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GLOBAL BEHAVIOUR OF A DISCRETE EPIDEMIC MODEL*

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In this paper, we study the asymptotic stability, the oscillation, and the monotonic convergence of the positive solution of the difference equation :

$$x_{n+1} = (1 - x_n - x_{n-1})(1 - e^{-Ax_n}),$$

where $A \in (0, +\infty)$ and the initial values x_{-1} and x_0 are arbitrary positive real numbers such that $0 < x_{-1} + x_0 < 1$.

Key Words : Difference Equations; Oscillation; Asymptotic Stability

1. INTRODUCTION

In the monograph of Kocic and Ladas[1], they give a Research Project (see [1, p169]). To this end, we consider the equation :

$$x_{n+1} = (1 - x_n - x_{n-1})(1 - e^{-Ax_n}) \text{ for } n = 0, 1, ..., \qquad ... (1)$$

where $A \in (0, +\infty)$ and the initial values x_{-1} and x_0 are arbitrary positive real numbers such that $0 < x_{-1} + x_0 < 1$. The aim of this paper is to investigate the oscillation and asymptotic stability of the nonnegative equilibrium of eq. (1).

2. SOME LEMMAS

Lemma 2.1 — *i*) Assume that $0 < A \le 1$. Then the equation :

$$x = f(x) = (1 - 2x)(1 - e^{-Ax}) \qquad ... (2)$$

has a unique nonnegative root $x = 0$ and $f(x) < x$ for $0 < x < \frac{1}{2}$;

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ii) Assume that $A > 1$. Then eq. (2) has a unique positive root $\bar{x}$ and $\bar{x} < \frac{1}{3}$.

PROOF : From $f(x) = (1 - 2x)(1 - e^{-Ax})$. Then we have

$$f'(x) = -2 + (2 + A - 2Ax)\, e^{-Ax}$$

and
$$f''(x) = -A\,(4 + A - 2Ax)\, e^{-Ax} < 0$$

for $0 < x < \frac{1}{2}$. So, we know that $f(x)$ is a convex function in $\left(0, \frac{1}{2}\right)$.

i) Note that $f'(0) = A \le 1$ for $0 < A \le 1$, $f(0) = 0$ and $f'(x) < f'(0) \le 1$. By the properties of $f(x)$, we obtain that Eq. (2) has a unique nonnegative root $x = 0$ and $f(x) < x$ for $0 < x < \frac{1}{2}$.

ii) Observe that $f'(0) = A > 1$ for $A > 1$ and $f(0) = f^{(1/2)} = 0$. By the properties of convex functions, we obtain that eq (2) has a unique positive $\bar{x}$ and $\bar{x} < \frac{1}{3}$.

Thus, the proof of Lemma 2.1 is complete.

Lemma 2.2 — The equation

$$1 + \frac{A}{2} = \frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right) + e^{\frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right)} \quad \dots (3)$$

has a unique root A^* for $A > 1$. Moreover

$$1 + \frac{A}{2} > \frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right) + e^{\frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right)} \quad \dots (4)$$

holds for $1 < A < A^*$ and

$$1 + \frac{A}{2} < \frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right) + e^{\frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right)} \quad \dots (5)$$

holds for $A > A^*$.

PROOF : Set $H(A) = 1 + \frac{A}{2}$ and

$$G(A) = \frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right) + e^{\frac{1}{12}\left(5A + 2 - \sqrt{A^2 + 20A + 4}\right)}$$

Observe that $H(1) = \frac{3}{2}$, $G(1) = \frac{1}{6} + e^{\frac{1}{6}} < \frac{3}{2}$, $H'(A) = \frac{1}{2}$ $H(0) = G(0) = 1$,

$$G'(A)=\frac{5}{12}-\frac{A+10}{12\sqrt{A^2+20A+4}}+e^{\frac{1}{12}(5A+2-\sqrt{A^2+20A+4})}\left(\frac{5}{12}-\frac{A+10}{12\sqrt{A^2+20A+4}}\right)>0$$

$$G''(A)=\frac{8}{(A^2+20A+4)^{\frac{3}{2}}}+e^{\frac{1}{12}(5A+2-\sqrt{A^2+20A+4})}\left(\frac{5}{12}-\frac{A+10}{12\sqrt{A^2+20A+4}}\right)^2$$

$$+\ e^{\frac{1}{12}(5A+2-\sqrt{A^2+20A+4})}\left[\frac{8}{(A^2+20A+4)^{\frac{3}{2}}}\right]>0.$$

Hence, $G(A)$ is a convex function.

By the properties of convex function, we know that (3)-(5) hold. This completes the proof.

Lemma 2.3 — Assume that $A = A^*$. Then eq. (2) has the same root as the following equation :

$$f'(x)=[(1-2x)(1-e^{-Ax})]'=0, \qquad \ldots (6)$$

i.e.,

$$\bar{x}=x^*=\frac{5A^*+2-\sqrt{A^{*2}+20A^*+4}}{12A^*},$$

where $\bar{x}$ or x^* is the root of eq. (2) or eq. (6), respectively.

PROOF : From (2) and (6), we have, respectively, that $e^{-Ax}=\frac{1-3x}{1-2x}$ and

$$e^{-Ax}=\frac{2}{2+A-2Ax}. \qquad \ldots (7)$$

Hence $\frac{1-3x}{1-2x}=\frac{2}{2+A-2Ax}$. It follows

$$x=\frac{5A+2-\sqrt{A^2+20A+4}}{12A} \qquad \ldots (8)$$

Substituting (8) into (7), we get

$$1+\frac{A}{2}=\frac{1}{12}(5A+2-\sqrt{A^2+20A+4})+e^{\frac{1}{12}(5A+2-\sqrt{A^2+20A+4})} \qquad \ldots (9)$$

Consequently, (9) holds for $A = A^*$ from Lemma 2.2. By virtue of Lemma 2.1 and the properties of $f(x)$ eq. (2) has the same root as eq. (6), i.e.,

$$\bar{x}=x^*=\frac{5A^*+2-\sqrt{A^{*2}+20A^*+4}}{12A^*}.$$

Hence, the proof is complete.

Lemma 2.4 — Assume that $\bar{x}$ is the root of eq (2) and x^* is that of eq. (6). Then

i) $\bar{x} < x^*$ holds for $1 < A < A^*$;

ii) $\bar{x} > x^*$ holds for $A > A^*$;

iii) If $1 < A < A^*$, then $f(x)$ is increasing and $f(x) > x$ for $0 < x < \bar{x}$, $f(x)$ is decreasing and $f(x) < x$ for $\bar{x} < x < \frac{1}{2}$;

iv) If $A = A^*$, then $f(x)$ is increasing and $f(x) > x$ for $0 < x < x^*$, and $f(x)$ is decreasing and $f(x) < x$ for $x^* < x < \frac{1}{2}$;

v) If $A > A^*$, then there exists an $\bar{x}^*$ which is a positive number and $0 < \bar{x}^* < x^*$ such that $f(\bar{x}^*) = f(\bar{x})$, $f(x)$ is strictly increasing in $(0, \bar{x}^*) \cup (\bar{x}^*, x^*)$, $f(x) > x$ holds. and $f(x)$ is strictly decreasing in $\left(\bar{x}, \frac{1}{2}\right) \cup (x^*, \bar{x})$, $f(x) < x$ holds for $\bar{x} < x < \frac{1}{2}$, $f(x) > x$ holds for $x^* < x < \bar{x}$.

PROOF : i) Set $a = \frac{5A + 2 - \sqrt{A^2 + 20A + 4}}{12A}$. Then we have $f'(a) = -2 + (2 + A - 2Aa)\, e^{-Aa}$. From (4), we obtain $f'(a) > -2 + 2 = 0$. Thus, by the properties of $f(x)$, we have

$$a < x^*. \qquad \text{... (10)}$$

From (4) and eq. (2), we have

$$f(a) - a = (1 - 2a)(1 - e^{-Aa})\, a < \frac{6Aa^2 - (5A + 2)\, a + A}{2 + A - 2Aa} = 0.$$

Hence, $f(a) < a$. This follows

$$\bar{x} < a \qquad \text{... (11)}$$

from Lemma 2.1. Therefore, $\bar{x} < x^*$ holds from (10) and (11). This completes the proof of i)

The proof of ii) is just the same as that of i). We omit it. Moreover, the proofs of iii), iv) and v) can be easily proved. We also omit them. The proof is complete.

Lemma 2.5 — Assume that $A \geq A^*$. Then $|\lambda| < 1$ holds, where λ is any root of the equation.

$$\lambda^2 - \frac{6A\bar{x}^2 - 5A\bar{x} - \bar{x} + A}{1 - 2\bar{x}}\lambda + \frac{\bar{x}}{1 - 2\bar{x}} = 0 \qquad \text{... (12)}$$

and $\bar{x}$ is the equilibrium of eq. (2).

PROOF : Set $\lambda = \frac{z + 1}{z - 1}$. Changing (12) into

$$\frac{1 - 2\bar{x} - [6A\bar{x}^2 - (5A + 2)\bar{x} + A]}{1 - 2\bar{x}} z^2 + \frac{2 - 6\bar{x}}{1 - 2\bar{x}} z + \frac{6A\bar{x}^2 - (5A + 2)\bar{x} + A + 1}{1 - 2\bar{x}} = 0. \qquad \text{... (13)}$$

By virtue of Lemmas 2.3 and 2.4, we have $6A\bar{x}^2-(5A+2)\bar{x}+A\le 0$. Thus

$$\frac{1-2\bar{x}-[6A\bar{x}^2-(5A+2)\bar{x}+A]}{1-2\bar{x}}>0.$$

From Lemma 2.1, we know that $\bar{x}<\frac{1}{3}$. So, $\frac{2-6\bar{x}}{1-2\bar{x}}>0$. Let $b=\frac{5A+2}{12A}$. Then $a<\bar{x}<\frac{1}{3}<b$, where a is as that in the above Lemma. Therefore, by the properties of function $6Ax^2-(5A+2x+A)$, we obtain

$$\frac{6A\bar{x}^2-(5A+20\bar{x}+A+1}{1-2\bar{x}}>\frac{\frac{6A}{9}-\frac{5A+2}{3}+A+1}{1-2\bar{x}}>0.$$

This follows that $Re\ z<0$. By the well-known Schur-Cohn Criterion, we know that $|\lambda|<1$. This completes the proof.

3. MAIN RESULTS

***Theorem* 3.1** — *Assume that $0<A\le 1$. If $\{x_n\}_{n=0}^{\infty}$ is a positive solution of eq. (1). Then there exists an $n_0>0$ such that $\{x_n\}_{n=n_0}^{\infty}$ is strictly decreasing and converges to zero.*

PROOF : Note that zero is the only nonnegative equilibrium of eq (1) and $\{x_n\}$ is a positive bounded sequence. Hence, there exists an $n_0>0$ such that $x_{n_0}\le x_{n_0-1}<\frac{1}{2}$. (Otherwise $\{x_n\}$ would be strictly increasing to a positive number or $x_n\ge\frac{1}{2}$, which is a contradiction).

Now, by observing that

$$x_{n_0+1}=(1-x_{n_0}-x_{n_0-1})(1-e^{-Ax_{n_0}})$$

$$\le(1-2x_{n_0})(1-e^{-Ax_{n_0}})$$

and $x_{n_0+1}\le x_{n_0}$ from Lemma 2.1, we have $x_{n+1}\le x_n$, for $n\ge n_0$ from induction. Hence, $L=\lim_{n\to\infty}x_n$ exists and we get $L=0$ from Lemma 2.1. This completes the proof.

***Theorem* 3.2** — *Assume that 1 < $A<A^*$ and $\{x_n\}_{n=0}^{\infty}$ is a nonoscillating solution of eq. (1). Then there exists an $n_0\ge 0$ such that $\{x_n\}_{n=n_0}^{\infty}$ monotonically converges to x, where A^*, is a constant which was derived in Lemmas 2.2 and 2.3, and x is the equilibrium of eq. (1).*

PROOF : Let $\{x_n\}_{n=n_0}^{\infty}$ be a nonoscillatory solution of eq (1). Then there exists an n_0' such that $x_n \le \bar{x}$ or $x_n \ge \bar{x}$ for $n \ge n_0'$.

First, we consider the case $x_n \le \bar{x}$ for $n \ge n_0'$.

Note that $\{x_n\}_{n=0}^{\infty}$ is a positive bounded sequence. So, there exists an $n_0 \ge n_0'$ such that $x_{n_0} \ge x_{n_0-1}$. (Otherwise, $\{x_n\}_{n=0}^{\infty}$ would be strictly decreasing to zero. But

$$x_{n+1} = (1 - x_n - x_{n-1})(1 - e^{-Ax_n}) > (1 - 2x_{n-1})(1 - e^{-Ax_n}) > \frac{1 - 2x_{n-1}}{1 - 2x_n} x_n.$$

Thus $\quad x_n - x_{n+1} < 2x_{n-1}x_n - 2x_n x_{n+1}$ holds for $n \ge n_0'$. Now, observing that

$$x_{n+1} - x_{n+2} < 2x_n x_{n+1} - 2x_{n+1}x_{n+2},$$

$$x_{n+2} - x_{n+3} < 2x_{n+1}x_{n+2} - 2x_{n+2}x_{n+3}$$

and $\quad x_{n+p} - x_{n+p+1} < 2x_{n+p-1}x_{n+p} - 2x_{n+p}x_{n+p+1},$

where p is a natural number. Hence

$$x_n - x_{n+p+1} < 2x_{n-1}x_n - 2x_{n+p}x_{n+p+1}. \qquad \text{... (14)}$$

From (14), we have $x_n \le 2x_{n-1}x_n$ as $p \to \infty$ and $x_{n-1} \ge \frac{1}{2}$. This is a contradiction)

Noting that $x_{n_0+1} = (1 - x_{n_0} - x_{n_0-1})(1 - e^{A x_{n_0}}) > (1 - 2x_{n_0})(1 - e^{-Ax_{n_0}})$, we have from Lemma 2.4 and induction that $x_{n+1} \ge x_n$ for $n \ge n_0$. Therefore, $\lim_{n \to \infty} x_n = L$ exists and $L = \bar{x}$ from Lemma 2.1.

Second, we consider the case $x_n \ge \bar{x}$ for $n \ge n_0'$.

Note that $\{x_n\}_{n_0'}^{\infty}$ is a positive bounded sequence. So, there exists an $n_0 \ge n_0'$ such that $\bar{x} \le x_{n_0} \le x_{n_0-1}$. (Otherwise, $\{x_n\}_{n_0'}^{\infty}$ would be strictly increasing to L and $L > \bar{x}$).

Owing to that $x_{n_0+1} = (1 - x_{n_0} - x_{n_0-1})(1 - e^{-Ax_{n_0}}) \le (1 - 2x_{n_0})(1 - e^{-A}x_{n_0})$, we obtain $x_{n_0+1} \le x_{n_0}$ from Lemma 2.4 and $x_{n+1} \le x_n$ for $n \ge n_0$ from induction. Hence, $\lim_{n \to \infty} x_n = L$ exists and $L = \bar{x}$ from Lemma 2.1. The proof is thus complete.

***Theorem* 3.3** — *Assume that* $A = A^*$. *Then*

i) the equilibrium x *of eq. (1) is asymptotically stable; and*

ii) if $\{x_n\}_{n=0}^{\infty}$ *is a nonoscillatory solution of eq. (1), then there exists an* $n_0 > 0$ *such that* $\{x_n\}_{n_0'}^{\infty}$ *is strictly increasing to* $\bar{x}$.

PROOF : *i*) The linearized equation associated with eq (1) about the equilibrium $\bar{x}$ is

$$y_{n+1} - \frac{6A\bar{x}^2 - 5A\bar{x} - \bar{x} + A}{1 - 2\bar{x}} y_n + \frac{\bar{x}}{1 - 2\bar{x}} y_{n-1} = 0. \qquad \text{... (15)}$$

So, the characteristic equation of eq. (15) is

$$\lambda^2 - \frac{6A\bar{x}^2 - 5A\bar{x} - \bar{x} + A}{1 - 2\bar{x}} \lambda + \frac{\bar{x}}{1 - 2\bar{x}} = 0. \qquad \text{... (16)}$$

From Lemma 2.5, we know that every solution of eq. (16) satisfies $|\lambda| < 1$. Therefore, from Corollary 1.32 [1, p14], we know that the equilibrium $\bar{x}$ of eq. (1) is asymptotically stable. This completes the proof of the first part.

ii) Let $\{x_n\}_{n=0}^{\infty}$ be a nonoscillatory solution of eq. (1). Then there exists an $n_0' > 0$ such that $x_n \le \bar{x}$ or $x_n \ge \bar{x}$ for $n \ge n_0'$.

First, we consider the case $x_n \le \bar{x}$ for $n \ge n_0'$.

The proof of this part is just as that of Theorem 3.2. So, we omit it.

Now, we consider the case $x_n \ge \bar{x}$ for $n \ge n_0'$.

Owing to that $\{x_n\}_{n=0}^{\infty}$ is a positive bounded sequence, there exists an $n_0 > n_0'$ such that $x_{n_0} \le x_{n_0-1}$. (Otherwise, $\{x_n\}_{n_0'}^{\infty}$ would be strictly increasing to $L > \bar{x}$)

Observing that $x_{n_0+1} = (1 - x_{n_0} - x_{n_0-1})(1 - e^{-Ax_{n_0}}) < (1 - 2x_{n0})(1 - e^{-Ax_{n_0}})$, we know $x_{n_0+1} < \bar{x}$ from Lemma 2.4. This is a contradiction. The proof is thus complete.

Theorem 3.4 — *Assume that* $A > A^*$. *Then*

i) every nontrivial solution of Eq (1) is oscillatory about $\bar{x}$*; and*

ii) the equilibrium $\bar{x}$ *of Eq (1) is asymptotically stable.*

PROOF : *i*) Suppose that $\{x_n\}_{n=0}^{\infty}$ is a nonoscillatory solution of eq (1). Then there exists an $n_0' \ge 0$ such that $x_n \ge \bar{x}$ or $x_n \le \bar{x}$ for $n \ge n_0'$.

First, we consider the case $x_n \ge \bar{x}$ for $n \ge n_0'$.

Noting that $\{x_n\}_{n_0'}^{\infty}$ is a positive bounded sequence, there exists an $n_0 > n_0'$ such that $x_{n_0-1} \le x_{n_0}$. (Otherwise, $\{x_n\}_{n_0'}^{\infty}$ would be strictly increasing to $L > \bar{x}$)

Observing that $x_{n_0+1} = (1 - x_{n_0} - x_{n_0-1})(1 - e^{-A x_{n_0}}) < (1 - 2x_{n_0})(1 - e^{-A} x_{n_0})$, we know $x_{n_0+1} < \bar{x}$ from Lemma 2.4. This is a contradiction.

Next, we consider the case $x_n \le \bar{x}$ for $n \ge n_0'$.

Noting that $\{x_n\}_{n=0}^{\infty}$ is a positive bounded sequence, there exists an $n_0 > n_0$ such that $x_{n_0-1} \le x_{n_0}$. (Otherwise, $\{x_n\}_{n_0}^{\infty}$, would be strictly decreasing to zero. This is a contradiction which was proved in Theorem 3.1).

Observing that $x_{n_0+1} = (1 - x_{n_0} - x_{n_{0-1}})(1 - e^{-Ax_{n_0}}) \ge (1 - 2x_{n_0})(1 - e^{-Ax_{n_0}})$, we know $x_{n_0} \le x_{n_0+1} < \bar{x}^*$ from Lemma 2.4.

Otherwise, if $\bar{x}^* < x_{n_0+1} < \bar{x}$, then $x_{n_0+2} \ge (1 - 2x_{n_0+1})(1 - e^{-Ax_{n_0}}) > \bar{x}$.

By induction, we have $x_n \le x_{n_0+1} < \bar{x}^*$. Thus $\{x_n\}_{n_0}^{\infty}$ is strictly increasing to $L \le \bar{x}^* < \bar{x}$. This is a contradiction. The proof is thus complete. The proof of part vi is very easy, we will omit it.

REFERENCES

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