{n} a_i z^i$ *such that*

$|a_n| \geq |a_{n-1}| \geq .... \geq |a_0| > 0$, where arg $[a_i] \leq \alpha < \frac{\pi}{2}$, then number of zeros in $|z| \leq \frac{1}{3}$ does not exceed $\frac{1}{\log 2} \log \left(\frac{k}{|a_0|} \right)$ where

$$k = |a_n| (\cos \alpha + \sin \alpha + 1) + 2 \sin \alpha \sum_{i=0}^{n-1} |a_i|$$

and
$$k_1 = |a_n| (1 + \cos \alpha + \sin \alpha) - |a_0| (\cos \alpha - \sin \alpha) + 2 \sin \alpha \sum_{i=1}^{n-1} |a_i|)$$

and $p(z)$ has nozero in $|z| < \frac{|a_0|}{k}$.

PROOF OF THE THEOREM : Consider $F(z) = (1 - z)\, p(z)$

$$= -a_n Z^{n+1} + (a_n - a_{n-1}) Z^n + (a_{n-1} - a_{n-2}) Z^{n-1} + ... +$$

$$(a_2 - a_1) Z^2 + (a_1 - a_0) Z + a_0 \qquad ... (1)$$

Now consider $|a_k - a_{k-1}|^2 = |a_k|^2 + |a_{k-1}|^2 - 2\, Re\, (a_k \bar{a}_{k+1})$

Now $a_k = |a_k| e^{i\alpha_k}, \bar{a}_{k-1} = |a_{k-1}| e^{-i\alpha_{k-1}}$

$$\Rightarrow Re\ (a_k \bar{a}_{k-1}) = Re\, (|a_k| e^{i\alpha_k} |a_{k-1}| e^{-i\alpha_{k-1}})$$

$$= Re\ (|a_k| |a_{k-1}| e^{i(\alpha_k - \alpha_{k-1})})$$

$$= |a_k| |a_{k-1} \cos (\alpha_k - \alpha_{k-1})$$

$$|a_k - a_{k-1}|^2 \leq |a_k|^2 + |a_{k-1}|^2 - 2 |a_k| |a_{k-1}| \cos 2\alpha$$

$$\leq [(|a_k| - |a_{k-1}|) \cos \alpha + (|a_k| + |a_{k-1}|) \sin \alpha]^2$$

$$\Rightarrow |a_k - a_{k-1}| \leq (|a_k| - |a_{k-1}|) \cos \alpha + (|a_k| + |a_{k-1}|) \sin \alpha$$

holds in general where $|a_k| \geq |a_{k-1}|$ and arg a_k, arg a_{k-1} is $\leq \alpha < \frac{\pi}{2}$.

On $|z| = 1$, we have

$$|F(z)| \leq |a_n| + \sum_{n=1}^{n} |a_i - a_{i-1}| + |a_0|$$

$$\leq [|a_n| + |a_0| + \sum_{i=1}^{n} (|a_i| - |a_{i-1}|)\cos\alpha + (|a_i| + |a_{i-1}|)\sin\alpha]$$

$$= |a_n|(1 + \cos\alpha + \sin\alpha) + (1 - \cos\alpha - \sin\alpha)|a_0| + 2\sin\alpha \sum_{i=0}^{n-1} |a_i|$$

$$\leq |a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{i=0}^{n-1} |a_i|$$

$\therefore\ |F(z)| \leq k$ for $|z| \leq 1$,

where $$k = |a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{i=0}^{n-1} |a_i|$$

Also $F(0) = a_0 \neq 0$

zeros of $F(z)$ in $|z| \leq \frac{1}{3}$ are

$\frac{1}{\log 2} \log \frac{k}{|a_0|}$, follows from Theorem B.

To prove that there exists no zero in $|z| < \frac{|a_0|}{k_1}$, we proceed as follows :

$$F(z) = (1 - z)\ (p(z) = -a_n z^{n+1} + \sum_{\gamma=1}^{n} (a_\gamma - a_{\gamma-1}) z^\gamma + a_0 = a_0 + h(z)$$

where $$h(z) = -a_n z^{n+1} + \sum_{\gamma=1}^{n} (a_\gamma - a_{\gamma-1}) Z^\gamma$$

On $|z| = 1$, $\max_{|z|=1} |h(z)| \leq |a_n| + \sum_{z=1}^{n} |a - a_{z-1}|$

$$\leq |a_n| + \sum_{\gamma=1}^{n} |a_y| - |a_{y-1}| \cos\alpha + (|a_y| + a_{y-1}|)\sin\alpha]$$

$$= |a_n| + \cos\alpha(|a_n| - |a_0|) + \sin\alpha \sum_{\gamma=1}^{n} (|a_y| + |a_{y-1}|)$$

$$= |a_n| + \cos\alpha(|a_n| - |a_0|) + \sin\alpha\left(|a_n| + a_0| + 2\sum_{\gamma=1}^{n-1} |a|\right)$$

$$|h(z)| \le |a_n|(1 + \cos\alpha + \sin\alpha) - |a_0| \cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{\gamma=1}^{n-1} |a_\gamma|$$

On $|z| = |h(z)| 1 \le k_1$

where $$k_l = |a_n|(1 + \cos\alpha + \sin\alpha) - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{\gamma=1}^{n-1} |a_\gamma|$$

$$\therefore |F(z)| = |a_0 + h(z)|$$

$$\ge |a_0| - |h(z)|$$

$$\ge |a_0| - |z| \max_{|z|=1} |h(z)| > 0$$

$$\ge |a_0| - |z| k_1$$

$$> 0 \text{ if } |z| < \frac{|a_0|}{k_1}$$

Thus $$|F(z)| > 0 \text{ if } |z| < \frac{|a_0|}{k_1}.$$

Hence the number of zeros of $p(z)$ does not exceed $\frac{1}{\log 2}\log\left(\frac{k}{|a_0|}\right)$ in $\frac{|a_0|}{k_1} \le |z| \le \frac{1}{3}$

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