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A RESULT IN BEST APPROXIMATION THEORY

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In this note we generalize a result of Saheb *et al.* [*J Approx. Theory* **55** (3) (1988), 349-351] in best approximation theory under some weaker conditions and using a fixed point theorem of Dhage [*Math Sci. Res. Hot-Line* **4** (2) (2000), 45-55].

Key Words : D-metric Space; Best Approximation and Fixed Point etc.

1. INTRODUCTION

Let X be a normed linear space with a norm $\|\cdot\|$ and let C be a non-empty subset of X. If $x_0 \in X$, then the set of all best C-approximants to x_0 in C to be a set $\mathcal{A}c(x_0)$ in C defined by

$$\mathcal{A}c(x_0) = \{x \in C \mid \| x_0 - y \| = d(x_0, C)\}, \qquad \text{... (1)}$$

where $\qquad d(x_0, C) = \inf \{\| x_0 - y \| : y \in C\}.$

Denote $\qquad \mathcal{A}c(x_0) = \mathcal{A}_C(x_0) \cup \{x_0\}. \qquad$ ... (2)

A subset C of X is said to be starshaped with respect to a point $q \in C$ if $(1-\lambda) q + \lambda x \in C$ for all $x \in C$ and for all $\lambda \in [0, 11]$. A convex set is starshaped with respect to each of its points.

A mapping $f : X \to X$ is called a *contraction* with respect to a map $g : X \to X$ if

$$\| fx - fy \| \leq \alpha \| gx - gy \| \; (\alpha < 1)$$

and *nonexpansive* with respect to a map $g : X \to X$ if

$$\| fx - fy \| \leq \| gx - gy \|$$

for all $x, y \in X$.

Two maps $f, g : X \to X$ are called commutative or commuting if $(fg)(x) = (gf)(x)$ for all $x \in X$. Generalizing the best approximation results in Brosowski[1] and Singh[6], Sahab *et al.*[5] proved the following result in the theory of best approximations.

Theorem 1.1 — *Let C be a non-empty subsect of the normed linear space X and let $x_0 \in X$, be a common fixed point of the mappings $f, g : X \to X$. Suppose that*

(i) f is nonexpansive on $\mathcal{A}'_c(x_0)$ w.r.t. the map g

(ii) $f : \partial c \to C$, where ∂c is a boundary of C,

(iii) g is linear and continuous on $\mathcal{A}_c(x_0)$ and

(iv) $\{f, g\}$ are commuting on $\mathcal{A}_c(x_0)$.

Further of (x_0) is non-empty, compact and starshaped with respect to a point $q = gq$ and if $f(\mathcal{A}_c(x_0)) = \mathcal{A}_c(x_0)$, then f and g have a common fixed point in C closest to x_0.

In this note we obtain an existence theorem similar to Theorem 1.1 concerning the existence of an element of best approximation from a subset to two points of a normed linear space under some weaker conditions.

2. PRELIMINARIES

Following Dhage[2], we have

Definition 2.1 — A non-empty set X together with a function $\rho : X \times X \times X \to (0, \infty)$ is called a D-metric space with a D-metric ρ, denoted by (X, ρ) if ρ satisfies

(i) $\rho(x, y, z) = 0 \Leftrightarrow x = y = z$ (coincidence)

(ii) $\rho(x, y, z) = \rho(p\{x, y, z\})$ (symmetry)

where ρ is a permutation of $\{x, y, z\}$

and (iii) $\rho(x, y, z) \le \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z)$

for all $x, y, z, a \in X$. (tetrachedral inequality).

A few details along with some specific examples and topological properties of a D-metric space appear in Dhage[2].

A sequence $\{x_n\} \subset X$ is called convergent and converges to a point $x \in X$ if $\lim\limits_{m,n} \rho(x_m, x_n, x) = 0$. Again a sequence $\{x_n\} \subset X$ is called *D-Cauchy* if $\lim\limits_{m,n,p} \rho(x_m, x_n, x_p) = 0$. A *complete* D-metric space is one which every D-Cauchy sequence converges to a point in it.

It is known that the D-metric ρ is a continuous function on X^3 in the topology of D-metric convergence which is Housdroff. See Dhage[2].

A mapping $f : (X, \rho) \to (X, \rho)$ is said to *D-contractive* with respect to a mapping $g : (X, \rho) \to (X, \rho)$ if

$$\rho(fx, fy, fz) < \rho(gx, gy, gz) \qquad \text{... (3)}$$

for all $x, y, z \in X$ for which $\rho(gx, gy, gz) \neq 0$, and *D-nonexpansive* w.r.t. the map $g : (X, \rho) \to (X, \rho)$ if

$$\rho(fx, fy, fz) \le \rho(gx, gy, gz) \qquad \text{... (4)}$$

for all $x, y, z \in X$.

We shall use a fixed point theorem of Dhage[2] in the proof of our main results. Also see Dhage[3]. We need the following definitions in the sequel[2].

Two maps $f, g : (X, \rho) \to (X, \rho)$ are called *coincident* if there is an $x \in X$ such that $fx = gx$ and *limit coincident* if there is a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n$. Similarly they are called *limit commutative* or *limit commuting* if there is a sequence $\{x_n\}$ in X s.t. $\lim_n (fg)(x_n) = \lim_n (gf)(x_n)$. Again two maps $f, g : (X, \rho) \to (X, \rho)$ are called *limit coincidentally commuting* if their limit coinicidence implies the limit commutingness on X, i.e. there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n$ implies $\lim_n (fg)(x_n) = \lim_n (gf)(x_n)$.

Similarly two maps $f, g : (X, \rho) \to (X, \rho)$ are called *coincidentally commuting* if they commute at coincidence points. Finally a mapping $f : (X, \rho) \to (X, \rho)$ is *continuous* if and only if for any sequence $\{x_n\}$ in X, $x_n \to x$ implies $fx_n \to fx$.

It is shown in Dhage[3] that every commuting pair of maps on a D-metric space is limit coincidentally commuting, but the converse may not be true. Similarly limit coincidentally commutativity implies coincidentally commutativity, but the converse may not be true.

We need the following fixed point theorem of Dhage[4] in the sequel. See also B.C. Dhage [*Math. Sci. Res. Hot-Line* **4**(2), (2000), 45-55].

Theorem 2.1[4] — *Let f and g be two continuous selfmaps of a compact D-metric space X satisfying (3). Further suppose that*

(i) $f(X) \subseteq g(X)$

and (ii) $\{f, g\}$ *are coincidentally commuting.*

Then f and g have a unique common fixed point.

Let $x_0, y_0 \in X$ and let us denote

$$D(x_0, y_0, C) = \inf \{\rho(x_0, y_0, c) \mid c \in C\}. \quad \text{... (5)}$$

An element $z \in C$ is said to be a best approximation to x_0 and y_0 from C or closest to x_0 and y_0 from C if

$$\rho(x_0, y_0, z) = D(x_0, y_0, C).$$

In this case the element z is called a best C-approximation to x_0 and y_0 from C and the set of all such best C-approximants to x_0 and y_0 from C is denoted by $\mathcal{A}_c(x_0, y_0)$. Thus we have

$$\mathcal{A}_c(x_0, y_0) = \{z \in C \mid \rho(x_0, y_0, z) = D(x_0, y_0, C)\}. \quad \text{... (6)}$$

Denote
$$\mathcal{A}_c(x_0, y_0) = \mathcal{A}_c(x_0, y_0) \cup \{x_0, y_0\}. \quad \text{... (7)}$$

Note that the notion of a closest element to x_0 and y_0 from a subset C is different from that of closest element to the set $\{x_0, y_0\}$ from C. See Dhage[5].

3. Main Results

Theorem 3.1 — *Let C be a non-empty subset of a D-metric space X and let $\{x_0, y_0\} \subset X$. Let $f, g : X \to X$ be two mappings satisfying (3). Suppose that*

(i) f is D-nonexpansive on $\mathcal{A}'_c(x_0, y_0)$ w.r.t. the map g,

(ii) $f : C \to C$,

(iii) f, g : $\{x_0, y_0\} \to \{x_0, y_0\}$ are injective,

(iv) $\{f, g\}$ are coincidentally commuting on $\mathcal{A}' c(x_0, y_0)$

and (v) g is continuous on X.

Further if $\mathcal{A}_c(x_0, y_0)$ is non-empty compact subset of X and $f(\mathcal{A}_c^{\cdot}(x_0, y_0)) = \mathcal{A}_c(x_0, y_0)$, then f and g have a common fixed point in C which is closest to x_0 and y_0.

Proof : Let $x \in \mathcal{A}'_c(x_0, y_0)$. Then by hypothesis, Let $gx \in \mathcal{A}'_c(x_0, y_0)$. Now from hypothesis (i) it follows that

$$\begin{aligned}\rho(fx, x_0, y_0) &= \rho(fx, fx_0, fy_0)\\ &\le \rho(gx, gx_0, gy_0)\\ &= \rho(gx, x_0, y_0)\\ &= D(x_0, y_0, C)\end{aligned}$$

and so $fx \in \mathcal{A}_c(x_0, y_0)$ because $fx \in C$. Since the hypothesis (ii) holds, $f(\mathcal{A}_c(x_0, y_0)) \subseteq \mathcal{A}_c(x_0, y_0)$. Now the desired conclusion follows from an application of Theorem 2.1.

Theorem 3.2 — *Let C be a non-empty subset of a normed linear space X and let $\{x_0, y_0\} \subset X$. Let $f, g : X \to X$ be two mappings satisfying the following conditions :*

(i) f is nonexpansive on $\mathcal{A}'_c(x_0, y_0)$ w.r.t. the map g,

(ii) $f : \partial C \to C$, where ∂C is a boundary of C,

(iii) f, g : $\{x_0, y_0\} \to \{x_0, y_0\}$ are injective,

(iv) $\{f, g\}$ are limit coincidentally commuting on $\mathcal{A}_c(x_0, y_0)$

and (v) g is uniformly continuous on X.

Further if $\mathcal{A}_c(x_0, y_0)$ is non-empty compact and starshaped w.r.t. a point $q \in \mathcal{A}_c(x_0, y_0)$ and if $g(\mathcal{A}_c(x_0, y_0)) \subseteq \mathcal{A}_c(x_0, y_0)$, then f and g have a common fixed point in C which is closest to x_0 and y_0 with respect to a D-metric ρ on X defined by $\rho(x, y, z) = \| x - y \| = \| y - z \| + \| z - x \|$.

Proof : Given D-metric ρ on the normed linear space X is

$$\rho(x, y, z) = \| x - y \| + \| y - z \| + \| z - x \| \qquad \dots (8)$$

for $x, y, z \in X$.

Since f is nonexpansive on X w.r.t.the map g, we have

$$\rho(fx, fy, fz) = \|fx - fy\| + \|fy - fz\| + \|fz - fx\|$$

$$\leq \|gx - gy\| + \|gy - gz\| + \|gz - gx\|$$

$$= \rho(gx, gy, gz)$$

for all $x, y, z \in X$ which shows that f is also D-nonexpansive on X w.r.t. the map g.

Let $y \in \mathcal{A}_c(x_0, y_0)$. Then $gy \in \mathcal{A}_c(x_0, y_0)$ since $g(\mathcal{A}_c(x_0, y_0)) \subseteq \mathcal{A}_c(x_0, y_0)$.

As it has been mentioned in Singh [7] that the element of best C-approximation need not belong to the interior of C, so $y \in \partial C$. Since $f(\partial C) \subset C$, we have $fy \in C$ and from (6), it follows that

$$\rho(fy, x_0, y_0) = \rho(fy, fx_0, fy_0)$$

$$\leq \rho(gy, gx_0, gy_0)$$

$$= \rho(gy, x_0, y_0)$$

$$= D(x_0, y_0, C)$$

and therefore $fy \in \mathcal{A}_c(x_0, y_0)$.

Let $\{t_n\}$ be a sequence of real numbers such that $0 \leq t_n < 1$ and for sufficiently large $n, t_n = 1$, i.e. $t_n \to 1$ as $n \to \infty$.

Define a sequence $\{f_k\}$ of mappings on $\mathcal{A}_c(x_0, y_0)$ by

$$f_k(x) = (1 - t_k)q + t_k fx$$

for each $k \in N$. From hypothesis (*i*) it follows that f and consequently each f_k is uniformly continuous on $\mathcal{A}_c(x_0, y_0)$.

Since $\mathcal{A}c(x_0, y_0)$ is starshaped w.r.t. the point $q \in \mathcal{A}_c(x_0, y_0)$, $f_k(x)\ \mathcal{A}_c(x_0, y_0)$ for each $k \in N$ and for each $x \in \mathcal{A}_c(x_0, y_0)$. Consequently $f_k(\mathcal{A}_c(x_0, y_0)) \subseteq \mathcal{A}_c(x_0, y_0)$ for each $k \in N$. Next we show that $\{f_{k,} g\}$ satisfy condition (3) and are limit coincidentally commuting on $\mathcal{A}_c(x_0, y_0)$ for sufficiently large value of k.

First we show that $\{f_k, g\}$ satisfy condition (3) on $\mathcal{A}_c(x_0, y_0)$ for each $k \in N$. Since f is nonexpansive on $\mathcal{A}_c(x_0, y_0)$ w.r.t. the map g, we have

$$\rho(f_k x, f_k y, f_k z) = \|f_k x - f_k y\| + \|f_k y - f_k z\| + \|f_k z - f_k x\|$$

$$= t_k \|fx - fy\| + t_k \|fy - fz\| + t_k \|fz - fx\|$$

$$\leq t_k \| gx - gy \| + t_k \| gy - gz \| + t_k \| gz - gx \|$$

$$= t_k [\| gx - gy \| + \| gy - hz \| + \| gz - gx \|]$$

$$= t_k \rho (gx, gy, gz)$$

$$< \rho (gx, gy, gz) \ (t_k < 1)$$

for all $x, y, z \in \mathcal{A}_c (x_0, y_0)$ for which $\rho (gx, gy, gz) \neq 0$.

This shows that $\{f_k, g\}$ satisfy condition (3) on $\mathcal{A}_c (x_0, y_0)$ and hence are *D*-contractive on $\mathcal{A}_c (x_0, y_0)$.

Next we show that $\{f_{k,} g\}$ are limit coincidentally commuting on $\mathcal{A}_c (x_0, y_0)$ for sufficiently large value of k. Assume that $\{f_{k,} g\}$ are limit coincident on $\mathcal{A}_c (x_0, y_0)$ for large value of k, that is, there is a sequence $\{x_n\}$ in $\mathcal{A}_c (x_0, y_0)$ such that

$$\lim_n f_k x_n = \lim_n gx_n$$

for large value of k.

By the definition of f_k,

$$\lim_n (\lim_k f_k x_n) = \lim_n f x_n = \lim_n gx_n$$

i.e.
$$\lim_n \left(\lim_k f_k x_n \right) = \lim_n gx_n.$$

Now
$$\lim_k f_k gx_n = \lim_k [(1 - t_k) q + t_k gx_n]$$

$$= \lim_k (1 - t_k) q + \lim_k t_k f gx_n$$

$$= f gx_n \ (t_k \to 1)$$

Therefore by the uniform continuity of g,

$$\lim_n \left(\lim_k f_k gx_n \right) = \lim_n f gx_n$$

$$= \lim_n g f x_n$$

$$= g \left(\lim_n f x_n \right)$$

$$= g \left(\lim_n \left(\lim_k f_k x_n \right) \right)$$

$$= \lim_{n} \left(g \lim_{k} f_k x_n \right)$$

$$= \lim_{n} \left(\lim_{k} g f_k x_n \right)$$

i.e.
$$\lim_{k} \left(\lim_{n} g f_k g x_n \right) = \lim_{k} \left(\lim_{n} g f_k x_n \right)$$

which shows that f_k and g are limit coincidentally commuting on $\mathcal{A}_c(x_0, y_0)$ for sufficiently large value of k.

It is known that the norm $\| \cdot \|$ and the D-metric ρ defined by (8) generate the equivalent topologies on X. Therefore, the compactness of $\mathcal{A}_c(x_0, y_0)$ and the continuity of g w.r.t. the norm $\| \cdot \|$ implies the compactness of $\mathcal{A}_c(x_0, y_0)$ and the continuity of g w.r.t. the D-metric ρ on X. See Dhage[2]. Now an application of Theorem 2.1 yields that f_k and g have a unique common fixed point x_k in $\mathcal{A}_c(x_0, y_0)$ for sufficiently large value of k, i.e. we have

$$f_k x_n = x_n \to g x_n$$

for sufficiently large value of k.

The compactness of $\mathcal{A}_c(x_0, y_0)$ implies that the sequence $\{x_n\}$ has a convergent subsequence, say $\{x_{k(i)}\}$ converging to a point $z \in \mathcal{A}_c(x_0, y_0)$.

By the definition of $f_{k(i)}$.

$$x_{k(i)} = f_{k(i)} x_{k(i)} = (1 - t_{k(i)})\, q + t_{k(i)} f x_{k(i)}. \quad \text{... (9)}$$

Since g is continuous, the D-nonexpansiveness of f w.r.t. g implies that f is also continuous on $\mathcal{A}_c(x_0, y_0)$ w.r.t the D-metric ρ on it. Therefore, taking the limit as $i \to \infty$ in (8) we obtain

$$z = \lim_{i} x_{k(i)} = \lim_{i} f_{k(i)} x_{k(i)} = \lim_{i} t_{k(i)} f x_{k(i)} = f z.$$

Similarly
$$g z = g \left(\lim_{i} x_{k(i)} \right) = \lim_{i} g x_{k(i)} = \lim_{i} x_{k(i)} = z.$$

Thus z is a common fixed point of f and g in C which is closest to. x_0 and y_0. This completes the proof.

Remark 2.1 : The conclusion of Theorem 2.2 also remains true if we replace the D-metric ρ given in (8) by

$$\rho\,(x, y, z) = \max\, \{\| x - y \|, \| y - z \|, \| z - x \|\}. \quad \text{... (10)}$$

Finally while conducting this paper, we mention that when $x_0 = y_0$ and $\rho\,(x, y, z\} = \max\, \{d\,(x, y), d(y, z), d(z, x)\}$, where d is an ordinary metric on the normed linear space X defined by $d(x, y) = \| x - y \|$, our Theorem 3.2 generalizes Theorem 1 of Sahab *et al.*[5] under the weaker conditions of the commutativity of f and g, the linearity of g and the starshapedness of $\mathcal{A}_c(x_0, y_0)$. Thus our result is much more general than that given in Brosowski[1], Singh[6] and Sahab *et al.*[5].

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