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ALMOST INCREASING SEQUENCES AND THEIR APPLICATIONS

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Recently, Bor[3] proved a theorem on $|\bar{N}, p_n|_k$ summability factors of an infinite series. In this paper, we prove a general theorem on $|\bar{N}, p_n, \phi_n|_k$ summability Factors of an infinite series, under weaker conditions by using an almost increasing sequence in place of an increasing sequence.

Key Words : Infinite Series; Summability; Summability Factors; Almost Increasing Sequence

1. INTRODUCTION

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with (s_n) as the sequence of its partial sums. Let (σ_n) and (t_n) denote the nth $(C, 1)$ means of the sequence (s_n) and (na_n) respectively. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|C, 1|_k, k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|_k < \infty. \quad \text{... (1)}$$

In view of the fact that $t_n = n(\sigma_n - \sigma_{n-1})$ (see [5]),

(1) can be written as

$$\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty. \quad \text{... (2)}$$

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty, \text{ as } n \to \infty \; (P_{-i} = P_{-i} = 0, i \geq 1). \quad \text{... (3)}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \qquad \text{... (4)}$$

defines the sequence (w_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [2]). The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty. \qquad \text{... (5)}$$

Let (ϕ_n) be any sequence of positive real constants. Then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|\overline{N}, p_n, \phi_n|_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |w_n - w_{n-1}|^k < \infty. \qquad \text{... (6)}$$

Clearly $\left|\overline{N}, p_n, \frac{P_n}{p_n}\right|_k = |\overline{N}, p_n|_k$, $|\overline{N}, p_n, 1|_1 = |\overline{N}, p_n|$, and $|\overline{N}, 1, n|_k = |C, 1|_k$.

We need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that

$$Ac_n \leq b_n \leq Bc_n \text{ (see [8]).}$$

Obviously every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = n \exp(-1)^n$.

Recently, Bor[3] has proved the following theorem on $|\overline{N}, p_n|_k$ summability factors of an infinite series.

Theorem A — *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(n p_n) \text{ as } n \to \infty. \qquad \text{... (7)}$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$\lambda_m X_m = O(1), \text{ as } m \to \infty, \qquad \text{... (8)}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1), \qquad \dots (9)$$

and
$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m), \text{ as } m \to \infty, \qquad \dots (10)$$

then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$.

2. Main Result

The aim of this paper is to generalize Theorem A for $|\overline{N}, p_n, \phi_n|_k$ summability. In addition we are also weakening the hypotheses of the theorem A by replacing the increasing sequence (X_n) by an almost increasing sequence.

In fact, we shall prove the following theorem :

Theorem — *Let (X_n) be an almost increasing sequence and the sequences (λ_n) and (p_n) are such that the conditions (7) — (9) of Theorem A are satisfied and*

$$\sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m), \text{ as } m \to \infty, \qquad \dots (11)$$

where (ϕ_n) be a sequence of positive real constants such that $\left(\frac{\phi_n p_n}{P_n}\right)$ is non-increasing sequence, then the series $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable $|\overline{N}, p_n, \phi_n|_k, k \geq 1$.

Remark : It should be noted that, if we take (X_n) as a positive non-decreasing sequence and $\phi_n = \frac{P_n}{p_n}$ in this theorem, then we get theorem A. In this case condition (11) reduces to the condition (10) and the condition $\left(\frac{\phi_n p_n}{P_n}\right)$ is non-increasing sequence becomes redundant.

We need the following lemma for the proof of our theorem.

Lemma[7] — Under the conditions on (X_n) and (λ_n) which are taken in the statement of our theorem, the following conditions hold :

$$(i)\ nX_n |\Delta \lambda_n| = O(1) \text{ as } n \to \infty, \qquad \dots (12)$$

$$(ii)\ \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty \qquad \dots (13)$$

and
$$(iii)\ X_n |\lambda_n| = O(1), \text{ as } n \to \infty. \qquad \dots (14)$$

3. Proof of the Theorem

Let (T_n) be the sequence of (N, p_n) means of the series $\sum_{n=1}^{\infty} a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v s_v = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{z=0}^{v} a_z \lambda_z = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v.$$

Then, for $n \geq 1$, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \lambda_v v a_v}{v}. \qquad \ldots (15)$$

Now applying Able's transformation to the right-hand side of (15) we get

$$T_n - T_{n-1} = \frac{(n+1) p_n t_n \lambda_n}{n P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v}$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \Delta \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v}.$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}$$

Since $\qquad |T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k$

$$\leq 4^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k),$$

to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} \phi_n^{k-1} \, |T_{n-z}|^k < \infty \text{ for } z = 1, 2, 3, 4. \qquad \ldots (16)$$

First we have

$$\sum_{n=1}^{m} \phi_n^{k-1} |T_{n,1}|^k = \sum_{n=1}^{m} \phi_n^{k-1} \left| \frac{(n+1) p_n t_n \lambda_n}{n P_n} \right|^k$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |t_n|^k |\lambda_n|^{k-1} |\lambda_n|$$

$$= O(1) \sum_{n=1}^{m} \phi_n^{k-1} |\lambda_n| \left(\frac{p_n}{P_n}\right)^k |t_n|^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \phi_v^{k-1} |t_v|^k \left(\frac{p_v}{P_v}\right)^k + O(1) |\lambda_m| \sum_{v=1}^{m} \phi_v^{k-1} |t_v|^k \left(\frac{p_v}{P_v}\right)^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| X_n + O(1) |\lambda_m| |X_m|, \text{ by } (11)$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| |X_m|$$

$$= O\ (1) \text{ as } m \to \infty, \text{ (by (13), (14))}.$$

Again for $k > 1$ and applying Holder's inequality with indices k and k and k', where $1/k + 1/k' = 1$, as in T_n, 1, we have

$$\sum_{n=2}^{m+1} \phi_n^{k-1} \quad |T_{n,2}|^k$$

$$= \sum_{n=2}^{m+1} \phi_n^{k-1} \quad \left| -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^k \sum_{n=1}^{v+1} \left(\frac{\phi_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v}\right)^{k-1} p_v |t_v|^k |\lambda_v|^k \sum_{n=1}^{v+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \phi_v^{k-1} |\lambda_n| \left(\frac{p_v}{P_v}\right)^k |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_v| \sum_{i=1}^{v} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k + O(1)\lambda_n \sum_{i=1}^{m} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_v| X_n + O(1) |\lambda_m| X_m, \text{ (by (11))}$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m$$

$$= O(1) \text{ as } m \to \infty, \text{ (by (13), (14)).}$$

Again we have

$$\sum_{n=2}^{m+1} \phi_n^{k-1} |T_{n,3}|^k$$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} v |\Delta\lambda_v|^k p_v |t_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v |\Delta\lambda_v|^k p_v |t_v|^k \sum_{n=1}^{v+1} \left(\frac{\phi_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{\phi_v p_v}{P_v}\right)^{k-1} v |\Delta\lambda_v| |p_v| t_v|^k \frac{1}{P_v}$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta(v|\Delta\lambda_v|)| \sum_{i=1}^{v} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$+ O(1)\, m |\Delta\lambda_m| \sum_{i=1}^{m} \phi_i^{k-1} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$= O(1) \sum_{v=1}^{m-1} v X_v |\Delta^2 \lambda_v| + O(1) \sum_{v=1}^{m-1} X_v |\Delta\lambda_{v+1}| + O(1)\, m |\Delta\lambda_m| X_m$$

$$= O(1) \text{ as } m \to \infty \text{ (by (9), (12) \& (13)).}$$

Finally, using the fact $P_n = O(np_n)$ by (7), as in $T_{n,1}$, we have that

$$\sum_{n=1}^{m+1} \phi_n^{k-1} \, |T_{n,4}|^k$$

$$= \sum_{n=1}^{m+1} \phi_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} \right|^k$$

$$= O(1) \sum_{v=1}^{m} \phi_v^{k-1} \, |\lambda_{n+1}| \left(\frac{p_v}{P_v} \right)^k |t_v|^k$$

$$= O(1) \text{ as } m \to \infty.$$

Therefore, we get

$$\sum_{n=1}^{\infty} \phi_n^{k-1} \, |T_{n,z}|^k < \infty \text{ for } z = 1, 2, 3, 4.$$

If we take $p_n = 1$ and $\phi_n = n$ for all values of n in this theorem, then we get a result concerning the $|C, 1|_k$ summability methods.

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