Indian J. pure appl. Math., 32(7) : 975-980, July 2001

ON GRACEFUL GRAPHS: ONE VERTEX UNIONS OF NON-ISOMORPHIC COMPLETE BIPARTITE GRAPHS

G. SETHURAMAN

School of Mathematics, Anna University, Chennai 600 025, *India*
(E-mail: sethu@annauniv.edu)

AND

P. SELVARAJU

Department of Mathematics, Jerusalem College of Engineering, Chennai 601 302, *India*
(E-mail : pselvar@yahoo.com)

(*Received* 31 *January* 2000; *after revision* 7 *August* 2000; *accepted* 8 *January* 2001)

Motivated by the result of Koh *et al.*[15], we prove that graphs that are the one vertex union of n complete bipartite graphs K_{2, m_i}, $1 \leq i \leq n$, are graceful if at least $n - 2$ of the K_{2, m_i}'s are non-isomorphic and we discuss a related problem.

Key Words : Graph Labelling; Graceful Labelling; One Vertex Union of Non-Isomorphic Complete Bipartite Graphs

1. INTRODUCTION

Let G be a graph with m edges. A graceful labelling of G is an injection from the set of its vertices into the set $\{0, 1, 2, ..., m\}$ such that the values of the edges are all the numbers from 1 to m, the value of an edge being the absolute value of the difference between the numbers attributed to its end vertices.

In the literature of graph labelling, it is interesting to observe that many mathematicians have shown much interest in constructing bigger type graceful graphs from popular or standard graphs by using various operations. Join and product operations are used extensively among graphs such as paths, cycles, stars, complete graphs, complete bipartite graphs, complements of complete graphs and graceful trees etc., to get bigger graceful or harmonious graphs etc.,[1, 2, 7, 8, 9, 12, 13 & 14]. On the other hand many copies of certain standard type of graphs, such as complete graphs, complete bipartite graphs, cycles etc., are adjoined at one common vertex and the resultant graphs, called one vertex union of the graphs are proved to be graceful[3, 4, 5 & 15]. For an exhaustive survey of these topics refer the excellent survey paper[11]. So far, no such attempt has been made to study the graceful labelling of one vertex union of n non-isomorphic graphs. Motivated by the result of Koh and others[15], [one vertex union of t copies of $K_{m, n}$ is graceful], we shall pose the question : Are graphs that are the one vertex union of n non-isomorphic graphs K_{p_i, q_i}, $1 \le i \le n$ graceful? In general, we ask whether graphs that are the one vertex union of n complete bipartite graphs K_{p_i, q_i}, for any p_i and for any q_i, $1 \le i \le n$ are graceful? In this direction, in this paper, we prove that graphs that are the one vertex union of K_{2, m_i}, $1 \le i \le n$ are graceful, if at least n-2 of the K_{2, m_i}'s are non-isomorphic.

2. THE ONE VERTEX UNION OF NON-ISOMORPHIC GRAPHS K_{2, m_i}'s, $1 \le i \le n$ ARE GRACEFUL

In the complete bipartite graph $K_{2, m}$ we call the part consisting of two vertices as 2-vertices part and the part consisting of m vertices as m-vertices part of $K_{2, m}$. Further, we consider one of the vertices of the 2-vertices part of $K_{2, m}$ as the root of $K_{2, m}$.

In this section we prove that graphs that are one vertex union of n complete bipartite graphs K_{2, m_i}'s are graceful if at least n-2 of the K_{2, m_i}'s are non-isomorphic, where the union is taken at the root of each K_{2, m_i} $1 \le i \le n$.

Case 1 — One Vertex Union of n-non-isomorphic Complete Bipartite Graphs K_{2, m_i}'s $1 \le i \le n$

Let G be the one vertex union of non-isomorphic complete bipartite graphs K_{2, m_i}, $1 \le i \le n$, such that $m_i < m_j$, for $1 \le i < j \le n$, where the union is taken at the root of each K_{2, m_i},

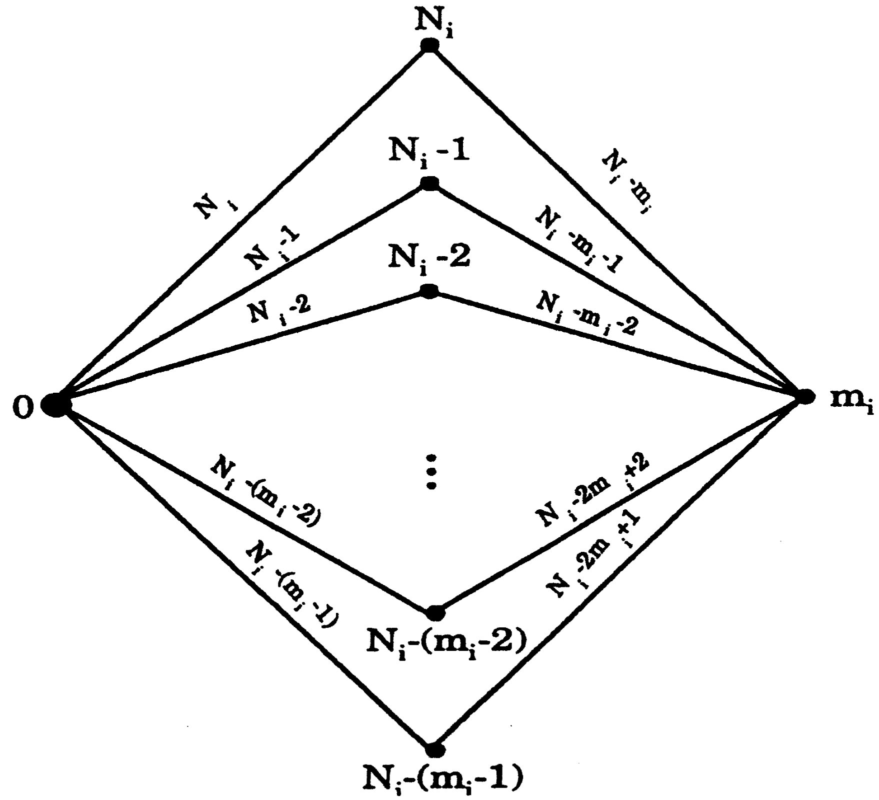

FIG. 1. Graceful labelling of K_{2, m_i} of G.

$1 \leq i \leq n$. Observe that G has $2(m_1 + m_2 + \ldots + m_n)$ edges. For convenience, we denote $|E(G)| = M$ and $N_i = M - 2(m_1 + m_2 + \ldots + m_{i-1})$, for $1 \leq i \leq n$. Now we shall give a graceful labelling for the vertices of the ith non-isomorphic complete bipartite graph K_{2, m_i}, $1 \leq i \leq n$ of the one vertex union.

Note that 0 is given to the root of K_{2, m_i} and the label m_i is given to the other vertex of the 2-vertices part of K_{2, m_i} and the distinct labels $N_i, N_i - 1, \ldots, N_i - (m_i - 1)$ are given to the vertices of m_i-vertices part of K_{2, m_i}. It is clear from the above labelling that the vertices of K_{2, m_i}, $1 \leq i \leq n$ get distinct labels and consequently the $2m_i$ distinct edge values $N_i, N_i - 1, N_i - 2, \ldots, N_i - 2m_i + 1$ are obtained at the $2m_i$ edges.

Hence, G is graceful.

Case 2 — One Vertex Union of K_{2, m_i}, $1 \leq i \leq n$, in which two K_{2, m_i}'s are isomorphic and Other $(n - 2)$ of the K_{2, m_i}'s are non-isomorphic

Let G be the one vertex union of n complete bipartite graphs K_{2, m_i}'s, $1 \leq i \leq n$ in which two K_{2, m_i}'s are isomorphic and other $(n - 2)$ of the K_{2, m_i}'s are non-isomorphic such that the union is taken at the root of each K_{2, m_i}, $1 \leq i \leq n$. For convenience, we may assume that the first and second K_{2, m_i} are isomorphic and other remaining $(n - 2)$ of the K_{2, m_i}'s are non-isomorphic. That is, G is one vertex union of n complete bipartite graphs $K_{2, m_1}, K_{2, m_1}, K_{2, m_i}, 3 \leq i \leq n$, such that $m_i < m_j$, for $3 \leq i < j \leq n$. Observe that G has $2(2m_1 + m_3 + \ldots + m_n)$ edges. Let $M = 2(2m_1 + m_3 + \ldots + m_n)$.

Now we shall give a graceful labelling to the vertices of the first two isomorphic copies

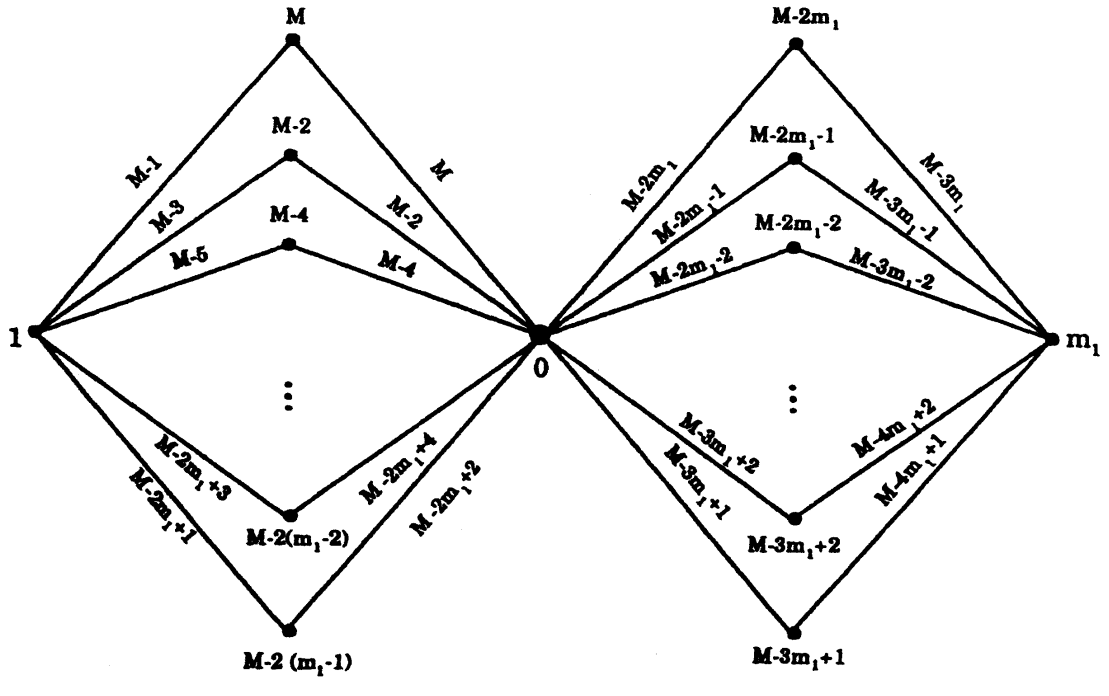

FIG. 2. Graceful labelling of two isomorphic copies K_{2, m_1} of G.

K_{2,m_1} of the one vertex union and that of the remaining $n - 2$ non-isomorphic K_{2,m_i}, $3 \le i \le n$ separately. A graceful labelling to the vertices of isomorphic copies in the one vertex union is given in Fig. 2.

Note that 0 is given to the root, 1 is given to the other vertex of the 2-vertices part of the first isomorphic copy and the m_1 is given to the other vertex of 2-vertices part of the second isomorphic copy of the one vertex union. From the above labeling, it is clear that in the first and second isomorphic copy of K_{2,m_1} the vertices get distinct labels and the $4m_1$ edge values are also distinct. Now for the vertices of K_{2,m_i}, $3 \le i \le n$, we give a graceful labelling as in the Case 1, by taking $N_i = M - 2(m_1 + m_2 + \ldots + m_{i-1})$. Thus, each K_{2,m_i}, $3 \le i \le n$ get distinct vertex labels and consequently distinct edge labels.

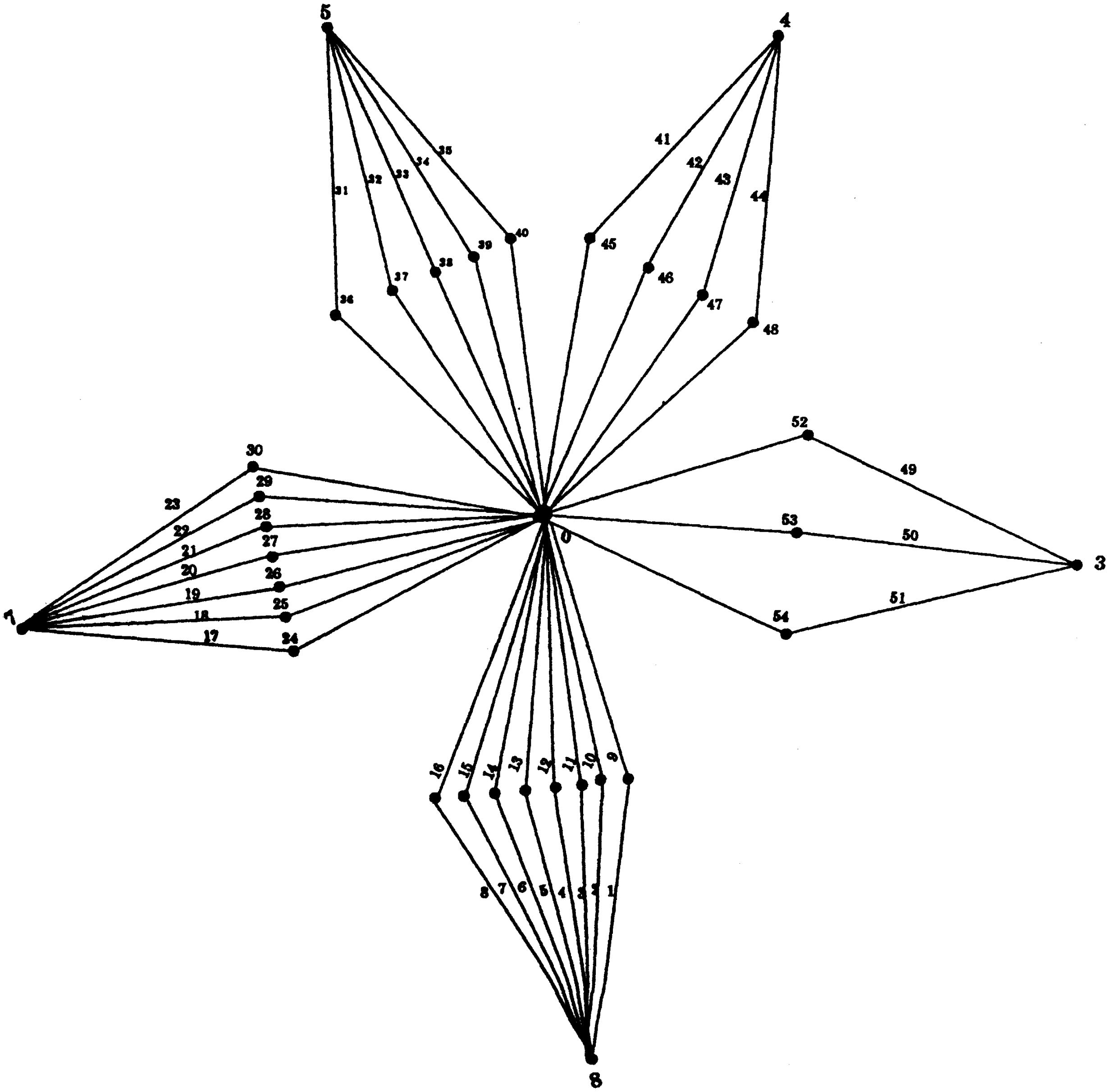

FIG. 3. Illustration for case 1, one vertex union of $K_{2,3}, K_{2,4}, K_{2,5}, K_{2,7}$ and $K_{2,8}$.

Hence, G is graceful.

3. Discussion

Koh *et al.*[15] have proved that the one vertex union of t copies of the complete bipartite graph $K_{2,m}$ is graceful. Their result motivates us to ask the question of whether the one vertex union of complete bipartite graphs K_{p_i, q_i} for any p_i and any q_i, $1 \leq i \leq n$ is graceful? Our result answers this question for the cases, $p_i = 2$, for $i = 1, 2, 3$ and q_i any positive integers for $i = 1, 2, 3$ and $p_i = 2$, $l \leq i \leq n$, and at least $(n - 2)$ of the q_i's are not equal. The result of Koh *et al.*[15] answers this question for the case $p_i = p$, $q_i = q$, $1 \leq i \leq n$. To completely answer this question, many such cases

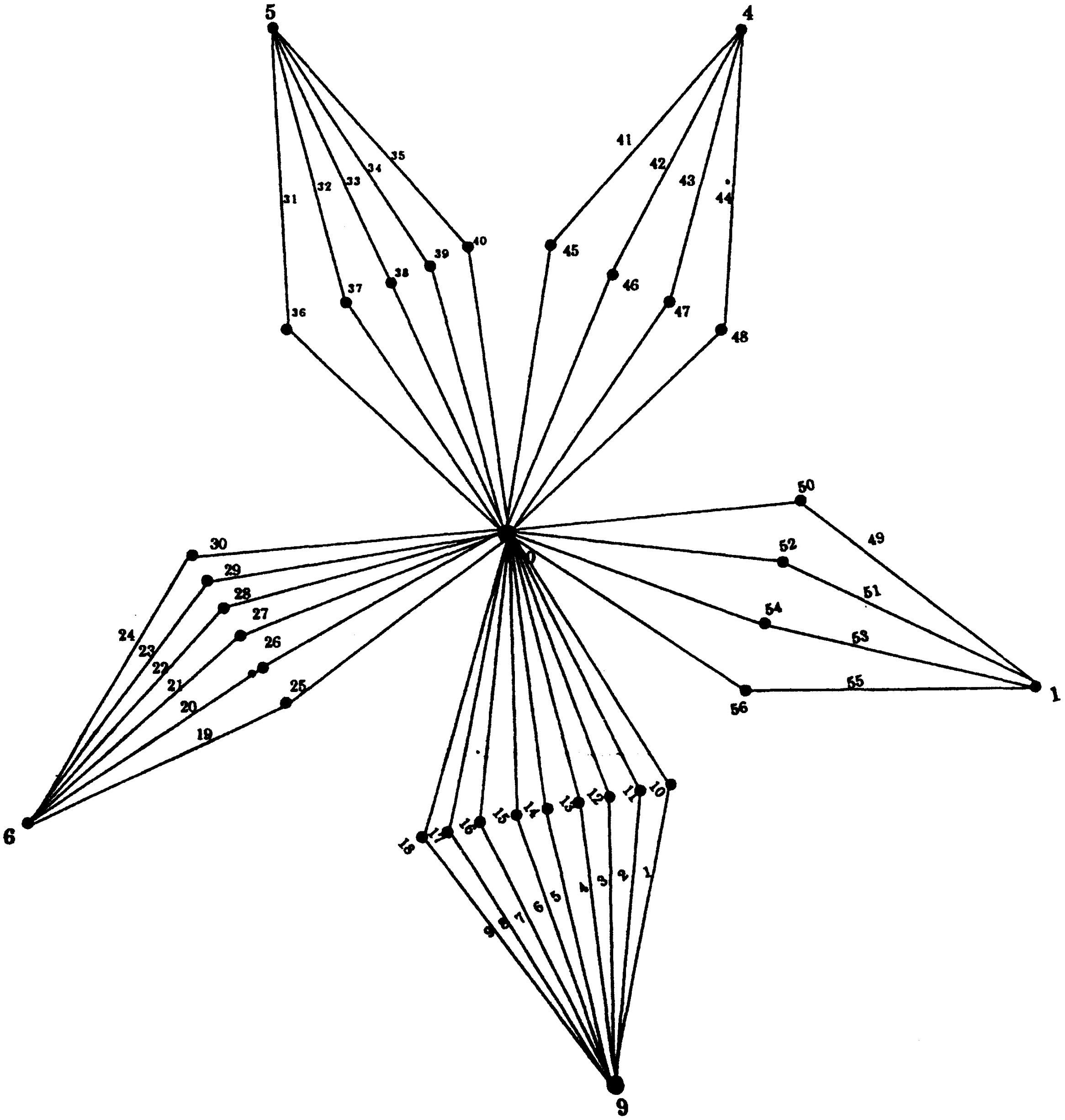

FIG. 4. Illustration for case 2, one vertex union of $K_{2,4}, K_{2,4}, K_{2,5}, K_{2,6}$ and $K_{2,9}$.

have to be settled. However, we strongly believe that one vertex union of K_{2,m_i}, $1 \leq i \leq n$, is graceful for all possible choices of m_i's. So, we conclude this paper with the following conjecture.

Conjecture

"The one vertex union of the complete bipartite graphs K_{2,m_i}, $1 \leq i \leq n$, [where the union is taken at one of the vertices of the partite sets with 2 vertices of each K_{2,m_i}, $1 \leq i \leq n$,], is graceful for all possible choices of the m_i's".

ACKNOWLEDGEMENT

The authors would like to thank the referees for their valuable comments and suggestions.

REFERENCES

1. B. D. Acharya and M. K. Gill, *Indian J. Math.*, **23** (1981) 81-94.
2. R. Balakrishnan and R. Kumar, *Utilitas Math.*, **46** (1994) 97-102.
3. R. Balakrishnan, A. Selvam and V. Yegnanarayanan, *Indian J. Pure and Appl. Math.*, **28** (1997) 905-16.
4. J. C. Bermond, *In : Graph Theory and Combinatorics*, Pitman, London (1979) 13-37.
5. J. C. Bermond, A. E. Browner and A. Germa, Systems de triplets et differences associeès, *Problems Combinatories et Thèrorie des Graphs, Colloq. Intern. du Centre National de la Rech. Scient.*, 260, Editions du Centre Nationale de la Recherche Scientifique, Paris, (1978) 35-38.
6. J. C. Bermond, A. Kotzig and J. Turgeon, *In : Combinatorics*, (*Eds* : A. Hajnal and V. T. Sos) *Colloq. Math. Soc.* János Bolyai, 18, 2 vols. North-Holland, Amsterdam, 1978, 135-49.
7. R. Bodendiek, H. Schumacher and H. Wegner, *Mitt. Math.* Gesellsch. Hamburg, **10** (1975) 241-48.
8. C. Bu, *J. Harbin Ship building Engng. Inst.*, **15** (1994) 91-93.
9. G. J. Chang, D. F. Hsu and D. G. Rogers, *Congr. Numer.*, **32** (1981) 181-97.
10. J. A. Gallian, *Discrete Appl. Math.*, **49** (1994) 213-29.
11. J. A. Gallian, *Electronic J. Combinatorics* 5 (1998), DS6 http://www.combinatorics.org.
12. T. Grace, *J. Graph Theory*, **7** (1983) 195-201.
13. J. Huang and S. Skiena, *Ars Combin* **38** (1994) 225-42.
14. D. Jungreis and M. Reid, *Ars Combin.*, **34** (1992) 167-82.
15. K. M. Koh, D. G. Rogers and P. Y. Lee and C. W. Toh, *Nanta Math.*, **12** (1979) 133-36.
16. K. M. Koh, D. G. Rogers, H. K. Teo and K. Y. Yap, *Congr. Numer.* **29** (1980) 559-71.