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EXISTENCE OF POSITIVE SOLUTIONS AND OSCILLATORY BEHAVIOUR FOR NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS*

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In this paper, the existence of positive solution and oscillatory behaviour of the neutral differential equation

$$(x(t) - C(t)\, x(t-r))' + P(t)x(t-\tau) - Q(t)\, x(t-\sigma) = 0$$

are discussed, where $P, Q, C \in C[(t_0, \infty), R^+]$, $r \in (0, \infty)$, $\tau, \sigma \in [0, \infty)$. Under condition $r \geq \sigma \geq \tau$, some results about sufficient conditions for the existence of positive solution and oscillatory behaviour are obtained.

Key Words : Positive Solutions; Oscillation; Neutral Differential Equations; Coefficients

1. INTRODUCTION

Consider the first order neutral differential equation with positive and negative coefficients

$$(x(t) - C(t)\, x(t-r))' + P(t)\, x(t-\tau) - Q(t)\, x(t-\sigma) = 0, \quad \text{... (1)}$$

where $P, Q, C \in C[t_0, \infty), R^+]$, $r \in (0, \infty)$, $\tau, \sigma \in [0, \infty)$, $\bar{P}(t) = P(t) - Q(t + \sigma - \tau) > 0$. ... (2)

In recent years, many papers investigated the existence of positive solutions and oscillatory theory of eq. (1), and obtained a lot of good results, see [1]-[10]. However, most of them studied eq. (1) under condition

$$0 \leq C(t) \leq 1, \ \tau \geq \sigma,$$

whereas there were few results obtained for the case

$$C(t) \geq 0, \ r \geq \sigma \geq \tau. \quad \text{... (3)}$$

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Farrell Grove and Ladas studied oscillatory behaviour of eq. (1) only for the case of constant coefficients. In this paper, we give some sufficient conditions for the existence of positive solution and oscillatory behaviour under assumption (3).

A function $x(t)$ is a solution of eq. (1) with initial function ϕ at T if $x \in C([T-r, \infty), R)$ such that $x(t) = \phi(t)$ for $T - r \leq t \leq T, x(t) - C(t)\,x(t-r)$ is continuously differentiable for $t \geq T$, and $x(t)$ satisfies equation (1) for all $t \geq T$.

Throughout this paper, we always assume that (2) and (3) holds and, unless specified, a functional inequality holds only for sufficiently large t.

2. Existence of Positive Solution

Lemma 1.1 — If $Z(t) : [T, \infty) \to (0, \infty)$ is a continuous positive solution of integral inequality

$$z(t) \geq \frac{1}{C(t+r)}\left(z(t+r) + \int_{t-\sigma+r}^{t-\tau+r} P(s+\tau)\, z(s)ds + \int_{T-\sigma+r}^{t-\sigma+r} \bar{P}(s+\tau)\, z(s)ds \right) \quad \ldots(4)$$

then corresponding integral equation

$$x(t) = \frac{1}{C(t+r)}\left(x(t+r) + \int_{t-\sigma+r}^{t-\tau+r} P(s+\tau)\, x(s)\, ds + \int_{T-\sigma+r}^{t-\sigma+r} \bar{P}(s+\tau)\, x(s)\, ds \right) \quad \ldots (5)$$

has also a continuous positive solution $X(t)$ and $X(t) \leq Z(t)$.

Proof : Define the set of function

$$\Omega = \{\omega \in C([T, \infty), R^{+}), 0 \leq \omega(t) \leq e^{-\lambda t}, t \geq T\}$$

and define an operator S on Ω as follows

$$(S\,\omega(t)) = \begin{cases} \dfrac{1}{C(t+r)z(t)\, e^{\lambda t}} \Big(z(t+r)\, e^{\lambda(t+r)}\, \omega(t+r) + \displaystyle\int_{t-\sigma+r}^{t-\tau+r} P(s+\tau)\, z(s)\, e^{\lambda s}\, \omega(s)ds \\ \qquad + \displaystyle\int_{T-\sigma+r}^{t-\sigma+r} \bar{P}(s+\tau)\, z(s)\, e^{\lambda s}\, \omega(s)ds \Big), \quad t \geq T + 2r, \\ (S\,\omega)(T+2r) + e^{-\lambda t} - e^{-\lambda(T+2r)}, \quad T \leq t < T+2r. \end{cases}$$

It is easy to get for $t \geq T + 2r$ that

$$(Sw)(t) \leq S(e^{-\lambda t})$$

$$= \frac{1}{C(t+r)\, e^{\lambda t}}\left(z(t+r) + \int_{t-\sigma+r}^{t-\tau+r} P(s+\tau)\, z(s)ds + \int_{T-\sigma+r}^{t-\sigma+r} \bar{P}(\tau+s)\, z(s)ds \right)$$

$$\leq e^{-\lambda t}.$$

Then, we have $S\,\Omega \subset \Omega$, choose $w_1, w_2 \in \omega$ and $\omega \leq \omega_2$. We can prove $S\,\omega_1 \leq S\,\omega_2$. Define sequence as

$$u_0 = e^{-\lambda t}, u_n = Su_{n-1}, n = 1, 2 \ \dots \ .$$

It is not difficult to see that

$$0 \leq u_n(t) \leq u_{n-1}(t) \leq \dots \leq e^{-\lambda t}, \ t \geq T_1.$$

Thus, $\lim\limits_{n \to \infty} u_n(t)$ exists. Set $u(t) = \lim\limits_{n \to \infty} u_n(t)$ for $t \geq T$. We have $u(t) > 0$ for $t \geq T$ since $u(t) > 0$ for $T = r - \sigma \leq t < T + 2r$. It follows from that Lebesgue's dominated convergence theorem that $u(t)$ satisfy $(Su)(t) = u(t)$. Set $X(t) = z(t)\,u(t)\,e^{\lambda t}$, one easily sees that $X(t)$ is a positive solution of (4) for $t \geq T + 2r$. The proof is complete.

Suppose that there exists $\lambda \in R$ such that

$$\sup_{t \geq T} \frac{1}{C(t+r)} \left(e^{\lambda r} + \int_{t-\sigma+r}^{t-\tau+r} p(s+\tau)\, e^{\lambda_0 (s-t)}\, ds + \int_{T-\sigma+r}^{t-\sigma+r} \overline{P}(s+\tau)\, e^{\lambda_0 (s-t)\, ds)} \right) \leq 1 \qquad \dots (6)$$

then next theorem is a immediate corollary since we see that $e^{\lambda t}$ is a positive solution of (4), and then eq. (1) has an eventually positive solution $X(t)$. It is easily to get that $X(t)$ is also an eventually positive solution.

Theorem 2.1 — *Assume that there exists $\lambda \in R$ such that (6) holds, then eq. (1) has a continuous eventually positive solution $X(t)$ and $X(t) \leq e^{\lambda t}$.*

Theorem 2.2 — *Assume that -*

(i) $C(t) \geq C > 1$ for $t \geq T$, and $|C(s_1) - C(s_2)| \leq M|s_1 - s_2|$, $s_1, s_2 \in [T, \infty)$.

(ii) There exists $\lambda_0 \in \left(0, \frac{1}{r} \ln C\right)$ such that (6) holds.

(iii) There exists $T_1 > T$ and $\lambda_1 \in (0, \lambda_0)$, such that

$$\inf_{t \geq T_1} \frac{1}{C(t+r)} \left(e^{\lambda_1 r} + \int_{t-\sigma+r}^{t-r+r} P(s+\tau)\, e^{\lambda_1 (s-t)}\, ds + \int_{T-\sigma+r}^{t-\sigma+r} \overline{P}(s+\tau)\, e^{\lambda_1 (s-t)}\, ds \right) \geq 1. \qquad \dots (7)$$

(iv) $\sup\limits_{t \geq T} P(t) = P < \infty$, $\sup\limits_{t \geq T} Q(t) = Q < \infty$, $\sup\limits_{t \geq T} \overline{P}(t) = \overline{P} < \infty$.

Then equation (1) has a positive solution which tends to ∞.

PROOF : Let E be set of all bounded continuous functions on $[T, \infty)$. When it is endowed with norm $\| \omega \| = \max_{t \geq T} \omega(t)$ for any $\omega \in E$, it is a Banach space. Take $\overline{T} > T_1$. Let Ω be a subset of E, and define it as

$$\Omega = \{\omega : \omega \in C([T, \overline{T}], R^{+}), e^{-\lambda_1 t} \leq \omega(t) \leq 1, t \in [T, \overline{T}], \text{ and}$$

$$|\omega(t_1) - \omega(t_2)| \leq K|t_1 - t_2|, \text{ for } t_1, t_2 \in [T, \overline{T}]\},$$

where positive constant K is chosen so large that

$$\frac{e^{\lambda_0 r}}{C} + \frac{M}{KC^2}\left(e^{\lambda_0 r} + P(\sigma - \tau) e^{\lambda_0 (r-r)} + \frac{1}{\lambda_0} \overline{P} e^{\lambda_0 (r-\sigma)} \right)$$

$$+ \frac{e^{\lambda_0 (r-r)}}{KC} (2P + P\lambda_0(\sigma - \tau) + 2\overline{P}) < 1$$

and $\lambda_0^2 < (1 - \beta) K$,

since $\lambda_0 \in \left(0, \frac{1}{r} \ln C\right)$, where $\beta \in (0, 1)$.

Define operator Φ on Ω as

$$(\Phi\omega)(t) = \begin{cases} \dfrac{1}{C(t+r)}\left(e^{\lambda_0 r}\omega(t+r) + \displaystyle\int_{t-\sigma+r}^{t-\tau+r} P(s+\tau) e^{\lambda_0 (s-t)} \omega(s)\,ds \right. \\ \qquad \left. + \displaystyle\int_{T-\sigma+r}^{t-\sigma r} \overline{P}(s+\tau) e^{\lambda_0 (s-t)} \omega(s) ds \right), & T_1 \leq t \leq \overline{T}, \\ \exp\left\{ \frac{t}{T} \ln(\Phi\omega)(T) \right\}, & T \leq t < T_1. \end{cases}$$

Next we will prove that Φ has a fixed point in Ω by Schauder's fixed point theorem. One easily sees that Ω is bounded, compact and convex set in $C([T, \overline{T}], R)$. Next, we show that $\Phi\Omega \subset \Omega$. Indeed, for $T_1 \leq t \leq \overline{T}$, by assumption (ii) and (iii) we have

$$(\Phi\omega)(t) \leq \frac{1}{C(t+r)}\left(e^{\lambda_0 r} + \int_{t-\sigma+r}^{t-r+r} p(s) e^{\lambda_0 (s-t)} ds + \int_{T-\sigma+r}^{t-\sigma+r} \overline{P}(s+\tau) e^{\lambda_0 (s-t)} ds \right) \leq 1$$

and $(\Phi\omega)(t)$

$$\geq \Phi(e^{-\lambda_1 t})$$

$$= \frac{e^{-\lambda_1 t}}{C(t+r)}\left((e^{(\lambda_0-\lambda_1)r} + \int_{t-\sigma+r}^{t-\tau+r} p(s)e^{(\lambda_0-\lambda_1)(s-t)}\,ds \right.$$

$$\left. + \int_{T-\sigma+r}^{t-\sigma+r} \overline{P}(s+\tau)\, e^{(\lambda_0-\lambda_1)(s-t)}\,ds \right)$$

$$\geq e^{-\lambda_1 t}.$$

It follows that

$$-\lambda_1 \leq \frac{\ln(\Phi\,\omega)(T_1)}{T_1} \leq 0.$$

For $T \leq t < T_1$, we have

$$e^{-\lambda_1 t} \leq \exp\left\{\frac{t}{T_1}\ln(\Phi\,\omega)(T_1)\right\} \leq 1.$$

This show that $e^{-\lambda_1 t} \leq (\Phi\,\omega)(t) \leq 1$ for $T \leq t \leq \overline{T}$. Choose $T \leq t_2 \leq t_1 \leq \overline{T}$ and $|t_1 - t_2| \leq \sigma - \tau$.

let $C_{t_1} \equiv C(t_1 + r)$, $C_{t_2} \equiv C(t_2 + r)$, when $|t_1 - t_2| \leq \sigma - \tau$,

$$|(\Phi\,\omega)(t_1) - (\Phi\,\omega)(t_2)|$$

$$\leq \frac{1}{C_{t_1}C_{t_2}}\{|C_{t_2} e^{\lambda_0 r} x(t_1+r) - C_{t_2} e^{\lambda_0 r} x(t_2+r)|$$

$$+ \left| C_{t_2} \int_{t_1-\sigma+r}^{t_1-\tau+r} P(s+\tau)\,x(s)\,e^{\lambda_0(s-t_1)}\,ds - C_{t_1}\int_{t_2-\sigma+r}^{t_2-\tau+r} p(s)x(s)\,e^{\lambda_0(s-t_2)}\,ds \right|$$

$$+ \left| C_{t_2} \int_{T-\sigma+r}^{t_1-\sigma+r} \overline{P}(s+\tau)\,x(s)\,e^{\lambda_0(s-t_1)}\,ds - C_{t_1}\int_{T-\sigma+r}^{t_2-\sigma+r} \overline{P}(s+\tau)\,x(s)\,e^{\lambda_0(s-t_2)}\,ds \right|$$

$$\leq + \frac{1}{C_{t_1}C_{t_2}}\{|C_{t_2} - C_{t_1}|\,e^{\lambda_0 r} x(t_1+r) + C_{t_1} e^{\lambda_0 r}|x(t_1+r) - x(t_2+r)|$$

$$+ |C_{t_2} - C_{t_1}| \int_{t_1-\sigma+r}^{t_1+\tau+r} P(s+\tau)\,x(s)\,e^{\lambda_0(s-t_1)}\,ds + C_{t_1}\int_{t_2-\tau+r}^{t_1-\tau+r} P(s+\tau)\,x(s)\,e^{\lambda_0(s-t_1)}\,ds$$

$$+ C_{t_1} \int_{t_1-\sigma+r}^{t_2-\tau+r} P(s+\tau)\, x(s)\, | e^{\lambda_0 (s-t_1)} - e^{\lambda_0 (s-t_2)} |\, ds + C_{t_1} \cdot$$

$$\int_{t_2-\sigma+r}^{t_1-\sigma+r} \overline{P}(s+\tau)\, x(s)\, e^{\lambda_0 (s-t_1)}\, ds$$

$$+ | C_{t_2} - C_{t_1} | \int_{T-\sigma+r}^{t_1-\sigma+r} \overline{P}(s+\tau)\, x(s)\, e^{\lambda_0 (s-t_1)}\, ds + C_{t_1} \int_{t_2-\sigma+r}^{t_1-\sigma+r} \overline{P}(s+\tau)\, x(s)\, e^{\lambda_0 (s-t_1)}\, ds$$

$$+ C_{t_1} \int_{T-\sigma+r}^{t_2-\sigma+r} \overline{P}(s+\tau)\, x(s)\, | e^{\lambda_0 (s-t_1)} - e^{\lambda_0 (s-t_2)} |\, ds \}$$

$$\le \frac{e^{\lambda_0 r}}{C^2} M |t_1 - t_2| + \frac{e^{\lambda_0 rk}}{C} |t_1 - t_2| + \frac{PM(\sigma-\tau)}{C^2} e^{\lambda_0 (r-\tau)} |t_1 - t_2| + \frac{P}{C} e^{\lambda_0 (r-\tau)} |t_1 - t_2|$$

$$+ \frac{\lambda_0 P(\sigma - r)}{C} e^{\lambda_0 (r-\tau)} |t_1 - t_2| + \frac{P}{C} e^{\lambda_0 (r-\tau)} |t_1 - t_2| + \frac{\overline{P} M}{C^2 \lambda_0} e^{\lambda_0 (r-\sigma)} |t_1 - t_2|$$

$$+ \frac{\overline{P}}{C} e^{\lambda_0 (r-\sigma)} |t_1 - t_2| + \frac{\overline{P}}{C} e^{\lambda_0 (r-\tau)} |t_1 - t_2| \}$$

$$\le K |t_1 - t_2| \left(\frac{e^{\lambda_0 r}}{C} + \frac{M}{KC^2} \left(e^{\lambda_0 r} + P(\sigma-\tau) e^{\lambda_0 (r-\tau)} + \frac{1}{\lambda_0} \overline{P} e^{\lambda_0 (r-\sigma)} \right) + \frac{e^{\lambda_0 (r-\tau)}}{KC} (2P + P\lambda_0 (\sigma-\tau) + 2\overline{P}) \right)$$

$$\le \beta K |t_1 - t_2|$$

$$< K |t_1 - t_2|.$$

For $T \le t_2 \le t_1 < T_1$, by mean value theorem we have

$$\left| \exp\left\{ \frac{t_1}{T_1} \ln \Phi \omega)(T_1) \right\} - \exp\left\{ \frac{t_2}{T_1} \ln (\Phi \omega)(T_1) \right\} \right| \le \left| \exp\left\{ \frac{t_2}{T_1} \ln (\Phi \omega)(T_1) \right\} \frac{\ln 2 (\Phi \omega)(T_1)}{T_1^2} \right| |t_1 - t_2|$$

$$\le \lambda_1^2 |t_1 - t_2|$$

$$\le (1-\beta) K |t_1 - t_2|$$

$$< K |t_1 - t_2|.$$

For $T \le t_2 < T_1$, $T_1 \le t_1 \le \overline{T}$ and $|t_1 - t_2| \le \sigma - \tau$

$$|(\Phi\,\omega)(t_1) - (\Phi\,\omega)(t_2)| \le |(\Phi\,\omega)(t_1) - \Phi\,\omega)(T)| + |(\Phi\,\omega)(T) - \Phi\,\omega)(t_2)|$$

$$\le \beta K |t_1 - t_2| + (1-\beta) K |t_1 - t_2|$$

$$= K |t_1 - t_2|.$$

When $|t_1 - t_2| > \sigma - \tau$, we can assume that $t_1 = t_2 + n(\sigma - \tau) + \theta$, where $n \in N, 0 \le \theta < \sigma - \tau$

$$|(\Phi\,\omega)(t_1) - (\Phi\,\omega(t_2) \le |(\Phi\,\omega)(t_1) - (\Phi\,\omega)(t_1 - (\sigma - \tau)| + |(\Phi\,\omega)(t_1 - (\sigma - \tau))$$

$$-\ (\Phi\,\omega)(t_1 - 2(\sigma - \tau))| + \ldots + |(\Phi\,\omega)(t_1 - n(\sigma - \tau)) - (\Phi\,\omega)(t_2)|$$

$$\le K(\sigma - \tau) + K(\sigma - \tau) + \ldots + K(t_1 - n(\sigma - \tau) - t_2)$$

$$= K |t_1 - t_2|.$$

Obviously, $\Phi\,\Omega \subset \Omega$. Next we prove that Φ is continuous, choose $\{\omega_n\} \in \Omega$, and $\omega_n \to \omega$, $\omega \in \Omega$.

For $T_1 \le t \le \overline{T}$, we have

$$|(\Phi\,\omega)(t) - (\Phi\,\omega_n)(t)| \le \frac{1}{C(t+r)} (e^{\lambda_1 r} |\omega_n(t) - \omega(t)| + \int_{t-\sigma+r}^{t-\tau+r} P(s+\tau)\, e^{\lambda_1 (s-t)}$$

$$|\omega_n(s) - \omega(s)|\, ds + \int_{T-\sigma+r}^{t-\sigma+r} \overline{P}(s+\tau)\, e^{\lambda_1 (s-t)} |\omega_n(s) - \omega(s)|\, ds$$

$$\le \|\omega_n(t) - \omega(t)\|.$$

For $T \le t < T_1$, by mean value theorem, we have

$$|(\Phi\,\omega)(t) - (\Phi\,\omega_n)(t)| \le |\ln(\Phi\,\omega_n)(T) - \ln(\Phi\,\omega)(T)| = \ln \frac{(\Phi\,\omega_n)(T)}{(\Phi\,\omega)(T)}.$$

Therefore, Φ is continuous. It follows from Schauder's fixed theorem that Φ has a fixed point $\omega(t)$ in Ω. Set $x(t) = \omega(t)\, e^{\lambda_0 t}$. One easily sees that $x(t)$ satisfies (1) for $T \le t \le \overline{T}$. And since $\overline{T}$ is arbitrarily chosen, $x(t)$ satisfy (1) for $t \ge T$ and $x(t) \ge e^{(\lambda_0 - \lambda_1)t}$. Then $x(t)$ tends ∞ when t tends ∞. The proof is complete.

Remark : Assume that $C(t) \equiv C, P(t) \equiv P, Q(t) \equiv Q$, then condition (ii) reduces to

$$F(\lambda) \equiv \lambda - Ce^{-\lambda r} + Pe^{-\lambda r} - Qe^{-\lambda \sigma}$$

has a $\lambda_0 \in \left(0, \frac{\ln C}{r}\right)$ such that $F(\lambda_0) \le 0$. At this time,

$$\frac{1}{C}\left\{e^{(\lambda-\lambda_1)r}+P\int_{t-\sigma+r}^{t-\tau+r}e^{(\lambda_0-\lambda_1)(s-t)}\,ds+\overline{P}\int_{T-\sigma+r}^{t-\sigma+r}e^{(\lambda_0-\lambda_1)(s-t)}\,ds\right\}$$

$$=\frac{1}{C}\left(e^{(\lambda_0-\lambda_1)r}+\frac{P}{\lambda_0-\lambda_1}\left(e^{(\lambda_0-\lambda_1)(r-\tau)}-e^{(\lambda_0-\lambda_1)(r-\sigma)}\right)\right.$$

$$\left.+\frac{\overline{P}}{\lambda_0-\lambda_1}\left(e^{(\lambda_0-\lambda_1)(r-\sigma)}-e^{(\lambda_0-\lambda_1)(T-t+r-\sigma)}\right)\right\}$$

We can choose a λ_1 sufficiently approaching λ_0 such that

$$\frac{1}{C}\left\{e^{(\lambda_0-\lambda_1)r}+\frac{P}{\lambda_0-\lambda_1}\left(e^{(\lambda_0-\lambda_1)(r-\tau)}-e^{(\lambda_0-\lambda_1)(r-\sigma)}\right)+\frac{\overline{P}}{\lambda_0-\lambda_1}e^{(\lambda_0-\lambda_1)(r-\sigma)}\right\}\geq\frac{3}{2}.$$

Since λ_1 is chosen, we can choose T_1 so large that

$$\frac{\overline{P}}{\lambda_0-\lambda_1}e^{(\lambda_0-\lambda_1)(T-t+r-\sigma)}\leq\frac{1}{2},\quad t\geq T_1.$$

So
$$\frac{1}{C}\left\{e^{(\lambda_0-\lambda_1)r}+P\int_{t-\sigma+r}^{t-\tau+r}e^{(\lambda_0-\lambda_1)(s-t)}\,ds+\overline{P}\int_{T-\sigma+r}^{t-\sigma+r}e^{(\lambda_0-\lambda_1)(s-t)}\,ds\right\}>\frac{3}{2}-\frac{1}{2}=1,$$

and then condition (9) holds. Thus eq. (1) has a positive solution. On the other hand, we notice that $F(\lambda)$ is the characteristic equation of equation (1), and there exists $\overline{\lambda}\in(0,\lambda_0)$ such that $F(\overline{\lambda})=0$, since that $F(\lambda_0)\leq 0$ and $F(0) = P - Q > 0$. This indicates that equation (1) has a positive solution $e^{\overline{\lambda}t}$ which tends to ∞. This shows that our result is available in a sense.

3. Oscillatory Behaviour

For convience, we introduce following mark :

$$P_1(t)=\overline{P}(t)\left(\frac{P(t-\tau)}{\overline{P}(t-\tau)}-\int_{t-\sigma+r}^{t}N'_-(s-\tau)\,ds\right),$$

$$Q_1(t)=\overline{P}(t)\left(\frac{Q(t-\tau)}{\overline{P}(t-\sigma)}+\int_{t-\sigma+r}^{t}N'_+(s-\tau)\,ds\right),$$

$$P_2(t)=\overline{P}(t)=\left(\frac{P(t-\tau)}{\overline{P}(t-\tau)}-\int_{t-\sigma+r}^{t}N'_+(s-\tau)\,ds\right),$$

$$Q_2(t)=\overline{P}(t)\left(\frac{Q(t-\tau)}{\overline{P}(t-\sigma)}+\int_{t-\sigma+\tau}^{t}N'_-(s-\tau)\,ds\right),$$

$$\overline{P}_t(t) = P_i(t) - q_i(t + \sigma - \tau),\ N(t) = \frac{Q(t - \tau + \sigma)}{\overline{P}(t)},$$

$$B_i(t) = 1 - C_i + \int_{t-\sigma+\tau}^{t} Q_i(s + \sigma - \tau)\,ds,$$

$$a_i(t) = 1 - C_i + \int_{t-\sigma+r}^{t} p_i(s + \sigma - \tau)\,ds,\ i = 1, 2,$$

$$r_+(t) = \max\{0, r(t)\},\ r_-(t) = \min\{0, r(t)\}$$

where $r(t)$ is a given function.

Lemma 3.1 — Assume that

$$C_1 \le C(t - \tau)\frac{\overline{P}(t)}{\overline{P}(t - r)} \ge C_2,\ Q_i(t) \text{ is bounded,} \qquad \text{... (8)}$$

$$\overline{P}_i(t) > 0,\ \int_T^{\infty} \overline{P}_i(t)\,dt = \infty. \qquad \text{... (9)}$$

If $x(t)$ is an eventually positive solution of eq. (1), then

(i) $\lim_{t\to\infty} y(t) = 0,$

or $\lim_{t\to\infty} y(t) = \infty.$

(ii) In addition if

$$B_1(t) < 0, \qquad \text{... (10)}$$

then it must be

$$\lim_{t\to\infty} y(t) = \infty,$$

where $y(t)$ is defined as

$$y(t) = x(t) - C(t)\,x(t - r) + \int_{t-\sigma+\tau}^{t} Q(s + \sigma - \tau)\,x(s - \tau)\,ds. \qquad \text{... (11)}$$

PROOF : Suppose that $x(t - \tau) > 0, t \ge T_1$, for some $T_1 \ge T + \tau$. From (11), we get

$$y'(t) = -\overline{P}(t)\,x(t - \tau)$$

$$= -\overline{P}(t)\left(y(t - \tau) + C(t - \tau)\,x(t - \tau - r) - \int_{t-\sigma+2}^{t} Q(s - 2\tau + \sigma)\,x(s - 2\tau)\,ds\right)$$

$$= -\overline{P}(t)\, y(t-\tau) - \overline{P}(t)\, C(t-\tau) \frac{y'(t-r)}{-\overline{P}(t-r)} + \overline{P}(t) \int_{t-\sigma+\tau}^{t} Q(s-2\tau+\sigma) \frac{y'(s-\tau)}{-\overline{P}(s-\tau)}\, ds$$

$$= -\overline{P}(t)\, y(t-\tau) - C(t-\tau) \frac{\overline{P}(t)}{\overline{P}(t-r)} y'(t-r) - \overline{P}(t) \int_{t-\sigma+\tau}^{t} N(s-\tau)\, y'(s-\tau)\, ds$$

so $$y'(t) < 0, \quad t \geq T_1 \qquad \dots (12)$$

and $$y'(t) - C(t-\tau) \frac{\overline{P}(t)}{\overline{P}(t-r)} y'(t-r) + \overline{P}(t) \frac{P(t-\tau)}{\overline{P}(t-\tau)} y(t-\tau) - \overline{P}(t) \frac{Q(t-\tau)}{\overline{P}(t-\sigma)} y(t-\sigma) \dots (13)$$

$$-\overline{P}(t) \int_{t-\sigma+\tau}^{t} N'(s-\tau)\, y(s-\tau)\, ds = 0.$$

From (12), it is easy to see that $y(t)$ is positive or negative eventually. If $y(t) > 0$, from (13) we get

$$(y(t) - C_1\, y(t-r))' + P_1(t)\, y(t-\tau) - Q_1(t)\, y(t-\sigma) \leq 0 \qquad \dots (14)$$

and if $y(t) < 0$, we have

$$(y(t) - C_2\, y(t-r))' + P_2(t)\, y(t-\tau) - Q_2(t)\, y(t-\sigma) \geq 0. \qquad \dots (15)$$

Let $$z_i(t) = y(t) - C_i\, y(t-r) + \int_{t-\sigma+r}^{t} Q_i(s+\sigma-\tau)\, y(s-\tau)\, ds \quad i = 1, 2. \qquad \dots (16)$$

From (14), (15) and $(16)_i$, we get

$$z_1'(t) \leq -\overline{P}_1(t)\, y(t-\tau) \qquad \dots (17)$$

and $$z_2'(t) \geq -\overline{P}_2(t)\, y(t-\tau). \qquad \dots (18)$$

It follows from (12) that $\lim_{t\to\infty} y(t) = l \in R$ or $\lim_{t\to\infty} y(t) = -\infty$. Next, we show that $l = 0$. If it is not true, we assume first of all that $l > 0$, $l > 0$, then for some $T_2 \geq T_1$, we have

$$y(t) \geq \frac{l}{2}, \quad t \geq T_2.$$

Integrate (17) from $T_2 + \tau$ to t

$$z_1(t) \leq z_1(t+\tau) - \int_{T_2}^{t} \overline{P}_2(s+\tau)\, y(s) ds \leq z_1(t+\tau) - \frac{1}{2} \int_{T_2}^{t} \overline{P}_2(s+\tau)\, ds.$$

This implies that $\lim_{t \to \infty} z_1(t) = -\infty$. On the other hand, we see that $z_1(t)$ is bounded from (8) and $(16)_1$, this is a contradiction.

If $l < 0$, in a similar way, we can get $\lim_{t \to \infty} z_2(t) = \infty$ from (18). This is also a contradiction with the fact $z_2(t)$ is bounded. Thus $1 = 0$.

(ii) On the contrary, suppose that $\lim_{t \to \infty} y(t) = 0$. Combining it with (12), we get $y(t) = 0$. It is easy to obtain from $(16)_1$.

$$z_1(t) \le \left(1 - C_1 + \int_{t-\sigma+\tau}^{t} Q(s+\sigma-\tau)\, ds \right) y(t-r) < 0. \qquad \text{... (19)}$$

On the other hand, from $(16)_1$ one easily sees that $\lim_{t \to \infty} z_1(t) = 0$, since that $Q_1(t)$ is bounded and $z_1'(t) \le 0$ from (17). This shows that $z_1(t) > 0$, which is a conraction with (19). The proof of this lemma is complete.

Lemma 3.2[9] — Assume

$$p \in C([T, \infty), R^+),\ \tau > 0$$

and

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} p(s)\, ds > \frac{1}{e}.$$

Then differential inequality

$$u'(t) - p(t)u(t+\tau) \ge 0$$

and

$$u'(t) + p(t)u(t-\tau) \le 0$$

has not eventually positive solution.

Theorem 3.1 — *Suppose that all conditions in Lemma 3.1 (include (10)) is satisfied, and*

$$\liminf_{t \to \infty} \int_{t+r-\tau}^{t} \frac{\overline{P}(s)}{-B_2(s+r-\tau)}\, ds > \frac{1}{e}. \qquad \text{... (20)}$$

Then every solution of equation (1) oscillates.

PROOF : Without loss of generality, on the contrary we suppose that equation (1) has a eventually positive solution $x(t)$. It follows from Lemma 3.1 that $z_2(t) > 0$. From $(16)_2$, we obtain

$$z_2(t) \le B_2(t)\, y(t-r).$$

On the other hand, (8) and (10) implies $B_2(t) < 0$. Substitute the preceding inequality into (18), we obtain

$$z_2'(t) - \frac{\overline{P}_2(t)}{-B(t+r-\tau)} z_2(t+(r-\tau)) \geq 0. \qquad \text{... (21)}$$

By Lemma 3.2 and (20), we see that differential inequality (21) has not positive solution, a contradiction to our assumation. The proof is complete.

Now, we show our result is available by a example.

Example — Consider

$$\left(x(t) - (8+\sin t)\, x\left(t - \frac{3\pi}{2}\right)\right)' + (9+\sin t)\, x\left(t-\frac{\pi}{2}\right) - x(t-\pi) = 0. \qquad \text{...(22)}$$

Notice that

$$\overline{P}(t) = 8 + \sin t,\ N(t) = \frac{1}{8+\sin t},\ |N'(t)| = \left| \frac{-\cos t}{(8+\sin t)^2} \right| \leq \frac{1}{49},$$

$$P_1(t) = (8+\sin t)\left(\frac{9+\sin\left(t-\frac{\pi}{2}\right)}{8+\sin\left(t-\frac{\pi}{2}\right)}\right) - \int_{t-\frac{\pi}{2}}^{t} \left[\frac{-\cos t}{(8+\sin t)^2}\right] dt \geq \frac{70}{9},$$

$$Q_1(t) = (8+\sin t)\left(\frac{1}{8+\sin(t-\pi)}\right) + \int_{t-\frac{\pi}{2}}^{t} \left[\frac{-\cos t}{(8+\sin t)^2}\right]_+ dt \leq \frac{19}{14},$$

$$P_2(t) = (8+\sin t)\left(\frac{9+\sin\left(t-\frac{\pi}{2}\right)}{8+\sin\left(t-\frac{\pi}{2}\right)}\right) - \int_{t-\frac{pi}{2}}^{t} \left[\frac{-\cos t}{(8=\sin t)^2}\right]_+ dt \geq 7,$$

$$Q_2(t) = (8+\sin t)\left(\frac{1}{9+\sin(t-\pi)}\right) + \int_{t-\frac{\pi}{2}}^{t} \left[\frac{-\cos t}{(8+\sin t)^2}\right] dt < \frac{9}{7},$$

$$\frac{49}{9} \leq C\left(t-\frac{\pi}{2}\right)\frac{\overline{P}(t)}{\overline{P}\left(t-\frac{3\pi}{2}\right)} \leq \frac{81}{7}.$$

Take $C_1 = \frac{49}{9},\ C_2 = \frac{81}{7}$, and then

$$B_1(t) = 1 - C_1 + \int_{t-\frac{\pi}{2}}^{t} Q_1\left(t+\frac{\pi}{2}\right) dt \leq 1 - \frac{49}{9} + \frac{19}{14} \times \frac{\pi}{2} < 0,$$

$$-\frac{74}{7} \leq B_2(t) = 1 - C_2 + \int_{t-\frac{\pi}{2}}^{t} Q_2\left(t + \frac{\pi}{2}\right) dt \leq 1 - \frac{81}{7} + \frac{18}{7} < 0,$$

$$\int_{t+r-\tau}^{t} \frac{\overline{P}_2(s)}{-B_2(s+r-\tau)} ds \geq \frac{7 - \frac{9}{7}}{\frac{74}{7}} \times \frac{\pi}{2} > \frac{1}{e}.$$

So, all conditions of Theorem 3.1 are satisfied, each solution of (22) is oscillatory.

Theorem 3.2 — *Suppose that all conditions in Lemma 3.1 (include (10)) are satisfied and $C(t)$ is bounded. Then each nonoscillation solution of equation (1) is unbounded.*

PROOF : By (11), we obtain $y(t) \geq -C(t)\, x(t-r)$, or $x(t-r) \geq \frac{y(t)}{-C(t)}$. This shows that $x(t)$ is unbounded. The proof is complete.

Theorem 3.3 — *Suppose that*

$$A(t) = 1 - C(t) + \int_{t-\sigma+\tau}^{t} P(s) \leq 0, \qquad \text{... (23)}$$

$$\int_{T}^{\infty} \overline{P}(s) ds = \infty. \qquad \text{... (24)}$$

If differential inequality

$$(x(t) - C(t)\, x(t-r))' + P(t)\, x(t-\tau) - Q(t)\, x(t-\sigma) \leq 0 \qquad \text{... (25)}$$

has a monotone continuous positive solution $X(t)$, then equation (1) has a eventually positive solution.

PROOF : Define

$$\upsilon(t) = x(t) - C(t)\, x(t-r) + \int_{t-\sigma+r}^{t} P(s)\, x(s-\tau)\, ds \qquad \text{... (26)}$$

we claim that $\upsilon(t)$ is eventually negative.

Case 1 — $x(t)$ is monotone decreasing. $\upsilon(t) \leq A(t)\, x(t-r) < 0, t \geq T + r$.

Case 2 — $x(t)$ is monotone increasing. There exists $m > 0$ such that $x(t) \geq m$ for $t \geq T$. From (25) and (26), we get

$$\upsilon'(t) \leq -\overline{P}(t-\sigma+\tau)\, x(t-\sigma).$$

Integrating the preceding inequality from $T + \sigma$ to t, we obtain

$$\upsilon(t) \leq \upsilon(T+\sigma) - \int_{T+\sigma}^{t} \overline{P}(s-\sigma+\tau)\, x(s-\sigma)\, ds \leq \upsilon(T+\tau) - m \int_{T+\sigma}^{t} \overline{P}(s-\sigma+\tau)\, ds.$$

Obviously, $\lim_{t \to \infty} \upsilon(t) = -\infty$, and there exists $T' \geq T$ such that $\upsilon(t) \leq 0$ for $t \geq T'$. Integrate (26) from $T'+r$ to t.

$$x(t) - C(t)\,x(t-r) - (x(T'+r) - C(T'+r)\,x(T)) + \int_{T'+r}^{t} P(s)\,x(s-\tau)\,ds - \int_{T'+r}^{t} Q(s)\,x(s-\sigma)\,ds \leq 0.$$

Further

$$\frac{1}{C(t+r)}\left\{x(t+r) + \int_{t-\sigma+r}^{t-\tau+r} P(s+\tau)\,x(s)ds + \int_{T'-\sigma+r}^{t-\sigma+r} \overline{P}(s+\tau)\,x(s)ds\right\} \leq x(t) + \frac{\upsilon(T'+r)}{C(t+r)} \leq x(t).$$

By Lemma 2.1, we see that integral eq. (5) has a positive solution $X(t)$, it is easily to see that $X(t)$ satisfies (1). The proof is complete.

***Theorem* 3.4** — *Assume that (8), (10), (23) hold and*

$$p_i(t) \leq p_i(t),\ q_i(t) \geq Q_i(t),\ i = 1, 2, \qquad \dots (27)$$

and

$$a_i(t) < 0,\ \int_T^{\infty} a_i(t)\,dt = \infty,\ i = 1, 2, \qquad \dots (28)$$

where $p_i, q_i \in C([T,\infty), R^+)$, $\overline{p}_i(t) > 0$. *If there exists* $p_i(t), q_i(t)$ *such that each solution of following equations*

$$(y(t) + C_i\,y(t-r))' + p_i(t)\,y(t-\tau) - q_i(t)\,y(t-\sigma) = 0,\ i = 1, 2. \qquad \dots (29)$$

oscillates, then all solution of eq. (1) are oscillatory.

PROOF : Without loss of generality, on the contrary, suppose that equation (1) has an eventually positive solution $x(t)$, that is, there exists $T_1 \geq T$ such that $x(t) > 0$ for $t \geq T_1$. From (11) and (23), we get $y'(t) < 0$ for $t \geq T_1 + \tau$. Consequently, from (14), (15) and $(27)_i$ we have

$$(y(t) + C_1\,y(t-r))' + p_1(t)\,y(t-\tau) - q_1(t)\,y(t-\sigma) \leq 0. \qquad \dots (30)$$

for $y(t) > 0$ and

$$(y(t) + C_2\,y(t-r))' + p_2(t)\,y(t-\tau) - q_2(t)\,y(t-\sigma) \geq 0. \qquad \dots (31)$$

for $y(t) < 0$. We set $w(t) = -y(t)$, for $y(t) < 0$. And then $w(t) > 0$ and $w'(t) > 0$, (31) reduces to

$$(w(t) - C_2\,w(t-r))' + p_2(t)\,(w(t-\tau) - q_2(t)\,w(t-\sigma) \leq 0. \qquad \dots (32)$$

By Theorem 3.3, we obtain that eqs. (29) have eventually positive solution. Clearly, this is a contradiction. The proof is complete.

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