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# A NECESSARY AND SUFFICIENT CONDITION FOR THE MINIMAL POINT BEING A CRITICAL POINT AND APPLICATIONS*

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In this paper, we give a necessary and sufficient condition for the existence of critical points. Then we apply it to nonlinear integral equations of Hammerstein type, and find at least three solutions under suitable conditions.

**Key Words : Critical Points; Minimal Point; Variational Method; Nonlinear Integral Equations of Hammerstein Type**

## 1. INTRODUCTION

We all know a basic and important conclusion in variational calculus.

***Theorem A*** — *Suppose that $E$ is a Banach space, $D \subset E$ has interior points, $f: D \to R$ is a functional. If $f$ attains its minimal value at some interior point $x_0 \in D$, and if $f$ has bounded linear Gâteaux differential at $x_0$, then $f'(x_0) = \theta$, i.e., $f(x_0)$ is a critical value.*

The basic conditions in Theorem A are that $D$ has interior points, and $f$ attains minimal value at some interior point of $D$. It is clear that we cannot conclude that the minimal value of $f$ in $D$ is a critical value if these conditions are not satisfied. Thus we put forward a question: when $D$ has no interior point (we often meet with the case, for example the cone $P_p$ in $L_p$ $(p \geq 1)$, see the application in this paper) and $f(x)$ attains minimal value at some $x_0 \in D$, or $D$ has interior points, but $f$ attains minimal value at some boundary point $x_0 \in D$, what conditions do we need to add to ensure that the minimal value is a critical value?

We solve the above problem when $D$ is a convex closed set in a Hilbert space, and give a necessary and sufficient condition for the minimal value being a critical value. As an application, we discuss number of solutions of nonlinear integral equations of Hammerstein type. The results of[1,2] can only ensure the existence of solutions of nonlinear integral equations of Hammerstein type. In this paper, we use the abstract results of this paper to get the existence of nontrivial solutions of nonlinear integral equations of Hammerstein type even if the nonlinear integral equation of Hammerstein type has a trivial solution.

In this paper, we always suppose that $H$ is a Hilbert space, and $f: H \to R$ is a functional, defined by

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$$f(x) = \frac{1}{2} \| x \|^2 - g(x). \qquad \text{... (1)}$$

Assume that $D \subset H$ is a convex closed set, and we don't assume that $D$ is bounded or has interior points.

For $x \in D$, let

$$I_D(x) = \{(1-\lambda)x + \lambda y \mid y \in D, \lambda \geq 0\},$$

then $I_D(x)$ is called an inward set of $x$ relative to $D$ (see [3] [4]).

A map $A : H \to H$ is called an inward map on $D$ (see [3] [4]), if

$$Ax \in I_D(x), \ \forall x \in D.$$

Let $E$ be a real Banach space, A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following two conditions :

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;

(ii) $x \in P, -x \in P$ implies $x = \theta$, where $\theta$ denotes the zero element of $E$ (see[6]).

## 2. Main Theorem

***Theorem* 1** — *Suppose that $f(x)$ attains its minimal value at $x_0 \in D$ (i.e., there exists a neighbourhood $U$ of $x_0$ in $D$, such that $f(x_0) \leq f(x)$, $\forall x \in U$), and $f(x)$ has bounded linear Gâteaux differential at $x_0$. Then $f(x_0)$ is a critical value of $f(x)$ if and only if*

$$g'(x_0) \in I_D(x_0), \qquad \text{... (2)}$$

*where $g(x)$ is defined by* (1).

PROOF : If $f(x_0)$ is a critical value of $f(x)$, then $f'(x_0) = \theta$. From (1), we know that $g'(x_0) = x_0 \in I_D(x_0)$. The necessity is proved. In the following, we try to prove the sufficiency. Suppose that $f(x)$ attains its minimal value at $x_0 \in D$, but $f'(x_0) \neq \theta$. From (2) and the definition of $I_D(x)$, there exist $y^* \in D$ and $\lambda^* \geq \theta$, such that

$$g'(x_0) = (1-\lambda^*)x_0 + \lambda^* y^*. \qquad \text{... (3)}$$

It is easy to see that $\lambda^* \neq 0$ (if $\lambda^* = 0$, then $g'(x_0) = x_0$, i.e., $f'(x_0) = \theta$, which contradicts with $f'(x_0) \neq \theta$), hence $\lambda^* > 0$. From (3), we have

$$y^* - x_0 = \frac{1}{\lambda^*}[g'(x_0) - x_0] \qquad \text{... (4)}$$

Since $D$ is a convex closed set and $x_0 \in D, y^* \in D$, we know that for $0 \leq t \leq 1$,

$$x_0 + t\,(y^* - x_0) \in D.$$

So
$$x_0 + \frac{t}{\lambda^*}\,[g'(x_0) - x_0] \in D,\ 0 \le t \le 1. \qquad \text{... (5)}$$

Let $h = \frac{1}{\lambda^*}\,[g'(x_0) - x_0]$. Because $f(x)$ has bounded linear Gâteaux differential at $x_0$, then

$$+\ f(x_0 \in th) - f(x_0) - (f'(x_0), th) + w(t), \qquad \text{... (6)}$$

where $w(t)$ satisfying $\lim_{t \to 0} \frac{w(t)}{t} = 0$. From (5), there exists $0 < \delta < 1$, such that for $0 < t < \delta$, we have $x_0 + th \in U$, and

$$\left| \frac{w(t)}{t} \right| \le \frac{1}{2\lambda^*}\, \| f'(x_0) \|^2. \qquad \text{... (7)}$$

Hence from (6), (7), we have

$$f(x_0 + th) - f(x_0) = -\frac{t}{\lambda^*}\, \| x_0 - g'(x_0) \|^2 + w(t) \le -\frac{t}{\lambda^*}\, \| f'(x_0) \|^2 + \frac{t}{2\lambda^*}\, \| f'(x_0) \|^2 < 0.$$

So $f(x_0 + th) < f(x_0)$ (when $0 < t < \delta$). On the other hand, $x_0 + th \in U$ (when $0 < t < \delta$), so $f(x_0 + th) \ge f(x_0)$. Which brings contradiction. The conclusion of Theorem 1 is proved.

*Remark* 1 : Theorem 1 gives a necessary and sufficient condition for the minimal point being a critical point, which is not always an interior point.

In the following, we always assume that $f(x)$ has bounded linear Gâteaux differential at every point of $H$, and let

$$A(x) = g'(x). \qquad \text{... (8)}$$

*Lemma* 1[5] — Let $E$ be a reflexive Banach space, $D \subset E$ be weakly closed, and let the functional $f: D \to R$ be weakly lower semicontinuous, and satisfying

$$\lim_{\substack{\|x\| + \infty \\ x \in D}} \to f(x) = +\infty,$$

then there exists $x_0 \in D$, such that $f(x_0) = \inf_{x \in D} f(x)$.

***Theorem*** 2 — *Suppose that $f$ is weakly lower semicontinuous. $A$ ($A$ is defined by (8)) is an inward map on $D$, $f(\theta) \neq \inf_{x \in D} f(x)$, and $f$ satisfies*

$$\lim_{\substack{\|x\| - +\infty \\ x \in D}} f(x) = +\infty, \qquad \text{... (9)}$$

then $f$ has at least a nontrivial critical point in $D$.

PROOF : Since $D$ is a convex closed subset of $H$, then $D$ is weakly closed. From (9) and Lemma 1, there exists $x_0 \in E$, such that $f(x_0) = \inf_{x \in D} f(x)$. On the other hand, $f(\theta) \neq \inf_{x \in D} f(x)$. Then $x_0 \neq \theta$.

Since $A$ is an inward map, then $A(x_0) \in I_D(x_0)$. By virtue of Theorem 1, we know that $x_0$ is a nontrivial critical point of $f$. The conclusion of Theorem 2 is proved.

## 3. APPLICATIONS

Consider the nonlinear integral equation of Hammerstein type

$$\varphi(x) = \int_G k(x, y) f(y, \varphi(y))\, dy, \qquad \dots (10)$$

where $G \subset R^n$ is measurable, $0 <$ mes $G < +\infty$, the kernel $k(x, y)$ is positive defined, symmetric, and measurable on $G \times G, k(x, y) \geq 0$, mes $\{(x, y) \in G \times G \mid k(x, y) = 0\} = d$, $f(x, u)$ $(x \in G, -\infty < u < +\infty)$ satisfies Caratheodory conditions.

Let $K\varphi(x) = \int_G k(x, y)\, \varphi(y)\, dy$.

***Theorem*** 3 — *Suppose that*

*($H_1$) there exist $p > 2$, $a(x) \geq 0, a(x) \in L_{\frac{p}{p-2}}, b > 0$, such that*

$$\int_G \int_G |k(x, y)|^p\, dx\, dy < +\infty, \qquad \dots (11)$$

*and*

$$|f(x, u)| \leq a(x) + b\, |u|^{p-1}; \qquad \dots (12)$$

*($H_2$) there exist $0 < \gamma < 2$, $b(x) \in L_{\frac{2}{\gamma}}$, $c(x) \in L$, and $0 \leq a < \lambda_1$ ($\lambda_1$ is the first eigenvalue of $K$ acting on $L_2$), such that*

$$\int_0^u f(x, v)\, dv \leq \frac{a}{2} u^2 + b(x)\, |u|^{2-\gamma} + c(x); \qquad \dots (13)$$

($H_3$) $f(x, u)\, u \geq 0, \ \forall\, u \in R^1$;

($H_4$) *there exists* $c_0 > 0$, *such that*

$$\liminf_{u \to 0} \frac{f(x, u)}{u} \geq \lambda_1 + c_0, \ \forall\, x \in G.$$

*Then eq. (10) has at least two nontrivial solutions in $L_p$.*

PROOF : From (11) and $0 <$ mes $G < +\infty$, we have $K$ is positive self-adjoint, and completely continuous from $L_2$ to $L_2$, completely continuous from $L_q$ to $L_p$. Then by the results of[1], there exists the positive square root $H$ of $K$, such that

$$K = HH^*$$

where $H : L_2 \to L_p$ is a linear completely continuous operator, and $H^*$ is the conjugate operator of $H$, $H^* : L_q \to L_2$ is completely continuous.

Using the regular method (see [2]), we know that the solvability of eq. (10) in $L_p$ is equivalent to the existence of critical points of function $\Phi$ in $L_2$, and if $\psi$ is a critical point of $\Phi$, then $H\psi$ is a solution of eq. (10), where

$$\Phi(\psi) = \frac{1}{2}\|\psi\|^2 - \int_G dx \int_0^{H\psi} f(x, u)\, du, \quad \psi \in L_2.$$

Let $$P_p = \{\varphi \in L_p \mid \varphi(x) \geq 0, x \in G\},$$

then $P_p$ is a cone of $L_p$. Define $\hat{P} = H^{-1} P_p$. Since $H$ is linear and continuous, we have $\hat{P}$ is a nonempty convex closed subset of $L_2$.

We can prove that $H^* : P_q \to \hat{P}$, i.e., $H H^* : P_q \to P_p$. Since $k(x, y) \geq 0$, then $K = HH^* : P_q \to P_p$, where $P_q = \{\varphi \in L_q \mid \varphi(x) \geq 0, x \in G\}$.

From (12) and ($H_3$), we know that Nemytsky operator $f : P_p \to P_q$ is continuous and bounded, where $fu = f(x, u(x)), x \in G$.

Let $$A = H^* f H,$$

then $A : \hat{P} \to \hat{P}$ and $\Phi' = I - A$ (see [1]).

From [2], we can prove that the functional $\Phi : H_2 \to R$ is weakly lower semicontinuous and satisfies

$$\lim_{\|\psi\|_2 \to \infty} \Phi(\psi) = +\infty.$$

Since $k(x, y) \geq 0$, symmetric, and $K$ is linear, completely continuous, then by Krein-Rutamann Theorem, there exists $\psi_1 \in L_2$, which is the positive eigenfunction of $K$ corresponding to $\lambda_1$, with $\|\psi_1\| = 1$, i.e., $\lambda_1 K \psi_1 = \psi_1$. Furthermore, we have $\sqrt{\lambda_1}\, H \psi_1 = \psi_1$, so $\psi_1 \in L_p$, and $\psi_1^2 \in L_{\frac{p}{2}}$.

From ($H_4$), there exist $r > 0$, $\varepsilon_0 \in (0, \lambda_1)$, such that

$$\frac{f(x, u)}{u} > \lambda_1 + \varepsilon_0; \quad \forall\, x \in G, 0 < u < r.$$

Since $\psi_1 \in L_2$, $\psi_1 \in L_p$, $\psi_1^2 \in L_{\frac{p}{2}}$, and $a(x) \in L_{\frac{p}{p-2}}$. By virtue of the absolutely continuity of Lebesgue integration, there exists $\delta > 0$, such that for $F \subset G$, with mes $F < \delta$, we have

$$\int_F \psi_1(x)^2\,dx < \frac{\varepsilon_0}{2\lambda_1},\ \int_F \psi_1(x)^p\,dx < \frac{p\,\varepsilon_0(\lambda_1-\varepsilon_0)}{8b\sqrt{\lambda_1^{4-p}}},$$

$$\int_F a(x)\,\psi_1(x)^2\,dx < \frac{r\,\varepsilon_0(\lambda_1-\varepsilon 0)}{16\,\lambda_1}.$$

Since $\psi_1 \in L_2 \subset L$, there exist $0 < t_0 < \min\ \{1, \sqrt{\lambda_1}\}$, such that for $0 < t < t_0$, we have mes $\left\{x \in G \mid \frac{\psi_1(x)}{\sqrt{\lambda_1}} > \frac{r}{t}\right\} < \delta.$

Let $$G_1 = \left\{x \in G \mid \frac{t\,\psi_1(x)}{\sqrt{\lambda_1}} \le r\right\},\ G_2 = G \setminus G_1 = \left\{x \in G \mid \frac{t\,\psi_1(x)}{\sqrt{\lambda_1}} > r\right\}.$$

In the following, we try to prove that $\Phi(\theta) \ne \inf_{\psi \in \hat{P}} \Phi(\psi)$.

$$\Phi(t\,\psi_1) = \frac{1}{2}t^2 - \int_{G_1} dx \int_0^{\frac{t\,\psi_1(x)}{\sqrt{\lambda_1}}} f(x, v)\,dv - \int_{G_2} \int_0^{\frac{t\,\psi_1(x)}{\sqrt{\lambda_1}}} f(x, v)\,dv$$

$$\le \frac{1}{2}t^2 - \int_{G_1} \frac{t^2[\psi_1(x)]^2}{2\lambda_1}(\lambda_1+\varepsilon_0)\,dx + \int_{G_2}\left[a(x)\frac{t\,\psi_1(x)}{\sqrt{\lambda_1}} + b\frac{t^p[\psi_1(x)]^p}{p\sqrt{\lambda_1^p}}\right]dx$$

$$\le \frac{1}{2}t^2 - \frac{\lambda_1+\varepsilon_0}{2\lambda_1}\int_{G_1}\psi_1(x)^2\,dx + \frac{t^2}{\lambda_1 r}\int_{G_2} a(x)\,[\psi_1(x)]^2\,dx + \frac{t^p b}{\sqrt{\lambda_1^p}\,p}\int_{G_2}[\psi_1(x)]^p\,dx$$

$$\le \frac{1}{2}t^2 - \frac{\lambda_1+\varepsilon_0}{2\lambda_1}\left(t^2 - \frac{\varepsilon_0}{2\lambda_1}\right) + \frac{t^2}{\lambda_1 r}\frac{r\,\varepsilon_0(\lambda_1-\varepsilon_0)}{16\,\lambda_1} + \frac{t^p b}{(\sqrt{\lambda_1})^p\,p}\frac{p\,\varepsilon_0(\lambda_1-\varepsilon_0)}{8b\,(\sqrt{\lambda_1})^{4-p}}$$

$$\le -\frac{\varepsilon_0(\lambda_1-\varepsilon_0)}{16\,\lambda_1^2}t^2 < 0.$$

It is obvious that $\Phi(\theta) = 0$, hence $\Phi(t\,\psi_1) < \Phi(\theta)$, i.e., $\Phi(\theta) \ne \inf_{\psi \in \hat{P}} \Phi(\psi)$. Then by Theorem 2, $\Phi$ has at least a nontrivial critical point $\varphi_1 \in \hat{P}$. From the above we know that $H\,\varphi_1$ is a solution of eq. (10). Since mes $\{(x, y) \in G \times G \mid k(x, y) = 0\} = 0$, we have $H\varphi_1 \ne \theta$.

In the similar way, we can prove that $\Phi$ has a nontrivial critical point $\varphi_2 \in -\hat{P}$, and $H\,\varphi_2$ is a nontrivial solution of eq. (10). It is obviously that $H\,\varphi_1 \ne H\,\varphi_2$, i.e. eq. (10) has at least two nontrivial solutions. The conclusion is proved.

*Remark* 2 : From condition $(H_3)$, we have $f(x, \theta) = \theta$, then $\varphi(x) \equiv 0$ is a solution of eq. (10). So under the assumptions of Theorem 3, eq. (10) has at least three solutions.

*Remark* 3 : We can not make sure that $\hat{P}$ in the proof of Theorem 3 has interior points, but we know there exists at least a critical point in $\hat{P}$, so we can see the significance of Theorem 1.

In paper [1], the author also discussed the existence of solutions of nonlinear integral equations of Hammerstein type when the kernel $k(x, y)$ is essentially bounded. We can use the abstract results of this paper to discuss the existence of nontrivial of eq. (10) when the kernel $k(x, y)$ is essentially bounded.

***Theorem*** 4 — *Suppose that*

(i) $\operatorname{ess\,sup}_{x, y \in G} |k(x, y)| = M < +\infty$;

(ii) there exists $p > 2$, such that for every $r > 0$, there exists $a_r(x) \in L_{\frac{p}{p-2}}$, satisfying

$$|f(x, u)| \le a_r(x), \; |u| \le r;$$

(iii) the conditions $(H_2)$-$(H_4)$ in Theorem 3 are valid.

Then eq. (10) has at least three bounded solutions in $L_p$.

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