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# COMPOSITION OPERATORS ON ORLICZ SPACES

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The compact, Fredholm and Isometric composition operators on Orlicz spaces are studied in this paper.

**Key Words : Composition Operator; Compact Operator; Fredholm Operator; Young Function**

## 1. INTRODUCTION

Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous convex function such that

i) $\phi(x) = 0$ iff $x = 0$

ii) $\lim_{x \to \infty} \phi(x) = \infty$.

Such a function $\phi$ is known as a young function. Let $(X, S, \mu)$ be a $\sigma$-finite measure space and let $L^{\phi}(\mu) = \{f : X \to C$ is measurable $\mid \int \phi(\varepsilon |f|)\, d\mu < \infty$ for some $\varepsilon > 0\}$. If we set $\|f\| \phi$ = Inf $\{\varepsilon > 0 : \int \phi(|f|)\, d\mu \le 1\}$, then $L^{\phi}(\mu)$ is a Banach space under the norm $\|\cdot\| \phi$. If $\phi(x) = x^p$, $1 \le p < \infty$, then $L^{\phi}(\mu) = L^p(\mu)$, the well-known Banach space of $p$-integrable functions on $X$.

A young function $\phi : \mathbb{R} \to \mathbb{R}^+$ is said to satisfy the $\Delta_2$-condition (globally) if $\phi(2x) < k\,\phi(x)$, $x \ge x_0 \ge 0$ $(x_0 = 0)$ for some absolute constant $k > 0$. If $\mu(X) = \infty$, then $\phi$ is called $\Delta_2$-regular. With each young function $\phi$ we can associate another convex function $\Psi : \mathbb{R} \to \mathbb{R}^+$ defined by $\Psi(y) = \sup\{x|y| - \phi(x) : x \ge 0\}$, $y \in \mathbb{R}$ which have similar properties.

The function $\Psi$ is called the complementary function to $\phi$. In general, simple functions are not dense in $L^{\phi}(\mu)$, but in case $\phi$ satisfy the $\Delta_2$ condition, then the class of simple functions becomes dense in $L^{\phi}(\mu)$. For more literature concerning orlicz spaces, we refer to Rao[6] Kufner[1], and Hudzik[3-4]. Throughout our paper we assume that $\phi$ satisfy $\Delta_2$ -condition.

Let $X$ and $Y$ be two non empty sets and let $F(X)$ and $F(Y)$ be two topological vector spaces of complex valued functions on $X$ and $Y$ respectively. Suppose $T: Y \to X$ is a mapping such that $f \circ T \in F(Y)$ whenever $f \in F(X)$. Then we can define a composition transformation $C_T: F(X) \to F(Y)$ by $C_T f = f \circ T$ for every $f \in F(X)$. If $C_T$ is continuous, we call it a composition operator induced by $T$.

A bounded linear operator $A: E \to E$ (where $E$ is a Banach space) is called compact if $A(B_1)$ has compact closure, where $B_1$ denotes the closed unit ball of $E$. A bounded linear operator $A: E \to E$ is called Fredholm if $A$ has closed range, dim ker$A$ and co dim ran $(A)$ are finite. The support of a function $f \in L^{\phi}(\mu)$ is denoted by suppf and the Randon Nikodym derivative of the measure $d\mu T^{-1}$ with respect to $\mu$ is denoted by $f_0$. In this paper we study composition operators on Orlicz spaces. It is proved that every composition linear transformation from an Orlicz space into an Orlicz space is bounded. The adjoint of a composition operator is obtained. The compact, Fredholm and isometric composition operators are also characterized.

## 2. Bounded Composition Operators on Orlicz Spaces

***Theorem* 2.1** — *If $C_T: L^{\phi}(\mu) \to L^{\phi}(\mu)$ is a linear transformation, then $C_T$ is continuous.*

PROOF : Let $\{f_n\}$ and $\{C_t f_n\}$ be sequences in $L^{\phi}(\mu)$ such that $f_n \to f$ and $C_T f_n \to g$ for some $f, g \in L^{\phi}(\mu)$. Then we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\phi(|f_{n_k} - f|)(x) \to 0 \text{ for } \mu\text{-almost all } x \in X.$$

From the non singularity of $T$,

$$\phi(|f_{n_k} - f) \mid T)(x) \to 0 \text{ for } \mu\text{-almost all } x \in X. \qquad \ldots (2)$$

Thus from (1) and (2), we conclude that $C_T f = g$. This proves that graph of $C_T$ is closed and hence by the closed graph theorem, $C_T$ is continuous.

## 3. Compact Composition Operators on Orlicz Spaces

***Theorem* 3.1** — *Let $C_T \in B(L^{\phi}(\mu))$. Then $C_T$ is compact if and only if $L^{\phi}(\chi_{\varepsilon}, \mu T^{-1})$ is finite dimensional for each $\varepsilon > 0$, where $\chi_{\varepsilon} = \left\{ x \in : \frac{d\mu T^{-1}}{d\mu}(x) \geq \varepsilon \right\}$.*

PROOF : For $f \in L^{\phi}(\mu)$, we have

$$\| C_T f \|_{\phi, \mu} = \inf \left\{ \varepsilon > 0 : \int \phi\left( \frac{|f \circ T|}{\varepsilon} \right) d\mu \leq 1 \right\}$$

$$= \inf . \left\{ \varepsilon > 0 : \int \phi\left( \frac{f}{\varepsilon} \right) d\mu T^{-1} \right\}$$

$$= \| If \|_{\phi, \mu T^{-1}}.$$

Thus $C_T$ is a compact operator if and only if $I : L^{\phi}(\chi_{\varepsilon}, \mu T^{-1}) \to L^{\phi}(\chi_{\varepsilon}, \mu T^{-1})$ is a compact operator if and only if $L^{\phi}(\chi_{\varepsilon}, \mu T^{-1})$ is finite dimensional, where $I$ is the identity operator.

*Corollary* 3.2 — If $(X, S, \mu)$ is a non atomic measure space, then no non zero composition operator on $L^{\phi}(\mu)$ is compact.

*Corollary* 3.3 — If $(X, S, \mu)$ is a $\sigma$-finite atomic measure space, then $C_T$ on $L^{\phi}(\mu)$ is compact if and only if the set $\left\{ n : \sum_{m \in T^{-1}(a_n)} \mu(a_m) \geq \varepsilon \mu(a_n) \right\}$ is a finite set, where $a_1, a_2$ ... are the atoms of the space.

## 4. FREDHOLM AND ISOMETRIC COMPOSITION OPERATORS ON ORLICZ SPACES

If we use the Holder's inequality for orlicz spaces, i.e., $\int fg d\mu \leq \| f \|_{\phi} \| g \|_{\psi}$, then by using Rao [6, prop. 1;. p. 100 & cor. 9; p. 111] we find that every $g \in L^{\psi}(\mu)$ gives rise to a bounded linear functional $F_g \in L^{\phi^*}(\mu)$ which is defined as

$$F_g(f) = \int f g \, d\mu \text{ for every } f \in L^{\phi}(\mu).$$

For each $f \in L^{\psi}(X, S, \mu)$ there exists a unique $T^{-1}(S)$ measurable function $E(f)$ such that $\int g f d\mu = \int gE(f) \, d\mu$ for $T^{-1}(S)$ measurable function $g$ for which the left integral exists. The function $E(f)$ is called the conditional expectation of $f$ with respect to the sigma algebra $T^{-1}(S)$. The operator $P_T : L^{\psi}(\mu) \to L^{\psi}(\mu)$ defined by $P_T f = f_0 E(f) \circ T^{-1}$ is called the Frobenius Perron operator where $E(f) \circ T^{-1} = g$ if and only if $E(f) = g \circ T$.

***Theorem*** 4.1 — *Let $C_T \in B(L^{\phi}(\mu))$. Then $C_T^* = P_T$*

PROOF : Take $A \in S$ to be a such that $0 < \mu(A) < \infty$. For $g \in L^{\psi}(\mu)$,

$$(C_T^* F_g)(\chi_A) = F_g(C_T \chi_A) = \int C_T \chi_A \cdot g \, d\mu = \int \chi_A \circ Tg \, d\mu$$

$$= \int \chi_A E(g) \circ T^{-1} f_0 \, d\mu = F_{E(g) \circ T^{-1} f_0}(\chi_A)$$

Hence, $C_T^* F_g = F_{E(g) \circ T^{-1} f_0}$. After indentifying $g \in L^\Psi$ with $F_g \in (L^\phi)^*$, we can write

$$C_T^* g = E(g) \circ T^{-1} \cdot f_0 = P_T g. \quad \text{... (3)}$$

***Theorem*** 4.2 — *Let $C_T \in B(L^\phi(\mu))$. Then $C_T$ has closed range if and only if there exists $\delta > 0$ such that $f_0(x) \geq \delta$ for $\mu$ almost all $x \in$ supp $f_0$ = 5..*

PROOF : If $f_0(x) \geq \delta$ for $\mu$ almost all $x \in s$, then for $\eta = \min(\delta, 1/\delta) \leq 1$

$$1 \geq \int \phi\left(\frac{C_T f}{\| C_T f \|_\phi}\right) d\mu = \int f_0 \, \phi\left(\frac{f}{\| C_T f \|_\phi}\right) d\mu$$

$$\geq \int \eta \, \phi\left(\frac{f}{\| C_T f \|_\phi}\right) d\mu$$

$$\geq \int \phi\left(\frac{\eta f}{\| C_T f \|_\phi}\right) d\mu$$

Hence, $\| C_T f \|_\phi \geq \eta \| f \|_\phi \ \forall \ f \in L^\phi(s)$ so that $C_T$ has closed range.

Conversely suppose that $C_T$ has closed range. Then there exists $\delta \geq 0$ such that

$$\| C_T f \|_\phi \geq \delta \| f \|_\phi \quad \text{... (4)}$$

for every $f \in L^\phi$ (supp $f_0$). Choose a positive integer $n$ such that $1/n < \delta$.

If the set $E = \{x \in X : f_0(x) < 1/n\}$ has positive measure, then for a given measurable subset $F \subset$ supp $f_0$ such that $0 < \mu(F) < \infty$, we have

$$\mu T^{-1}(E) = \int_E f_0 \, d\mu < 1/n \, \mu(E)$$

which implies that

$$\phi^{-1}\left(\frac{1}{\mu T^{-1} d(E)}\right) \geq \phi^{-1}\left(\frac{n}{\mu(E)}\right) \geq n \, \phi^{-1}\left(\frac{1}{\mu(E)}\right)$$

or equivalently

$$\| C_T \chi_E \|_\phi \leq 1/n \, \| \chi_E \| \, \phi$$

This contradicts the inequality (1). Hence, $f_0$ is bounded away from zero on supp $f_0$.

***Theorem*** 4.3 — *Let $C_T \in B(L^\phi(\mu))$. Then Ker $C_T^*$ is either zero dimensional or infinite dimensional.*

PROOF : Suppose $0 \neq g \in$ Ker $C_T^*$ . Then $E$ = supp $g$ is a set of non zero measure. Now we can partition $E$ into a sequence $\{E_n\}$ of disjoint measurable sets, $0 < \mu(E_n) < \infty$. We show that

$g\,\chi_{E_{n\circ T}} \in$ Ker $C_T^*$ . Consider

$$C_T^*(g\,\chi_{E_{n\circ T}})(f) = \int (g\cdot\chi_{E_{n\circ T}})(C_Tf)\,d\mu$$

$$= \int g\cdot C_T(\chi_{E_n}f)\,d\mu = 0$$

Hence, if Ker $C_T^*$ is not zero dimensional, it is infinite dimensional.

*Corollary* 4.4 — Let $C_T\in B(L^\phi(\mu))$. Then $C_T$ is injective if and only if $T$ is surjective.

*Corollary* 4.5 — Let $C_T\in B(L^\phi(\mu))$. Then $C_T$ has dense range if and only if $T^{-1}(S)=S$.

PROOF : Suppose $C_T$ has dense range. Let $E\in S$ be such that $\chi_E\in L^\phi(\mu)$.

Then there exists $\{f_n\}\subset L^\phi(\mu)$ such that $C_tf_n\to\chi_E$. Now we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $C_Tf_{n_k}\to\chi_E$ a.e.. Now each $C_Tf_{n_k}$ is measurable with respect to $T^{-1}(S)$. Therefore, $\chi_E$ is measurable with respect to $T^{-1}(S)$ so that $\chi_E=\chi_{T^{-1}(F)}$. Hence $T^{-1}(S) = S$ a.e. .

Conversely, suppose $T^{-1}(S) = S$ a.e. If $E\in S, 0<\mu(E)<\infty$, then there exists $F\in S$ such that $\mu(T^{-1}(F)\,\Delta\,E)=0$. Since $X$ is $\sigma$-finite, we can find an increasing sequence $\{F_n\}$ of sets of finite measure $F_n\uparrow F$ or $T^{-1}(F)\setminus T^{-1}(F_n)\downarrow\phi$. Hence for given $\varepsilon>0$ there exists a positive integer $n_0$ such that

$$\mu T^{-1}(F\setminus F_n) < \frac{1}{\phi(1/\varepsilon)} \text{ for every } n\geq n_0. \text{ Hence,}$$

$$\| C_T\chi_F - C_T\chi_{F_n}\|_\phi = \| C_t(\chi_{F\setminus F_n})\|_\phi$$

$$= \|\chi_{T^{-1}(F\setminus F_n)}\|_\phi$$

$$= \frac{1}{\phi^{-1}\left(\dfrac{1}{\mu T^{-1}(F\setminus F_n)}\right)} < \varepsilon$$

for all $n\geq n_0$. Then $\chi_E=\chi_{T^{-1}(F)}\,\varepsilon\,\overline{\text{ran } C_T}$. This proved that $C_T$ has dense range.

We are now ready to present a criterion for Fredholm composition operators.

***Theorem*** 4.6 — *Let $C_T\in B(L^\phi(\mu))$. Then $C_T$ is fredholm if and only if $C_T$ is invertible.*

PROOF : Assume that $C_T$ is fredholm. In view of theorem 4.3, ker $C_T$ and ker $C_T^*$ are zero dimensional so that $C_T$ is injective and $T^{-1}(S)=S$ a.e. Therefore by corollary 4.5, $C_T$ has dense range. Since ran $C_T$ is closed, so $C_T$ is surjective. This proves the invertibility of $C_T$. The proof of the converse is obvious.

*Proposition* 4.7 — Let $C_T \in B(L^{\Psi}(\mu))$. Then

$C_T^* C_T = M_{f_0}$, where $f_0 = \frac{d\mu T^{-1}}{d\mu}$, the Radon Nikodym derivative of the measure $\mu T^{-1}$ with respect to the measure $\mu$.

PROOF : Replacing $g$ by $C_T g$ in condition (1) of theorem 4.1 we get

$$C_T^*(C_T g) = E(g \circ T) \circ T^{-1} \cdot f_0 = g \cdot f_0 = M_{f_0} g$$

for every $g \in L^{\Psi}(\mu)$. Hence $C_T^* C_T = Mf_0$.

*Corollary* 4.8 : Let $C_T \in B(L^{\phi}(\mu))$. Then $C_T$ is an isometry if and only if $T$ is measure preserving.

*Example* — Let $T$ : [0, 1] → [0, 1] be defined by

$$T(x) = \left\{ \begin{array}{ll} 2x, & 0 \leq x \leq 1/2 \\ 2x - 1, & 1/2 \leq x \leq 1 \end{array} \right\}$$

Then $$C_T^* f(x) = 1/2 f(1/2\, x)\, \chi_{[0,\, 1]} + 1/2 f(1 - 1/2\, x)\, \chi_{[0,\, 1]}$$

$$= 1/2\ f(1/2\ x) + 1/2\ f\ (1 - 1/2\ x)$$

Then ker $C_T$ is infinite dimensional. Hence, $C_T$ is not Fredholm.

*Example* Let $T$: [0, 1] → [0, 1] be defined by

$$T(x) = \left\{ \begin{array}{ll} x/1 - x, & 0 \leq x \leq 1/2 \\ 2(1 - x), & 1/2 \leq x \leq 1 \end{array} \right\}$$

Then $C_T^*$, the adjoint of $C_T$ is given by

$$C_T^* f(x) = \frac{f(x/1 + x)}{(1 + x)^2} + \frac{f(1 + x/2)}{2}$$

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