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EXISTENCE AND STABILITY OF LIBRATION POINTS IN THE RESTRICTED THREE BODY PROBLEM WHEN THE PRIMARIES ARE TRIAXIAL RIGID BODIES AND SOURCE OF RADIATIONS

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This paper deals with the stationary solutions of the planar restricted three body problem when the primaries are triaxial rigid bodies and source of radiations with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. It is seen that there are five libration points, two triangular and three collinear. It is further observed that the collinear points are unstable, while the triangular points are stable for the mass parameter $0 \le \mu < \mu_{crit}$ (the critical mass parameter). It is further seen that the triangular points have long or short periodic elliptical orbits in the same range of μ.

Key Words : Restricted Three Body Problem; Libration Points; Rigid Body; Source of Radiation; Stability

1. INTRODUCTION

It is well known that the classical planar restricted three body problem possesses five libration points, two triangular and three collinear. The collinear libration points L_1, L_2, L_3 are unstable, while the two equilateral libration points L_4, L_5 are stable for $\mu < \mu_{crit}$ = 0.0385208965... (Szebehely), Wintner[2] showed that the stability of the two equilateral points is due to the existence of coriolis terms in the equations of motion written in a synodic co-ordinate system.

In recent times many perturbing forces i.e., oblateness and radiation forces of the primaries, coriolis and centrifugal forces, variation of the masses of the primaries and of the infinitesimal mass etc., have been included in the study of the restricted three body problem. In the case of restricted three body problem where both the primaries are oblate spheroids whose equatorial plane coincides with the plane of motion, the location of libration points and their stability in the Liapunov sense has been studied by Vidyakin[3]. For the case, where the bigger primary is an oblate spheroid whose equatorial plane coincides with the plane of motion, Subba Rao and Sharma[4] have studied the stability of libration points. A similar problem has been studied by El-Shaboury[5]. Khanna and Bhatnagar[6&7] have studied the problem when the smaller primary is a triaxial rigid body. Sharma, Taqvi and Bhatnagar[8] have studied the problem when the bigger primary is a triaxial rigid body as well as source of radiation.

In this paper, we consider the primaries as triaxial rigid bodies and source of radiations with one of the axes as the axis of symmetry and their equatorial plane coinciding with the plane of motion. Further we assume that the primaries are moving without rotation about their centre of mass in circular orbits. An attempt is made to study the existence and stability of libration points.

2. Equations of Motion

We shall adopt the notation and terminology of Szebehely[1]. As a consequence, the distance between the primaries does not change and is taken equal to one; the sum of the masses of the primaries is also taken as one. The unit of time is so chosen as to make the gravitational constant unity. Using dimensionless variables, the equations of motion of the infinitesimal mass m_3 in a synodic co-ordinate system (x, y) are

$$\ddot{x} - 2n\dot{y} = \frac{\partial \Omega}{\partial x}\,.$$

and $$\ddot{y} + 2n\dot{x} = \frac{\partial \Omega}{\partial y}, \qquad \dots (1)$$

where $$\Omega = \sum_{i=1}^{2} \left[\frac{1}{2} n^2 \mu_i r_i^2 + (1-p_i) \left\{ \frac{\mu_i}{r_i} + \frac{\mu_i}{2m_i r_i^3} (I_{1i} + I_{2i} + I_{3i} - 3I_i) \right\} \right], \qquad \dots (2)$$

(McCusky)

$$\left.\begin{aligned} &\mu_1 = 1-\mu, \mu_2 = \mu \\ &r_i^2 = (x-x_i)^2 + y^2, (i = 1, 2), \\ &x_1 = \mu, x_2 = -1+\mu, \end{aligned}\right] \qquad \dots (3)$$

$$p_1 = \frac{\text{Radiation pressure due to bigger primary}}{\text{Gravitation force due to bigger primary}} \ll 1,$$

$$p_2 = \frac{\text{Radiation pressure due to smaller primary}}{\text{Gravitation force due to smaller primary}} \ll 1.$$

Here μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 < \mu \le \frac{1}{2}$. That is, $\mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2}$ with $m_1 \ge m_2$ being the masses of the primaries.

I_{1i}, I_{2i}, I_{3i} $(i = 1, 2)$ are the principal moments of inertia of the triaxial rigid body of mass m_i $(i = 1, 2)$ at its centre of mass, with a_i, b_i, c_i $(i = 1, 2)$ as lengths of its semi-axes. I_i $(i = 1, 2)$ is the moment of inertia about a line joining the centre of the rigid body of mass m_i $(i = 1, 2)$ and the infinitesimal body of mass m_3 and is given by

$$I_i = I_{1i} l_{1i}^2 + I_{2i} m_{1i}^2 + I_{3i} n_{1i}^2 \ (i = 1, 2),$$

where l_{1i}, m_{1i}, n_{1i} $(i = 1, 2)$ are the direction cosines of the line with respect to its principal axes.

Here, we have also assumed that the principal axes of m_1 and m_2 are parallel to the synodic axes $O\,(xyz)$.

The axes $O\ (xyz)$ have been defined by Szebehely.

The mean motion, n, is given by

$$n^2 = 1 + \sum_{i=1}^{2} \frac{3}{2}(2A_{1i} - A_{2i} - A_{3i}), \qquad \dots (4)$$

where $A_{1i} = \frac{a_i^2}{5R^2}, A_{2i} = \frac{b_i^2}{5R^2}, A_{3i} = \frac{c_i^2}{5R^2}, \ (i = 1, 2),$

and R is the distance between the primaries.

Here we are neglecting the perturbation in the potential between m_1 and m_2 due to radiation pressure because m_1 and m_2 are supposed to be sufficiently large.

Ω in eq. (2) can also be written as

$$\Omega = \sum_{i=1}^{2} \left[\frac{1}{2} n^2 \mu_i r_i^2 + \frac{\mu_i}{r_i} + \frac{\mu_i}{2r_i^3}(2\sigma_{1i} - \sigma_{2i}) - \frac{3\mu_i}{2r_i^5}(\sigma_{1i} - \sigma_{2i})y^2 - p_i \frac{\mu_i}{r_i} \right],$$

where $\sigma_{1i} = A_{1i} - A_{3i}$

and $\sigma_{2i} = A_{2i} - A_{3i} \ (i = 1, 2).$

We assume that σ_{1i} and $\sigma_{2i} << 1 \ (i = 1, 2)$.

The mean motion given in eq. (4), becomes

$$n^2 = 1 + \sum_{i=1}^{2} \frac{3}{2}(2\sigma_{1i} - \sigma_{2i}). \qquad \dots (5)$$

It may be noted that the mean motion, n, is independent of the solar radiation pressure p_i $(i = 1, 2)$.

3. Location of Libration Points

Eqs. (1) permit an integral analogous to Jacobi integral

$$\dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0.$$

The libration points are the singularities of the manifold

$$F(x, y, \dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0.$$

Therefore, these points are the solutions of the equations

$$\Omega_x = 0, \Omega_y = 0.$$

Two cases arise :

Case (a) — Triangular Libration Points ($y \neq 0$),

The triangular libration points are the solutions of the equations

$$n^2 x + \sum_{i=1}^{2} \left[(p_i - 1) \frac{\mu_i}{r_i^3} (x - x_i) - \frac{3\,\mu_i}{2r_i^5} (2\sigma_{1i} - \sigma_{2i}) (x - x_i) + \frac{15\,\mu_i}{2r_i^7} (\sigma_{1i} - \sigma_{2i}) (x - x_i) y^2 \right] = 0,$$

$$n^2 + \sum_{i=1}^{2} \left[(p_i - 1) \frac{\mu_i}{r_i^3} - \frac{3\,\mu_i}{2r_i^5} (4\,\sigma_{1i} - 3\,\sigma_{2i}) + \frac{15\,\mu_i}{2r_i^7} (\sigma_{1i} - \sigma_{2i}) y^2 \right] = 0. \quad \text{... (6)}$$

If we take $p_i = \sigma_{1i} = \sigma_{2i} = 0$, ($i$ = 1, 2), the solution of eq. (6) is given by $r_1 = r_2 = 1$ and from the eq. (5), n = 1.

Now, we suppose that the solution for the eq. (6) when p_i, σ_{1i}, σ_{2i}, (i = 1, 2), are not equal to zero as

$$r_1 = 1 + \alpha,\; r_2 = 1 + \beta, \text{ where } \alpha, \beta << 1. \quad \text{... (7)}$$

Putting the values of r_1 and r_2 from the eq. (7) in the eq. (3), we get

$$x = \mu - \frac{1}{2} + \beta - \alpha,$$

and

$$y = \pm \frac{\sqrt{3}}{2} \left[1 + \frac{2}{3} (\alpha + \beta) \right]. \quad \text{... (8)}$$

Putting the values of r_1, r_2 from eq. (7) and x, y from eq. (8) in eq. (6), rejecting higher order terms, we get

$$\alpha = -\frac{1}{3} p_1 - \frac{11}{8} \sigma_{11} + \frac{11}{8} \sigma_{21} + \frac{1}{2(1-\mu)} (-2 + 3\,\mu)\, \sigma_{12} + \frac{1}{2(1-\mu)} (1 - 2\,\mu)\, \sigma_{22},$$

$$\beta = -\frac{1}{3} p_2 - \frac{1}{2\mu} (3\,\mu - 1)\, \sigma_{11} - \frac{1}{2\,\mu} (1 - 2\,\mu)\, \sigma_{21} - \frac{11}{8} \sigma_{12} + \frac{11}{8} \sigma_{22}.$$

Then we get the co-ordinates (x, y) of the libration points $L_{4,5}$ as

$$x = \mu - \frac{1}{2} + \frac{1}{3} (p_1 - p_2) + \frac{1}{8\,\mu} (4 - \mu)\, \sigma_{11} - \frac{1}{8\,\mu} (4 + 3\,\mu)\, \sigma_{21} - \frac{1}{8(1-\mu)} (3 + \mu)\, \sigma_{12} + \frac{1}{8(1-\mu)} (7 - 3\,\mu)\, \sigma_{22},$$

and
$$y=\pm\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{-\frac{1}{3}(p_1+p_2)+\frac{1}{8\mu}(4-23\mu)\sigma_{11}+\frac{1}{8\mu}(-4+19\mu)\sigma_{21}\right.\right.$$
$$\left.\left.+\frac{1}{8(1-\mu)}(-19+23\mu)\sigma_{12}+\frac{1}{8(1-\mu)}(15-19\mu)\sigma_{22}\right\}\right].$$

Case (b) — The Collinear Libration Points are the solutions of the equations

$$y = 0$$

and
$$f(x)=n^2x+\sum_{i=1}^{2}\left[(p_i-1)\frac{\mu_i}{r_i^3}(x-x_i)-\frac{3\mu_i}{2r_i^5}(2\sigma_{1i}-\sigma_{2i})(x-x_i)\right]=0, \quad \text{... (9)}$$

where $r_i=|x-x_i|, (i=1,2)$.

Obviously they lie on the x-axis and their abscissae are the roots of eq. (9). Since $f(x) > 0$ in each of the open intervals $(-\infty, \mu-1), (\mu-1, \mu)$ and (μ, ∞), the function f is strictly increasing in each of the them.

Also,

$$f(x)\to-\infty \text{ as } x\to-\infty, (\mu-1)+0 \text{ or } \mu+0,$$

and
$$f(x)\to\infty \text{ as } x\to\infty, (\mu-1)-0 \text{ or } \mu-0.$$

Therefore, there exists one and only one value of x in eachof the above intervals such that $f(x) = 0$. Further, $f(\mu-2)<0, f(0)\geq 0$ and $f(\mu+1)>0$. Therefore, there are only three real roots of eq. (9), one lying in each of the intervals $(\mu-2, \mu-1), (\mu-1, 0)$ and $(\mu, \mu+1)$. Thus there are three collinear libration points.

4. Stability of Libration Points

Case (a) — Stability of Triangular Libration Points

Now, we write the variational equations by putting $x = a + \xi$ and $y=b+\eta$ in the equations of motion (1), where (a, b) are the co-ordinates of L_4 (or L_5) and $\xi, \eta << 1$.

The variational equations of motion are

$$\ddot{\xi}-2n\dot{\eta}=\Omega_{xx}^{0}\xi+\Omega_{xy}^{0}\eta,$$

and
$$\ddot{\eta}+2n\dot{\xi}=\Omega_{xy}^{0}\xi+\Omega_{yy}^{0}\eta. \quad \text{... (10)}$$

Here we have taken only linear terms in ξ and η. The subscript in Ω indicates the second partial derivative of Ω and superscript indicates that the derivative is to be evaluated at the libration point (a, b).

The characteristic equation corresponding to eq. (10) is

$$\lambda^4+(4n^2-\Omega_{xx}^{0}-\Omega_{yy}^{0})\lambda^2+\Omega_{xx}^{0}\Omega_{yy}^{0}-(\Omega_{xy}^{0})^2=0, \quad \text{... (11)}$$

where
$$\Omega_{xx}^{0} = \frac{3}{4}\left[1 + \frac{2}{3}(-1 + 3\mu) p_1 + \frac{2}{3}(2 - 3\mu) p_2 + \frac{1}{4\mu}(-8 + 19\mu + 15\mu^2)\sigma_{11} + \frac{1}{4\mu}(8 - \mu - 31\mu^2)\sigma_{21} + \frac{1}{4(1-\mu)}(26 - 49\mu + 15\mu^2)\sigma_{12} + \frac{1}{4(1-\mu)}(-24 + 63\mu - 31\mu^2)\sigma_{22}\right],$$

$$\Omega_{xy}^{0} = \frac{3}{2}\sqrt{3}\left[\mu - \frac{1}{2} + \frac{1}{9}(1 + \mu) p_1 + \frac{1}{9}(-2 + \mu) p_2 + \frac{1}{24\mu}(8 - 47\mu + 89\mu^2)\sigma_{11} + \frac{1}{24\mu}(-8 + 9\mu - 37\mu^2)\sigma_{21} + \frac{1}{24(1-\mu)}(-50 + 131\mu - 89\mu^2)\sigma_{12} + \frac{1}{24(1-\mu)}(36 - 65\mu + 37\mu^2)\sigma_{22}\right],$$

$$\Omega_{yy}^{0} = \frac{9}{4} + \frac{1}{2}(1 - 3\mu) p_1 + \frac{1}{2}(-2 + 3\mu) p_2 + \frac{3}{16\mu}(8 + 29\mu - 15\mu^2)\sigma_{11} + \frac{3}{16\mu}(-8 - 7\mu + 15\mu^2)\sigma_{21} + \frac{3}{16(1-\mu)}(22 + \mu - 15\mu^2)\sigma_{12} + \frac{3}{16(1-\mu)}(-23\mu + 15\mu^2)\sigma_{22}.$$

Replacing λ^2 by Λ in eq. (11), we get

$$\Lambda^2 + P\Lambda + Q = 0, \qquad \text{... (12)}$$

where
$$P = 1 + 3\sigma_{11} + \frac{3}{2}(-3 + 2\mu)\sigma_{21} + 3\sigma_{12} - \frac{3}{2}(1 + 2\mu)\sigma_{22} > 0,$$

$$Q = \frac{27}{4}\mu(1-\mu) + \frac{3}{2}\mu(1-\mu) p_1 + \frac{3}{2}\mu(1-\mu) p_2 + \frac{9}{16}(1-\mu)(-10 + 89\mu)\sigma_{11} + \frac{9}{16}(1-\mu)(10 - 37\mu)\sigma_{21} + \frac{9}{16}\mu(79 - 89\mu)\sigma_{12} + \frac{9}{16}\mu(-27 + 37\mu)\sigma_{22}. \qquad \text{... (13)}$$

and
$$\Lambda_{1,2} = \frac{1}{2}[-P \pm \sqrt{P^2 - 4Q}].$$

Consequently, the roots $\lambda_1 = +\Lambda_1^{1/2}$, $\lambda_2 = -\Lambda_1^{1/2}$, $\lambda_3 = +\Lambda_2^{1/2}$ and $\lambda_4 = -\Lambda_2^{1/2}$ depend, in a simple manner, on the value of the mass parameter μ, p_i, σ_{1i} and σ_{2i} $(i = 1, 2)$.

Now the discriminant of eq. (12) is zero if $P^2 - 4Q = 0$. That is,

$$1 - 27\,\mu(1-\mu) - 6\,\mu(1-\mu)\,p_1 - 6\,\mu(1-\mu)\,p_2 - \frac{3}{4}[-38 + 297\,\mu - 267\,\mu^2]\,\sigma_{11}$$

$$-\frac{3}{4}[42 - 149\,\mu + 111\,\mu^2]\,\sigma_{21} + \frac{3}{4}[8 - 237\,\mu + 267\,\mu^2]\,\sigma_{12}$$

$$+\frac{3}{4}[-4 + 73\,\mu - 111\,\mu^2]\,\sigma_{22} = 0. \qquad \ldots (14)$$

If p_i, σ_{1i} and σ_{2i}, $(i = 1, 2)$, are equal to zero, then $\mu = \mu_0$ is a root of the eq. (14) where $\mu_0 = 0.0385208965$ ... (Szebehely)[1]. When p_i, σ_{1i} and σ_{2i}, $(i = 1, 2)$, are not equal to zero, we suppose $\mu_{crit} = \mu_0 + r_1 p_1 + r_2 p_2 + r_3\,\sigma_{11} + r_4\,\sigma_{21} + r_5\,\sigma_{12} + r_6\,\sigma_{22}$ as the root of the eq. (14), where $r_1, r_2, r_3, r_4, r_5, r_6$ are to be determined in such a manner such that $P^2 - 4Q = 0$. Therefore, we have,

$$r_1 = -\frac{2(1-\mu_0)}{9(1-2\,\mu_0)}\,\mu_0,$$

$$r_2 = -\frac{2(1-\mu_0)}{9(1-2\,\mu_0)}\,\mu_0,$$

$$r_3 = \frac{1}{36(1-2\,\mu_0)}[38 - 297\,\mu_0 + 267\,\mu_0^2],$$

$$r_4 = \frac{1}{36(1-2\,\mu_0)}[-42 + 149\,\mu_0 - 111\,\mu_0^2],$$

$$r_5 = \frac{1}{36(1-2\,\mu_0)}[8 - 237\,\mu_0 + 267\,\mu_0^2],$$

$$r_6 = \frac{1}{36(1-2\,\mu_0)}[-4 + 73\,\mu_0 - 111\,\mu_0^2].$$

$\mu_{crit} = 0.0385208965... - 0.0089174706\,p_1 - 0.0089174706\,p_2 + 0.81126474\,\sigma_{11} - 1.09626653\,\sigma_{21} - 0.02206859\,\sigma_{12} - 0.04071097\,\sigma_{22}$. ... (15)

Now we shall treat the three regions of the values of μ separately.

(i) $0 \le \mu < \mu_{crit}$

We have, $-\frac{1}{2}P < \Lambda_1 \le 0$ and $-\frac{1}{2}P > \Lambda_2 \ge -P$.

But $P > 0$, therefore, Λ_1 and Λ_2 are negative. Therefore, in this case, the four roots of the characteristic equation are written as

$$\lambda_{1,2} = \pm i\,(-\Lambda_1)^{1/2} = \pm i s_1$$

and

$$\lambda_{3,4} = \pm i\,(-\Lambda_2)^{1/2} = \pm i s_2. \qquad \text{... (16)}$$

This shows that the libration points are stable.

The solution is given by

$$\xi = C_1 \cos s_1 t + S \sin s_1 t + C_2 \cos s_2 t + S_2 \sin s_2 t,$$

and

$$\eta = \overline{C}_1 \cos s_1 t + \overline{S}_1 \sin s_1 t + \overline{C}_2 \cos s_2 t + \overline{S}_2 \sin s_2 t$$

where

$$\overline{C}_i = \Gamma_i\,(2n\, S_i\, s_i - \Omega^0_{xy}\, C_i),$$

$$\overline{S}_i = -\Gamma_i\,(2n\, C_i\, S_i - \Omega^0_{xy}\, S_i),$$

and

$$\Gamma_i = \frac{1}{S_i^2 + \Omega^0_{yy}} > 0,\ i = 1, 2. \text{ (Szebehely}^1\text{, pp. 251)}$$

Also, for $\mu \cong 0$, from eq: (16), we get,

$$s_1 = \frac{3}{2}\left[\, 3\,\mu - \frac{5}{2}\,\sigma_{11} + \frac{5}{2}\,\sigma_{21}\right]^{1/2},$$

and

$$s_2 = 1 - \frac{27}{8}\,\mu + \frac{69}{16}\,\sigma_{11} - \frac{81}{16}\,\sigma_{21} + \frac{3}{2}\,\sigma_{12} - \frac{3}{4}\,\sigma_{22}. \qquad \text{... (17)}$$

Evidently $s_1 \leq s_2$.

Therefore, the terms with the coefficients $C_1, S_1, \overline{C}_1, \overline{S}_1$ are called long period terms and the terms with the coefficients $C_2, S_2, \overline{C}_2, \overline{S}_2$ are the short period terms.

The expression of Ω around $L_{4,5}$ is

$$\Omega = \Omega^0 + \Omega^0_{xx}\,\frac{\xi^2}{2} + \Omega^0_{xy}\,\xi\,\eta + \Omega^0_{yy}\,\frac{\eta^2}{2} + O\,(3),$$

or

$$\Omega = \frac{3}{2} + (-1+\mu)\,p_1 - \mu\,p_2 + \frac{1}{8}\,(11+\mu)\,\sigma_{11} - \frac{1}{8}\,(1+5\,\mu)\,\sigma_{21}$$

$$+ \frac{1}{8}\,(12-\mu)\,\sigma_{12} + \frac{1}{8}\,(-6+5\,\mu)\,\sigma_{22}$$

$$+ \frac{3}{8}\,\xi^2 + \frac{1}{4}\,(-1+3\,\mu)\,p_1\,\xi^2 + \frac{1}{4}\,(2-3\,\mu)\,p_2\,\xi^2 + \frac{3}{32\,\mu}\,(-8+19\,\mu+15\,\mu^2)\,\sigma_{11}\,\xi^2$$

$$+ \frac{3}{32\,\mu}(8-\mu-31\,\mu^2)\,\sigma_{21}\,\xi^2+\frac{3}{32\,(1-\mu)}(26-49\,\mu+15\,\mu^2)\,\sigma_{12}\,\xi^2$$

$$+ \frac{3}{32\,(1-\mu)}(-24+63\,\mu-31\,\mu^2)\,\sigma_{22}\,\xi^2$$

$$+ \frac{3}{2}\sqrt{3}\left(\mu-\frac{1}{2}\right)\xi\,\eta+\frac{\sqrt{3}}{6}(1+\mu)\,p_1\,\xi\,\eta+\frac{\sqrt{3}}{6}(-2+\mu)\,p_2\,\xi\,\eta$$

$$+ \frac{\sqrt{3}}{16\,\mu}(8-47\,\mu+89\,\mu^2)\,\sigma_{11}\,\xi\,\eta+\frac{\sqrt{3}}{16\,\mu}(-8+9\,\mu-37\,\mu^2)\,\sigma_{21}\,\xi\,\eta$$

$$+ \frac{\sqrt{3}}{16\,(1-\mu)}(-50+131\,\mu-89\,\mu^2)\,\sigma_{12}\,\xi\,\eta+\frac{\sqrt{3}}{16\,(1-\mu)}(36-65\,\mu+37\,\mu^2)\,\sigma_{22}\,\xi\,\eta$$

$$+ \frac{9}{8}\eta^2+\frac{1}{4}(1-3\,\mu)\,p_1\,\eta^2+\frac{1}{4}(-2+3\,\mu)\,p_2\,\eta^2+\frac{3}{32\,\mu}(8+29\,\mu-15\,\mu^2)\,\sigma_{11}\,\eta^2$$

$$+ \frac{3}{32\,\mu}(-8-7\,\mu+15\,\mu^2)\,\sigma_{21}\,\eta^2+\frac{3}{32\,(1-\mu)}(22+\mu-15\,\mu^2)\,\sigma_{12}\,\eta^2$$

$$+ \frac{3}{32\,(1-\mu)}(-23\,\mu+15\,\mu^2)\,\sigma_{22}\,\eta^2.$$

Now, let us introduce the variables $\overline{\xi}$ and $\overline{\eta}$ by the transformation

$$\xi=\overline{\xi}\cos\alpha-\overline{\eta}\sin\alpha$$

and

$$\eta=\overline{\xi}\sin\alpha+\overline{\eta}\cos\alpha.$$

This is equivalent to a rotation of the co-ordinate system by an angle α. We choose α in such a way that the term containing $\overline{\xi}\,\overline{\eta}$ in Ω = 0.

The new quadratic form becomes

$$\Omega=\overline{l}\,\overline{\xi}2+\overline{m}\,\overline{\eta}^{2}+\overline{n},$$

where

$$\overline{l}=\frac{3}{8}+\frac{3}{4}\sin^2\alpha-\frac{3}{8}\sqrt{3}\,(1-2\,\mu)\sin 2\,\alpha$$

$$+\frac{1}{4}\left[(-1+3\,\mu)\cos 2\alpha+\frac{\sqrt{3}}{3}(1+\mu)\sin 2\,\alpha\right]p_1$$

$$+\frac{1}{4}\left[(2-3\,\mu)\cos 2\,\alpha+\frac{\sqrt{3}}{3}(-2+\mu)\sin 2\,\alpha\right]p_2$$

$$+\frac{3}{32\,\mu}\left[19\,\mu+10\,\mu\sin^2\alpha+(-8+15\,\mu^2)\cos 2\,\alpha+\frac{\sqrt{3}}{3}(8-47\,\mu+89\,\mu^2)\sin 2\,\alpha\right]\sigma_{11}$$

$$+\frac{3}{32\,\mu}\left[-\mu-16\,\mu^2\cos^2\alpha-6\,\mu\sin^2\alpha+(8-15\,\mu^2)\cos 2\,\alpha+\frac{\sqrt{3}}{3}(-8+9\,\mu-37\,\mu^2)\sin 2\,\alpha\right]\sigma_{21}$$

$$+ \frac{3}{32(1-\mu)}\left[22 - 15\mu^2 + 4(1-12\mu)\cos^2\alpha - \mu\cos 2\alpha + \frac{\sqrt{3}}{3}(-50 + 131\mu - 89\mu^2)\sin 2\alpha\right]\sigma_{12}$$

$$+ \frac{3}{32(1-\mu)}\left[8(1-\mu)(-3+2\mu)\cos^2\alpha + (23\mu - 15\mu^2)\cos 2\alpha + \frac{\sqrt{3}}{3}(36 - 65\mu + 37\mu^2)\sin 2\alpha\right]\sigma_{22}$$

$$\overline{m} = \frac{3}{8} + \frac{3}{4}\cos^2\alpha + \frac{3}{8}\sqrt{3}(1-2\mu)\sin 2\alpha$$

$$+ \frac{1}{4}\left[(1-3\mu)\cos 2\alpha - \frac{\sqrt{3}}{3}(1+\mu)\sin 2\alpha\right]p_1 + \frac{1}{4}\left[(-2+3\mu)\cos 2\alpha - \frac{\sqrt{3}}{3}(-2+\mu)\sin 2\alpha\right]p_2$$

$$+ \frac{3}{32\mu}\left[19\mu + 10\mu\cos^2\alpha + (8 - 15\mu^2)\cos 2\alpha - \frac{\sqrt{3}}{3}(8 - 47\mu + 89\mu^2)\sin 2\alpha\right]\sigma_{11}$$

$$+ \frac{3}{32}\mu\left[-\mu - 6\mu\cos^2\alpha - 16\mu^2\sin^2\alpha + (-8 + 15\mu^2)\cos 2\alpha - \frac{\sqrt{3}}{3}(-8 + 9\mu - 37\mu^2)\sin 2\alpha\right]\sigma_{21}$$

$$+ \frac{3}{32(1-\mu)}\left[22 - 15\mu^2 + 4(1-12\mu)\sin^2\alpha + \mu\cos 2\alpha - \frac{\sqrt{3}}{3}(-50 + 131\mu - 89\mu^2)\sin 2\alpha\right]\sigma_{12}$$

$$+ \frac{3}{32(1-\mu)}\left[8(1-\mu)(-3+2\mu)\sin^2\alpha + (-23 + 15\mu^2)\cos 2\alpha - \frac{\sqrt{3}}{3}(36 - 65\mu + 37\mu^2)\sin 2\alpha\right]\sigma_{22}$$

and $\overline{n} = \frac{3}{2} + (-1+\mu)p_1 - \mu p_2 + \frac{1}{8}(11+\mu)\sigma_{11} - \frac{1}{8}(1+5\mu)\sigma_{21}$

$$+ \frac{1}{8}(12-\mu)\sigma_{12} + \frac{1}{8}(-6+5\mu)\sigma_{22} \qquad \ldots (18)$$

and $\tan 2\alpha = \frac{N}{D}$,

where $N = -\frac{3}{2}\sqrt{3}\left[\mu - \frac{1}{2} + \frac{1}{9}(1+\mu)p_1 + \frac{1}{9}(-2+\mu)p_2 + \frac{1}{24\mu}\{8 - 47\mu + 89\mu^2\}\ \sigma_{11}\right.$

$$+ \frac{1}{24\mu}\{-8 + 9\mu - 37\mu^2\}\ \sigma_{21} + \frac{1}{24(1-\mu)}\{-50 + 131\mu - 89\mu^2\}\ \sigma_{12}$$

$$\left. + \frac{1}{24(1-\mu)}(36 - 65\mu + 37\mu^2)\sigma_{22}\right]$$

$$D = \frac{3}{4} + \frac{1}{2}(1-3\mu)p_1 + \frac{1}{2}(-2+3\mu)p_2 + \frac{3}{16\mu}\{8 + 5\mu - 15\mu^2\}\ \sigma_{11}$$

$$+ \frac{3}{16\mu}\{-8 - 3\mu + 23\mu^2)\sigma_{21} + \frac{3}{16(1-\mu)}(-2+25\mu)\sigma_{12}$$

$$+ \frac{3}{16(1-\mu)}(12 - 43\mu + 23\mu^2)\sigma_{22}.$$

Also, using the Jacobian constant, we have

$$C = 2\,\Omega = 2\,\bar{l}\,\bar{\xi}^2 + 2\,\bar{m}\,\bar{\eta}^2 + 2\,\bar{n}.$$

Hence, it follows that the above curve is an ellipse and the direction 'α' of the major axis is given by eq. (19). The lengths of the semi-major and semi-minor axes are given by

$$a_{sM} = \left[\frac{C - 2\,\bar{n}}{2\,\bar{l}}\right]^{1/2} \text{ and } b_{sm} = \left[\frac{C - 2\,\bar{n}}{2\,\bar{m}}\right]^{1/2}, \quad \ldots (20)$$

where $\bar{l}, \bar{m}$ and $\bar{n}$ are given by eq. (18) and C depends on the initial conditions.

(ii) $\mu_{crit} < \mu < \frac{1}{2}$:

When $\mu_{crit} < \mu < \frac{1}{2}$, the discriminant of the characteristic equation is negative.

Also, $\Lambda_{1,2} = \frac{1}{2}[-P \pm \sqrt{d}]$

where P is given by eq. (13) and $d = P^2 - 4Q$.

Therefore, $\Lambda_{1,2} = \frac{1}{2}[-P \pm i\,\delta]$,

where $$0 < \delta = +\sqrt{d}$$

$$= \left[27\,\mu\,(1-\mu) - 1 + 6\,\mu\,(1-\mu)\,p_1 + 6\,\mu\,(1-\mu)\,p_2 - \frac{3}{4}(38 - 297\,\mu + 267\,\mu^2)\,\sigma_{11} \right.$$
$$- \frac{3}{4}(-42 + 149\,\mu - 111\,\mu^2)\,\sigma_{21} - \frac{3}{4}(8 - 237\,\mu + 267\,\mu^2)\,\sigma_{12}$$
$$\left. - \frac{3}{4}(-4 + 73\,\mu - 111\,\mu^2)\,\sigma_{22}\right]^{1/2} \quad \ldots (21)$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \pm\,\Lambda_1^{1/2},\ \lambda_{3,4} = \pm\,\Lambda_2^{1/2}.$$

or $$\lambda_1 = \frac{1}{\sqrt{2}}\sqrt{-P + i\,\delta} = \alpha_1 + i\,\beta_1,$$

$$\lambda_2 = -\frac{1}{\sqrt{2}}\sqrt{-P + i\,\delta} = \alpha_2 + i\,\beta_2,$$

$$\lambda_3 = \frac{1}{\sqrt{2}}\sqrt{-P - i\,\delta} = \alpha_3 + i\,\beta_3,$$

and $$\lambda_4 = -\frac{1}{\sqrt{2}}\sqrt{-P - i\,\delta} = \alpha_4 + i\,\beta_4.$$

The lengths of these roots are equal and given by

$$|\lambda| = |\lambda_{1,2,3,4}| = \frac{1}{\sqrt{2}} \sqrt[4]{P^2 + \delta^2},$$

where P and δ are given by the eqs. (13) and (21).

The principal argument of the first root is

$$\theta = \theta_1 = arc \tan\left[\frac{P \pm \sqrt{P^2 + \delta^2}}{\delta}\right].$$

The arguments of the four roots are related by

$$\theta = \theta_1 = \theta_2 - \Pi = 2\,\Pi - \theta_3 = \Pi - \theta_4.$$

The real and imaginary parts of the roots, α_1 and β_1, are related by

$$\alpha = \alpha_1 = -\alpha_2 - \alpha_3 = -\alpha_4$$

and

$$\beta = \beta_1 = -\beta_2 = -\beta_3 = \beta_4,$$

where,

$$\alpha = \frac{\delta}{2\sqrt{2|\lambda|^2 + P}} > 0,$$

and

$$\beta = \frac{\sqrt{P + 2|\lambda|^2}}{2} > 0.$$

Therefore, it follows that the real parts of two of the characteristic roots are positive (and equal) and so the equilibrium point in this case is unstable.

(*iii*) $\mu = \mu_{crit}$:

When $\mu = \mu_{crit}$, $d = 0$.

Consequencly, $\Lambda_{1,2} = -\frac{1}{2} P$ and $\lambda_1 = \lambda_3 = i\sqrt{\frac{1}{2}P}$, $\lambda_2 = \lambda_4 = -i\sqrt{\frac{1}{2}P}$.

The double roots give sècular terms in the solution of the equations of motion and so the equilibrium point is unstable.

Case (*b*) — Stability of Collinear Libration Points

First we consider the point lying in $(\mu - 2, \mu - 1)$.

For this point, $r_2 < 1, r_1 > 1$, we have

$$\Omega^0_{xy} = 0,$$

$$\Omega^0_{xx} = n^2 + 2(1 - p_1)\frac{1-\mu}{r_1^3} + 2(1 - p_2)\frac{\mu}{r_2^3} + 6\frac{1-\mu}{r_1^5}(2\,\sigma_{11} - \sigma_{21})$$

$$+\ 6\frac{\mu}{r_2^5}\,(2\,\sigma_{12}-\sigma_{22})>0,$$

$$\Omega_{yy}^0=\mu\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\left(r_2-\frac{1}{r_2^2}\right)+\frac{\mu}{r_2}\left(-\frac{1}{r_1}+\frac{1}{r_2}\right)p_2+\frac{3\,\mu}{2\,r_1}\,(2\,\sigma_{11}-\sigma_{21}+2\,\sigma_{12}-\sigma_{22})$$

$$+\ \frac{3\,(1-\mu)}{2\,r_1^5}\,(-2\,\sigma_{11}+2\,\sigma_{21})+\frac{3\,\mu}{2r_1r_2^4}\,(2\,\sigma_{12}-\sigma_{22})-\frac{3\,\mu}{2\,r_2^5}\,(4\,\sigma_{12}-3\,\sigma_{22})<0.$$

Similarly, for the points lying in $(\mu-1,0)$ and $(\mu,\mu+1)$,

$$\Omega_{xy}^0=0,\ \Omega_{xx}^0>0\ \text{and}\ \Omega_{yy}^0<0.$$

Because $\Omega_{xx}^0\,\Omega_{yy}^0-(\Omega_{xy}^0)^2<0$, the discriminant is positive and the four roots of the characteristic equation can be written as $\lambda_1=s$, $\lambda_2=-s$, $\lambda_3=it$ and $\lambda_4=-it$ (s and t are real). So the motion around the collinear points is unbounded and consequently, the collinear points are unstable.

5. Conclusion

In the restricted three body problem, when the primaries are triaxial rigid bodies as well as source of radiations, there are five libration points, three collinear and two triangular. The collinear points are unstable for all values of the mass parameter μ such that $0<\mu\leq\frac{1}{2}$ and the triangular points are stable for $\mu<\mu_{crit}$ where μ_{crit} is given by eq. (15).

(i) For $0\leq\mu<\mu_{crit}$

where μ_{crit} = 0.0385208965... − 0.0089174706 p_1 − 0.0089174706 p_2

+ 0.81126474 σ_{11} − 1.09626653 σ_{21} − 0.02206859 σ_{12} − 0.04071097 σ_{22}.

(*a*) L_4 is stable.

(*b*) s_1, s_2, the short periodic term and long periodic term frequencies are found out in eq. (17).

(*c*) the lengths of the semi-major axis 'a_{sM}' and the semi-minor axis 'b_{sm}' of the ellipse around L_4 are determined in the eq. (20) and the direction 'α' of the major axis is given by eq. (19).

(*ii*) *For* $\mu_{crit}<\mu<\frac{1}{2}$

L_4 is unstable.

(*iii*) *For* $\mu=\mu_{crit}$

L_4 is unstable.

When both the primaries are spheres or point masses i.e., $p_i = \sigma_{1i} = \sigma_{2i} = 0$, $(i = 1, 2)$, the results obtained are in agreement with the classical problem (Szebehely[1]). When both the primaries are oblate spheroids i.e., $p_1 = p_2 = 0$, $\sigma_{11} = \sigma_{21}$ and $\sigma_{12} = \sigma_{22}$, the results are in agreement with those of Vidyakin. When the bigger primary is an oblate spheroid and the smaller one a sphere i.e., $p_1 = p_2 = \sigma_{12} = \sigma_{22} = 0$ and $\sigma_{11} = \sigma_{21}$, the results obtained are in agreement with those of Subba Rao and Sharma[4]. When the bigger primary is a sphere and the smaller one an oblate spheroid i.e., $p_1 = p_2 = \sigma_{11} = \sigma_{21} = 0$ and $\sigma_{12} = \sigma_{22}$, the results obtained are in agreement with those of Bhatnagar and Hallan[10]. When the bigger primary is a sphere and the smaller one a triaxial rigid body i.e., $p_1 = p_2 = \sigma_{11} = \sigma_{21} = 0$, the results obtained are in agreement with those of Khanna and Bhatnagar[6]. When the bigger primary is an oblate spheroid and the smaller one a triaxial rigid body i.e. $p_1 = p_2 = 0$, $\sigma_{11} = \sigma_{21}$, the results obtained are in agreement with those of Khanna and Bhatnagar[7]. When the bigger primary is a triaxial rigid body as wel as source of radiation and the smaller one a sphere i.e., $p_2 = \sigma_{12} = \sigma_{22} = 0$, the results obtained are in agreement with those of Sharma, Taqvi and Bhatnagar[8]. When the primaries are traxial rigid bodies i.e. $p_1 = p_2 = 0$ the result obtained are in agreement with those of Sharma *et al.* If we put $p_2 = 0$ and $\sigma_{12} = \sigma_{22}$ we get the desired results for the case when the bigger primary is a triaxial rigid body as well as source of radiation and the smaller an oblate spheroid. If we put $\sigma_{12} = \sigma_{22}$ we get the desired results for the case when the bigger primary is a triaxial rigid body as well as source of radiation and the smaller an oblate spheroid as well as source of radiation. If we put $\sigma_{11} = \sigma_{21}$ and $\sigma_{12} = \sigma_{22}$ we get the desired results for the case when both the primaries are oblate spheroids as well as source of radiations.

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REFERENCES

1. V. Szebehely, *Theory of Orbits*, Academic Press, New York, 1967.
2. A. Wintner, 1941, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton, 1941, pp. 372.
3. V. V. Vidyakin, (*in Russian*) : 1974, *Astron Zh*, **51** (1974), 5, 1087.
4. P. V. Subba Rao and R. K. Sharma, *Astron. Astrophys.* **43** (1975), 381.
5. S. M. El-Shaoury, *Celest. Mech. Dyn. Astron.* **50** (1991) 199-208.
6. M. Khanna and K. B. Bhatnagar, *Indian J. pure appl. Math.* **29** (10) (1998)
7. M. Khanna and K. B. Bhatnagar, *Indian J. pure appl. Math.* **30** (7) (1999)
8. R. K. Sharma, Z. A. Taqvi and K. B. Bhatnagar, *Indian J. pure appl. Math.* **32(6)** 2001.
9. S. W. McCusky, *Introduction to Celestial Mechanics*, Addision-Wesley (1963).
10. K. B. Bhatnagar and P. P. Hallan, *Celest. Mech.* **20** (1979) 95-103.
11. R. K. Sharma, Z. A. Taqvi and K. B. Bhatnagar, *Celest. Mech.* (Accepted)