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CRITICAL VALUES OF DEFORMED OSCULATING HYPERRULED SURFACES

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As is well known, the most useful method of studying the properties of a curve in a Euclidean space, from the standpoint of differential geometry, is making use of the Frenet formulas, in which the curvatures are the essential quantities for the curve. So, the motivation of the present work is to develop the variational problem in our work[1] by using an auxiliary formula of Frenet.[2, 6, 7 & 12]. Hence, the invariants of the of hyperruled surfaces generated by the osculating space in E^{n+1} (osculating hyperruled surfaces) are interpreted in terms of the curvatures for the base curve. Furthermore, the variation of these invariants are calculated. The variation of the curvatures for the base curve and the Frenet-frame are obtained. The necessary and sufficient condition for the stability of the osculating hyperruled surfaces in terms of the curvatures for the base curve are derived. Finally, the solution of the differential equation which is produced from stability condition, for example in E^3 and E^4, is obtained.

Key Words : Stability; Osculating Hyperruled Surfaces

1. INTRODUCTION

Here, and in the sequel, we assume that the indices $\{\nu, \mu\}$, $\{\gamma, \lambda\}$ and $\{i, j, k\}$ run over the ranges $\{2, ..., n-1\}$, $\{1, ... n-1\}$ and $\{0, ... n-1\}$ respectively unless otherwise stated.

Let M be an oriented n-dimensional hyperruled surface is an Euclidean $(n+1)$-space E^{n+1}, with a base curve $r :]a, b[\to E^{n+1}, r = r(u^0)$ and u^0 is the arc length. The Frenet-frame is dentoed by $\{e_i(u^0)\}$, $1 \leq i \leq n+1$, where $e_1(u^0)$ is the unit tangent vector. The n-dimensional hyperruled surface in E^{n+1} generated by $OS_p(r(u^0))$. Then $(n-1)$-dimensional linear osculating space of $r = r(u^0)$ is denoted by $OS_p(r(u^0))$ and generated by $\{e_\gamma(u^0)\}$. Then M can be represented locally by :

$$X(u^i) = r(u^0) + u^1 (e_1(u^0) + \sum_\nu u^\nu e_\nu(u^0), u^0 \in]a, b[, u^\gamma \in R, \quad ... (1)$$

which is called an osculating n-hyperruled surface in E^{n+1}. The tangent space at a point p of the hyperruled surface is spanned by the generating $(n-1)$-dimensional linear osculating space $OS_p(r(u^0))$ through p and $X_0 := \frac{\partial X}{\partial u^0}$. The unit normal vector field on M at a point p is $e_{n+1}(u^0)$. Thus, the representation (1) is a regular parameterization for a regular base curve $r = r(u^0)$.

Throughout the rest of this section we would like to mention of the following definition which are very important in the sequel [2], [6], [7] and [12] :

Definition 1.1 — If $r(u^0)$ is a curve E^{n+1}, parametrized by arc length u^0 and unit tangent vector $e_1(u^0)$, we say that r is a Frenet curve of osculating order $n+1$ when there exist orthonormal vector fields $\{e_i(u^0)\}$ along r such that :

$$\left.\begin{array}{l} \dot{r}(u^0) = e_1(u^0),\ \nabla_{e_1} e_1 = \kappa_1 e_2,\ \nabla_{e_1} e_2 = -\kappa_1 e_1 + \kappa_2 e_3, \dots \\ \nabla_{e_1} e_n = -\kappa_{n-1} e_{n-1} + \kappa_n e_{n+1},\ \nabla_{e_1} e_{n+1} = -\kappa_n e_n, \end{array}\right\} \quad \dots (2)$$

where $\kappa_1, \dots \kappa_{n-1}$ are positive C^∞ functions of u^0 and $\kappa_n \neq 0$ is C^∞ function of u^0 and ∇ the Riemannian connection. The eqs. (2) are the **Frenet formulas in E^{n+1}.**

2. The Fundamental Quantities

In the following, the fundamental quantities g_{ij}, g^{ij}, h_{ij} and h_i^j of M are derived in terms of the curvatures κ_i, $1 \leq i \leq n$, for the base curve $r(u^0)$.

From (1) and (2) we have :

$$\left.\begin{array}{l} \nabla_{e_1} X = X_0 = \dfrac{\partial X}{\partial u^0} = e_1 + u^1 \kappa_1 e_2 + \sum\limits_{\nu} u^\nu [-\kappa_{\nu-1} e_{\nu-1} + \kappa_\nu e_{\nu+1}], \\ X_\gamma = e_\gamma(u^0), \end{array}\right\} \quad \dots (3)$$

$$\left.\begin{array}{l} \nabla_{e_1} X_0 = -(u^1 \kappa_1^2) e_1 + (\kappa_1 + u^1 \dot{\kappa}_1) e_2 + (u^1 \kappa_1 \kappa_2) e_3 \\ \qquad + \sum\limits_{\nu} u^\nu [(\kappa_{\theta-1} \kappa_{\nu-2}) e_{\nu-2} - (\dot{\kappa}_{\nu-1}) e_{\nu-1} \\ \qquad - (\kappa_{\nu-1}^2 + \kappa_\nu e_\nu + (\dot{\kappa}_\nu) e_{\nu+1} + (\kappa_\nu \kappa_{\nu+1}) e_{\nu+\dot{2}}] \\ \nabla_{e_1} X_\nu = -\kappa_{\nu-1} e_{\nu-1} + \kappa_\nu e_{\nu+1}, \\ \nabla_{e_1} X_1 = \kappa_1 e_2,\ X_{\gamma\gamma} = X_{\gamma\lambda} = 0,\ \forall \gamma, \lambda, \end{array}\right\} \quad \dots (4)$$

where $$X_i = \frac{\partial X}{\partial u^i},\ \nabla_{e_1} X_i = \frac{\partial X_i}{\partial u^0},\ \cdot = \frac{\partial}{\partial u^0}.$$

From (3) we have :

$$g_{00} = \langle \nabla_{e_1} X, \nabla_{e_1} X \rangle = 1 - 2u^2 \kappa_1 + u^1 \kappa_1 (u^1 \kappa_1 - 2u^3 \kappa_2) + \sum_\nu (u^\nu)^2 (\kappa_{\nu-1}^2 + \kappa_\nu^2), \quad \dots (5)$$

$$g_{01} = \langle \nabla_{e_1} X, X_1 \rangle = 1 - u^2 \kappa_1, \qquad \text{... (6)}$$

$$g_{0\nu} = \langle \nabla_{e_1} X, X_\nu \rangle = u^{\nu-1} \kappa_{\nu-1} - u^{\nu+1} \kappa_\nu, \qquad \text{... (7)}$$

$$g_{\gamma\gamma} = 1, \; g_{\gamma\nu} = g_{\nu\mu} = 0, \; \forall \; \gamma, \nu, \mu. \qquad \text{... (8)}$$

Thus, using mathematical induction we can see that :

$$g = \det(g_{ij}) = g_{00} - \sum_{\gamma} g_{0\gamma}^{2} \; . \qquad \text{... (9)}$$

From (5), (6) and (7) we have :

$$g = [(u^1)^2 - (u^2)^2]\, \kappa_1^2 - 2u^1 u^3 \kappa_1 \kappa_2 + \sum_{\nu} [(u^\nu)^2 (\kappa_{\nu-1}^2 + \kappa_\nu^2)$$

$$- (u^{\nu-1} \kappa_{\nu-1} - u^{\nu+1} \kappa_\nu)^2]. \qquad \text{... (10)}$$

After little calculations we can see that the inverse $(n \times n)$ matrix (g^{ij}) of $(n \times n)$ matrix (g_{ij}) are given by:

$$g^{00} = \frac{1}{g}, \; g^{0\gamma} = g^{\gamma 0} = \frac{-g_{0\gamma}}{g}, \; g^{\nu\mu} = \frac{g_{0\nu} g_{0\mu}}{g}, \; g^{\gamma\gamma} = 1 - g^{0\gamma} g_{0\gamma} . \qquad \text{... (11)}$$

where g_{ij} and g are given from (5), (6), (7), (8) and (10) respectively.

From (4), we have :

$$h_{00} = u^{n-1} \kappa_n \kappa_{n-1}, \; h_{i\gamma} = h_{\gamma\lambda} = 0, \; \forall \; i, \gamma, \lambda, \; \det(h_{ij}) = 0, \text{ if } n \geq 2, \qquad \text{... (12)}$$

where, $h_{ij} = \langle e_{n+1}, X_{ij} \rangle$.

Thus, the 1-*st* and 2-*nd* fundamental forms of M are given by :

$$I = g_{00} = (du^0)^2 + 2 \sum_{\gamma} g_{0\gamma} du^0 du^\gamma + \sum_{\nu} g_{\nu\nu} (du^\nu)^2, \qquad \text{... (13)}$$

and $$II = h_{00} (du^0)^2, \qquad \text{... (14)}$$

respectively, where g_{ij} and h_{00} are given from (5), (6), (7), (8) and (12).

From Weingarten equations[1 & 10] using (6), (7), (11) and (12) we have :

$$\left.\begin{aligned} &h_0^0 g^{00} h_{00} = g^{00} (u^{n-1} \kappa_n \kappa_{n-1}), \\ &h_0^1 = g^{01} h_{00} = - g^{00} (1 - u^2 \kappa_1)(u^{n-1} \kappa_n \kappa_{n-1}), \\ &h_0^{\nu} = g^{0\nu} h_{00} + - g^{00} (u^{\nu-1} \kappa_{\nu-1} - u^{\nu+1} \kappa_{\nu})(u^{n-1} \kappa_n \kappa_{n-1}), \\ &h_{\gamma}^i m = h_{\gamma}^{\gamma} = h_{\gamma}^{\lambda} = 0, \quad i, \gamma, \lambda. \end{aligned}\right\} \quad \text{... (15)}$$

From (11) and (12) we have the following :

Corollary 2.1 — The mean curvature function H of M is given by :

$$H = \frac{1}{n} g^{00} h_{00} = \frac{1}{n} g^{00} (u^{n-1} \kappa_n \kappa_{n-1}), \quad n \geq 2. \quad \text{... (16)}$$

Corollary 2.2 — The Gaussian curvature G of M is given by :

$$G = 0, \quad n \geq 2. \quad \text{... (17)}$$

Using (12) and (16), we can see that :

Corollary 2.3 — The norm of the 2-*nd* fundamental form of M is given by :

$$S^2 + (h_0^0)^2 = (g^{00} h_{00})^2 = n^2 H^2 \quad \text{... (18)}$$

$$= (g^{00})^2 (u^{n-1} \kappa_n \kappa_{n-1})^2. \quad \text{... (19)}$$

It is important to remark that the foregoing results are considered as a generalization of the well-known results for the 2-dimensional ruled surfaces in E^3 which confirm that the envelope of the osculating plane of a space curve is the tangential developable[8, 9, 13 & 17].

3. The Variation of the Fundamental Quantities

In this section, the variation of the volume element, mean curvature and the norm of the second fundamental form are obtained in terms of the curvatures k_i for the base curve $r(u^0)$. For this purpose we give the following definition[1, 3 & 11].

Definition 3.1 — Let M be a compact hyperruled surface with piecewise smooth boundary ∂M, let $\varphi \in \overset{\circ}{\wedge}(M)$ be a continuous function vanishing identically on the boundary ∂M and satisfies the condition $\int_M \varphi(u^i)\, W\, du^0 \wedge du^1 \wedge \ldots du^{n-1} = 0$ (where $W = \sqrt{g}$). We consider a smooth map $F : J \times M \to E^{n+1}$ such that for $t \in J = [0, 1]$, the map $F_t : M \to E^{n+1}$, where $F_t(p) = F(t, p)$ for $p \in M$, is an immersion such that $F_0 = M$ with local representation (1) and $F_t = F_0$ on the boundary ∂M. The image $F(p)$ is represented by the parametrization :

$$\overline{X}(u^i, t) = X(u^i) + t\, \varphi(u^i)\, e_{n+1}(u^0). \quad \text{... (20)}$$

This representation defines a normal variation of M in E^{n+1} associated with φ and the family of hyperruled surfaces represented by $\overline{X} = \overline{X}(u^i, t)$ is called a deformable hyperruled surfaces resulting from $X = X(u^i)$ by the normal variation such that that the variation vector field $\delta X = \varphi(u^i)\, e_{n+1}(u^0)$ and the operator δ is defined as $(\partial/\partial t)|_{t=0}$.

From (13), the 1st fundamental form of the variation $\overline{X}$ (deformed family of surfaces) is $\overline{I} = \sum_{i,j} \overline{g}_{ij}\, du^i\, du^j$, where $\overline{g}_{ij} = \langle \overline{X}_i, \overline{X}_j \rangle$, using (2), (3) and (20) we have :

$$\overline{g}_{00} = g_{00} - 2\, t\, \varphi\, u^{n-1}\, \kappa_n\, \kappa_{n-1} + t^2\, [(\nabla_{e_1} \varphi)^2 + \kappa_n^2\, \varphi^2],$$

$$\overline{g}_{0\,\gamma} = g_{0\,\gamma} + t^2\, (\varphi_\gamma\, \nabla_{e_1} \varphi),$$

$$\overline{g}_{\gamma\gamma} = 1 + t^2\, \varphi_\gamma^2,$$

$$\overline{g}_{\gamma\lambda} = t^2\, \varphi_\gamma\, \varphi_\lambda, \;\; \forall\; \gamma \neq \lambda.$$

Thus, it is easy to see that the 1-*st* and 2-*nd* variation of g_{ij} are :

$$\delta\, g_{00} = -2\, \varphi\, u^{n-1}\, \kappa_n\, \kappa_{n-1}, \qquad \text{... (21)}$$

$$\delta\, g_{i\,\gamma} = 0, \;\; \forall\; i, \lambda, \qquad \text{... (22)}$$

and

$$\delta^2\, g_{00} + 2\left[\, (\nabla_{e_1} \varphi)^2 + \kappa_n^2\, \varphi^2 \,\right] > 0, \qquad \text{... (23)}$$

$$\delta^2\, g_{i\,\gamma} = 2\, \varphi_i\, \varphi_\gamma, \;\; \forall\; i, \gamma. \qquad \text{... (24)}$$

Thus, we have :

Lemma 3.1 — The 1*st* and 2*nd* variation of the metric tensor g_{ij} are given from (21), (22), (23) and (24).

From (9), (21) and (22), we have :

Corollary 3.1 —

$$\delta\, g = \delta\, g_{00} = -2\, \varphi\, u^{n-1}\, \kappa_n\, \kappa_{n-1}. \qquad \text{... (25)}$$

Using (6), (7), (8), (11), (22) and (25) we have the following :

$$\left.\begin{aligned}
&\delta\, g^{00} = 2\, \varphi\, (g^{00})^2\, (u^{n-1}\, \kappa_n\, \kappa_{n-1}),\\
&\delta\, g^{01} = -2\, \varphi\, (g^{00})^2\, (1 - u^2\, \kappa_1)\, (u^{n-1}\, \kappa_n\, \kappa_{n-1}),\\
&\delta\, g^{11} = 2\, \varphi\, (g^{00})^2\, (1 - u^2\, \kappa_1)^2\, (u^{n-1}\, (\kappa_n\, \kappa_{n-1}),\\
&\delta\, g^{0\,\nu} = -2\, \varphi\, (g^{00})^2\, (u^{\nu-1}\, \kappa_{\nu-1} - u^{\nu+1}\, \kappa_\nu)\, (u^{n-1}\, \kappa_n\, \kappa_{n-1}),\\
&\delta\, g^{\nu\,\mu} = 2\, \varphi\, (g^{00})^2\, (u^{\nu-1}\, \kappa_{\nu-1} - u^{\nu+1}\, \kappa_\nu)\, (u^{\mu-1}\, \kappa_{\mu-1} - u^{\mu+1}\, \kappa_\mu)\, (u^{\nu-1}\, \kappa_n\, \kappa_{n-1})\\
&\delta\, g^{\nu\,\nu} = 2\, \varphi\, (gh^{00})^2\, (u^{\nu-1}\, \kappa_{\nu-1} - u^{\nu+1}\, \kappa_\nu)^2\, (u^{n-1}\, \kappa_n\, \kappa_{n-1}).
\end{aligned}\right\} \qquad \text{... (26)}$$

From the foregoing results, we have :

Lemma 3.2 — The 1*st* variations of the metric tensors $g^{ij^{\cdot}}$ are given from (26).

The volume element $\bar{d}A$ of the variation $\overline{X} = \overline{X}(u^i, t)$ is :

$$\bar{d}A = \overline{W}\frac{dA}{W}, \; W = \sqrt{g}, \; \overline{W} = \sqrt{\bar{g}} \text{ and } \bar{g} = Det\,(\bar{g}_{ij}).$$

Thus and using (25) we have :

Corollary 3.2 — The variation of the volume element dA is given by :

$$\delta(dA) = -(\varphi\, g^{00}\, u^{n-1}\, \kappa_n\, \kappa_{n-1})\, dA. \qquad \text{... (27)}$$

From [1], the variations of the 2*nd* fundamental quantities h_{ij} are given by :

$$\delta h_{ij} = \nabla_i \nabla_j \varphi - \varphi \sum_k h_i^k h_{kj}. \qquad \text{... (28)}$$

Thus, using (12) and (15), we have in more explicitly :

$$\delta h_{ij} = \nabla_i \nabla_j \varphi - \delta_0^i\, \varphi\, g^{00}\, (u^{n-1}\, \kappa_n\, \kappa_{n-1})^2, \; \delta_0^i = \begin{cases} 1, i = 0 \\ 0, i \neq 0. \end{cases} \qquad \text{... (29)}$$

From [1], using (11) and (16) we have :

$$n\, \delta H = \Delta\varphi + \varphi\, (g^{00})^2\, (u^{n-1}\, \kappa_n\, \kappa_{n-1})^2. \qquad \text{... (30)}$$

Thus, we have :

Theorem 3.1 — *The variation of the mean curvature function H is given from (30).*

Using (16), (22), (25), (26) and (30) one can easily obtain $\delta^2 g$, $\delta^2 g^{00}$, $\delta^2 g^{01}$, $\delta^2 g^{11}$, $\delta^2 g^{0\nu}$, $\delta^2 g^{\nu\mu}$ and $\delta^2 g^{\nu\nu}$ in terms of the Laplace operator of φ and the curvatures for the base curve $r(u^0)$, as follows :

$$\left.\begin{aligned}
&\delta^2 g = \frac{-2}{3}\varphi\, g\,(4\Delta\varphi - \xi),\\
&\delta^2 g^{00} = 2\,\varphi\, g^{00}\,\xi,\\
&\delta^2 g^{01} = -2\,\varphi\, g^{00}\,(1 - u^2\,\kappa_1)\,\xi,\\
&\delta^2 g^{11} = 2\,\varphi\, g^{00}\,(1 - u^2\,\kappa_1)^2\,\xi,\\
&\delta^2 g^{0\nu} = -2\,\varphi\, g^{00}\,(u^{\nu-1}\,\kappa_{\nu-1} - u^{\nu+1}\,\kappa_\nu)\,\xi,\\
&\delta^2 g^{\nu\mu} = 2\,\varphi\, g^{00}\,(u^{\nu-1}\,\kappa_{\nu-1} - u^{\nu+1}\,\kappa_\nu)\,(u^{\mu-1}\,\kappa_{\mu-1} - u^{\mu+1}\,\kappa_\mu)\,\xi,\\
&\delta g^{\nu\nu} = 2\,\varphi\, g^{00}\,(u^{\nu-1}\,\kappa_{\nu-1} - u^{\nu+1}\,\kappa_\nu)^2\,\xi,
\end{aligned}\right\} \qquad \text{... (31)}$$

where, $\xi = \Delta\varphi + 3\,\varphi\,(g^{00})^2\,(u^{n-1}\,\kappa_n\,\kappa_{n-1})^2.$

From the foregoing results, we have :

Corollary 3.3 — The 2-*nd* variations of the determinant g and the metric tensors g^{ij} are given from (31).

The variation of the norm S of the 2*nd* fundamental form (18) is given by using (16) and (30) as :

Corollary 3.4 —

$$\delta S^2 = 2\, g^{00} (u^{n-1} \kappa_n \kappa_{n-1}) [\Delta \varphi + \varphi (g^{00})^2 (u^{n-1} \kappa_n \kappa_{n-1})^2]. \quad \text{... (32)}$$

4. Variation of the Curvatures for the Base Curve and the Frenet-Frame

Here, the 1*st* and 2*nd* variations of the curvatures κ_i for the base curve are derived. The variations of the Frenet-frame for the base curve are obtained.

From (6) and (22), we have :

$$\delta \kappa_1 = 0. \quad \text{... (33)}$$

From (7) and (22), we have :

$$u^{\nu+1} \delta \kappa_\nu = u^{\nu-1} \delta \kappa_{\nu-1}; \quad \text{... (34)}$$

Using (33), we have :

$$\delta \kappa_\nu = 0. \quad \text{... (35)}$$

From (16), (26), (30), and (35), we can obtain :

$$\delta \kappa_n = \frac{1}{(g^{00} u^{n-1} \kappa_{n-1})} [\Delta \varphi - \varphi (g^{00})^2 (u^{n-1} \kappa_n \kappa_{n-1})^2]. \quad \text{... (36)}$$

From the foregoing results, we have :

Corollary 4.1 — The 1*st* variation of the curvatures κ_i for the base curve are given from (33), (35) and (36).

From (6) and (24), we have :

$$\delta^2 \kappa_1 = \frac{-2}{u^2} \varphi_0 \varphi_1. \quad \text{... (37)}$$

From (7) and (24), we have :

$$\delta^2 \kappa_\nu = \frac{1}{u^{\nu+1}} [u^{\nu-1} \delta^2 \kappa_{\nu-1} - 2 \varphi_0 \varphi_\nu]. \quad \text{... (38)}$$

From (37) and (38), we have :

$$\delta^2 \kappa_2 = \frac{-2 \varphi_0}{u^2 u^4} [u^1 \varphi_1 + u^2 \varphi_2 + u^3 \varphi_3]. \quad \text{... (39)}$$

$$\delta^2 \kappa_3 = \frac{-2\varphi_0}{u^3 u^4}[u^1 \varphi_1 + u^2 \varphi_2 + u^3 \varphi_3] \qquad \ldots (40)$$

From (38) and (40), we have :

$$\delta^2 \kappa_4 = \frac{-2\varphi_9}{u^4 u^5}[u^1 \varphi_1 + u^2 \varphi_2 + u^3 \varphi_3 + u^4 \varphi_4] \qquad \ldots (41)$$

and

$$\delta^2 \kappa_n = \frac{-2\varphi_0}{u^n u^{n+1}}[u^1 \varphi_1 + u^2 \varphi_2 + \ldots + u^n \varphi_n]. \qquad \ldots (42)$$

From the foregoing results, we have :

Corollary 4.2 — The 2-*nd* variation of the curvatures κ_i for the base curve are given by :

$$\delta^2 \kappa_i = \frac{-2\varphi_0}{u^i, u^{i+1}} \sum_{i=1}^{n} u^i \varphi_i, \; \varphi_i = \frac{\partial \varphi}{\partial u^i}. \qquad \ldots (43)$$

From (3) and (20), we have :

$$\delta e_\gamma = \varphi_\gamma e_{n+1}. \qquad \ldots (44)$$

From (2), (4), (20), (35) and (44), we can see that :

$$\delta e_{v+1} = \frac{1}{k_v}[(\varphi_{0\,v} + \varphi_{v-1} \kappa_{v-1}) e_{n+1} - \varphi_v \kappa_n e_n]. \qquad \ldots (45)$$

From the foregoing results, we have :

Corollary 4.3 — The variation of the Frenet-frame for the base curve is given from (44) and (45).

5. Stability Condition

Here, the necessary and sufficient condition for the stability of an immersion X in terms of the curvatures for the base curve, with respect to $I = \int_M H^c \, dA, c \geq 0$ are derived. The solution of the differential equation which is produced from the stability condition, for example in E^3 and E^4 is obtained. For this purpose we give the following definition :[1, 3-5, 11,14 & 15]

Definition 5.1 — A closed hyperruled surface M^n in E^{n+1} is called an S-hyperruled surface if it is stable with respect to the integral $I(M^n) = \int_M H^n \, dA$, i.e., for any normal variation of M^n in E^{n+1}, we have $\delta(I(M)) = 0$.

Thus using (16), we have :

$$I(M) = \int_M \left(\frac{1}{n} g^{00} u^{n-1} \kappa_n \kappa_{n-1} \right)^c dA, \; c \geq 0. \qquad \ldots (46)$$

From (27), we have :

$$\delta I(M) = \int_M \left[\frac{c}{n^c} (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c-1} u^{n-1} (g^{00} \kappa_n \delta \kappa_{n-1} + g^{00} \kappa_{n-1} \delta \kappa_n + \kappa_n \kappa_{n-1} \delta g^{00}) \right] dA - \int_M \frac{\varphi}{n^c} (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c+1} dA.$$

Using (26), (35) and (36), we have :

$$\delta I(M) = \int_M \left[\frac{c}{n^c} (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c-1} (\Delta \varphi + \varphi (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^2) \right] dA - \int_M \frac{\varphi}{n^c} (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c+1} dA.$$

Suppose that M is closed; then, by Green's theorem, we have :

$$\delta I(M) = \frac{\varphi}{n^c} \int_M [c \Delta (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c-1} + (c-1) (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c+1}] dA. \quad \dots (47)$$

From (47), we see that M is stable with respect to the integral (46), if and only if the right-hand side of (47) is identically zero for all differentiable functions φ on M, that is :

$$S_c : c \Delta (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c-1} + (c-1) (g^{00} u^{n-1} \kappa_n \kappa_{n-1})^{c+1} = 0. \quad \dots (48)$$

Thus, we reach to the proof of the main theorem :

Theorem 5.1 — *The oriented closed osculating hyperruled surface $X : M \to E^{n+1}$ is stable with respect to the integral (46) if and only if the condition (S_c) is valid for the curvatures κ_i of the base curve.*

Now we shall try to find a solution of the differential eq. (48) in :

Case when $n = 2$ and $n = 3$.

(I) We put $c = a \geq 2, n = 2$ in (48); using (10) and (11), we have :

$$S_a : a \Delta \left[\frac{\kappa_2 (u^0)}{u^1 \kappa_1 (u^0)} \right]^{a-1} + (a-1) \left[\frac{\kappa_2 (u^0)}{u^1 \kappa_1 (u^0)} \right]^{a+1} = 0, \kappa_1 \neq 0. \quad \dots (49)$$

Let us denote :

$$F(u^0, u^1) = \left[\frac{\kappa (u^0)}{u^1 \kappa_1 (u^0)} \right]. \quad \dots (50)$$

Using the definition of Laplace operator of any differentiable function on M [1] we have

$$\Delta F^{a-1} = \frac{-1}{\sqrt{g}} \left\{ \begin{array}{c} (\sqrt{g}\, g^{00}) \frac{\partial^2 F^{a-1}}{\partial (u^0)^2} + \frac{\partial F^{a-1}}{\partial u^0} \frac{\partial (\sqrt{g}\, g^{00})}{\partial u^0} + (\sqrt{g}\, g^{11}) \frac{\partial^2 F^{a-1}}{\partial (u^1)^2} \\ + \frac{\partial F^{a-1}}{\partial u^1} \frac{\partial (\sqrt{g}\, g^{11})}{\partial u^1} + 2 (\sqrt{g}\, g^{01}) \frac{\partial^2 F^{a-1}}{\partial u^0 \partial u^1} \\ + \frac{\partial F^{a-1}}{\partial u^1} \frac{\partial (\sqrt{g}\, g^{01})}{\partial u^0} + \frac{\frac{\partial F^{a-1}}{\partial u^0} \partial (\sqrt{g}\, g^{01})}{\partial u^1} \end{array} \right\} \quad \dots (51)$$

where, from (6), (10), (11) and (50) we have :

$$\left. \begin{array}{l} \frac{\partial F^{a-1}}{\partial u^0} = -(a-1) \left(\frac{u^1}{\kappa_2^2} \right) (\kappa_2 \dot{\kappa}_1 - \kappa_1 \dot{\kappa}_2) F^a, \frac{\partial F^{a-1}}{\partial u^1} = -(a-1) \left(\frac{1}{u^1} \right) F^{a-1}, \\ \frac{\partial^2 F^{a-1}}{\partial (u^0)^2} (a-1) \left(\frac{u^1}{\kappa_2^2} \right)^2 \left[\begin{array}{c} (a-2)\, \kappa_1^2 \dot{\kappa}_2^2 + a\, \kappa_2^2 \dot{\kappa}_1^2 - \kappa_2^2 \kappa_1 \ddot{\kappa}_1 \\ + (2-2a)\, \kappa_1 \kappa_2 \dot{\kappa}_1 \dot{\kappa}_2 + \kappa_1^2 \kappa_2 \ddot{\kappa}_2 \end{array} \right] F^{a+1}, \\ \frac{\partial^2 F^{a-1}}{\partial (u^1)^2} = a\,(a-1) \left(\frac{1}{u^1} \right)^2 F^{a-1}, \frac{\partial^2 F^{a-1}}{\partial u^0 \partial u^1} = (a-1)^2 \left(\frac{1}{\kappa_2} \right)^2 [\kappa_2 \dot{\kappa}_1 - \kappa_1 \dot{\kappa}_2] F^a, \\ \frac{\partial (\sqrt{g}\, g^{00})}{\partial u^0} = \frac{\partial (\sqrt{g}\, g^{01})}{\partial u^0} = \frac{-\dot{\kappa}_1}{u^1 \kappa_1^2}, \frac{\partial (\sqrt{g}\, g^{11})}{\partial u^1} = \frac{(u^1)^2 \kappa_1^2 - 1}{(u^1)^2 \kappa_1}, \frac{\partial (\sqrt{g}\, g^{01})}{\partial u^1} = \frac{1}{(u^1)^2 \kappa_1}. \end{array} \right\} \quad \dots (52)$$

Using (50) and (51), the condition (S_a) takes the form :

$$(a+a^2)\, \kappa_1^2 \kappa_2^2 + u^1 \left[(2a^2 - a)\, \kappa_1^2 \kappa_2 \frac{\partial \kappa_2}{\partial u^0} - 2a^2 \kappa_1 \kappa_2^2 \frac{\partial \kappa_1}{\partial u^0} \right] + (u^1)^2 [(a^2 - a)\, \kappa_1^4 \kappa_2^2$$

$$- \kappa_1^2 \kappa_2^4 + (a+a^2)\, \kappa_2^2 \left(\frac{\partial \kappa_1}{\partial u^0} \right)^2 + (a - 2a^2)\, \kappa_1 \kappa_2 \frac{\partial \kappa_1}{\partial u^0} \frac{\partial \kappa_2}{\partial u^0} + (a^2 - 2a)\, \kappa_1^2 \left(\frac{\partial \kappa_2}{\partial u^0} \right)^2$$

$$\left. - a\, \kappa_1 \kappa_2^2 \frac{\partial^2 \kappa_1}{\partial (u^0)^2} + a\, \kappa_1^2 \kappa_2 \frac{\partial^2 \kappa_2}{\partial (u^0)^2} \right] = 0.$$

Thus, $\forall\, u^1 \in R$, we have :

$$(a+a^2)\, \kappa_1^2 \kappa_2^2 = 0, \quad \dots (53)$$

$$(2a^2 - a)\, \kappa_1^2 \kappa_2 \frac{\partial \kappa_2}{\partial u^0} - 2a^2 \kappa_1 \kappa_2^2 \frac{\partial \kappa_1}{\partial u^0} = 0 \quad \dots (54)$$

and
$$(a^2 - a)\,\kappa_1^4\,\kappa_2^2 - \kappa_1^2\,\kappa_2^4 + (a + a^2)\,\kappa_2^2\left(\frac{\partial \kappa_1}{\partial u^0}\right)^2 + (a - 2a^2)\,\kappa_1\,\kappa_2\,\frac{\partial \kappa_1}{\partial u^0}\frac{\partial \kappa_2}{\partial u^0} \qquad \text{... (55)}$$

$$+\ (a^2 - 2a)\,\kappa_1^2\left(\frac{\partial \kappa_2}{\partial u^0}\right)^2 - a\,\kappa_1\,\kappa_2^2\,\frac{\partial^2 \kappa_1}{\partial (u^0)^2} + a\,\kappa_1^2\,\kappa_2\,\frac{\partial^2 \kappa_2}{\partial (u^0)^2} = 0.$$

From (53), we have $a \neq 0$ $(a \geq 2)$, $\kappa_1 \neq 0$, thus :

$$\kappa_2 = 0. \qquad \text{... (56)}$$

Thus, we see that (54) and (55) are vanished. So, we have

Theorem 5.2 — *The condition of stability* (S_a) *is valud for the closed osculating ruled surface, for which the base curve is a plane curve with respect to the integral (46) in* E^3 $(c = a \geq 2)$.

(II) We put $c = a \geq 2$, $n = 3$ in (48); using (10) and (11), we have :

$$S_a : a\,\Delta\left[\frac{\kappa_3(u^0)}{u^2\,\kappa_2(u^0)}\right]^{a-1} + (a-1)\left[\frac{\kappa_3(u^0)}{u^2\,\kappa_2(u^0)}\right]^{a+1} = 0,\ \kappa_2 \neq 0. \qquad \text{... (57)}$$

After a long straight-forward computation similar to (49), we can see that the condition (57) is split to four conditions :

$$\left.\begin{aligned} &a\,\kappa_1\,\kappa_2^2\,\kappa_3^2 = 0,\\ &(a + a^20\,\kappa_1^2\,\kappa_2^2\,\kappa_3^2 = 0, \end{aligned}\right\} \qquad \text{... (58)}$$

$$a\,\kappa_1^2\,\kappa_2^2\,\kappa_3^2 + (a^2 - a)\,\kappa_2^4\,\kappa_3^2 - \kappa_2^2\,\kappa_3^4 + (a + a^2)\,\kappa_3^2\left(\frac{\partial \kappa_2}{\partial u^0}\right)^2 +\ (a - 2a)^2\,\kappa_2\,\kappa_3\,\frac{\partial \kappa_2}{\partial u^0}\frac{\partial \kappa_3}{\partial u^0}$$

$$+\ (a^2 - 2a)\,\kappa_2^2\left(\frac{\partial \kappa_3}{\partial u^0}\right)^2 - a\,\kappa_2\,\kappa_3^2\,\frac{\partial^2 \kappa_2}{\partial (u^0)^2} + a\,\kappa_2^2\,\kappa_3\,\frac{\partial^2 \kappa_3}{\partial (u^0)^2} = 0, \qquad \text{... (59)}$$

$$a\,\kappa_2\,\kappa_3\,\frac{\partial \kappa_1}{\partial u^0} - 2a^2\,\kappa_1\,\kappa_3\,\frac{\partial \kappa_2}{\partial u^0} + (a + 2a^2)\,\kappa_1\,\kappa_2\,\frac{\partial \kappa_3}{\partial u^0} = 0. \qquad \text{... (60)}$$

Since $a \neq 0\ (a \geq 2)$, $\kappa_1 \neq 0$, $\kappa_2 \neq 0$ and from (58) we have :

$$\kappa_3 = 0. \qquad \text{... (61)}$$

Thus, Also (59) and (60) are satisfied. So, we have

Theorem 5.3 — *The closed osculating hyperruled surface in* E^4, *for which the base curve is a hyperplanar curve, is stable with* $c = a \geq 2$.

Using (10) and (11), the condition (S_c) takes the form :

$$S_c : c\,\Delta\left[\frac{\kappa_n(u^0)}{u^{n-1}\,\kappa_{n-1}(u^0)}\right]^{c-1} + (c-1)\left[\frac{\kappa_n(u^0}{u^{n-1}\,\kappa_{n-1}(u^0)}\right]^{c+1} = 0,\ \kappa_{n-1} \neq 0,$$

where $c \geq 0, n \geq 2$.

Thus, in the case when $c = 0$, the condition (S_c) is degenerate to $k_n = 0$. So, we have :

Corollary 5.1 — The differential equation of stability S_0 $(c = 0)$ has a solution within an osculating hyperruled surface in E^{n+1}, for which the base curve is a hyperplanar curve.

In the case when $c = 1$, we have :

Corollary 5.2 — The condition (S_1) $(c = 1)$ is valid for an osculating hyperruled surface in E^{n+1}, for which the base curve is not a hyperplanar curve.

It is important to remark that the foregoing results are a confirmation of our preceding examples in [1], whereas well-known for any plane curve in E^3, $\kappa_2 = 0, \kappa_1 \neq 0$ and for any hyperplanar curve in E^4, $(\kappa_1, \kappa_2) \neq (0, 0)$ and $\kappa_3 = 0$.

REFERENCES

1. N. H. Abdel-All and H. N. Abd-Ellah, Stability of closed hyper ruled surfaces (*to appear*).
2. C. Baikoussis and D. E. Blair, *Hokkaido Math. J.* **24** (1995) 473-90.
3. B. Y. Chen, *J. London Math. Soc* **6** (2) 321-25 (1973).
4. B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific Publishing Co. Ltd, 1984.
5. B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.
6. H. W. Guggenheimer, *Differential Geometry*. Mc Graw-Hill Book Company, Inc. New York, 1963.
7. W. Klingenberg, *A Course in Differential Geometry*, Springer-Verlag, New York, 1978.
8. D. Laugwitz, *Differential and Riemannian Geometry*, Academic Press, Inc., New York, 1965.
9. N. Prakash, *Differential Geometry, An Integrated Approach*, Tata Mc Graw-Hill Publishing Company Limited, New Delhi, 1981.
10. M. A. Soliman, N. H. Abdel-All and H. N. Abd-Ellah, Geometric invariants on an m-dimensional ruled manifold. Presented at International Conference of the Tensor Society, Japan, 1996.
11. M. A. Soliman, N. H. Abdel-All and R. A. Hussien, *Bull. Fac. Sci., Assiut Univ.*, **24** (2-c) (1995) 189-203.
12. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish Inc., Boston, Vol IV, 1970.
13. D. J. Struik, *Lectures on Classical Differential Geometry*, Addison-Wesley Publishing Company, Inc., U.S.A., 1961.
14. T. J. Willmore, *Total Curvature in Riemannian Geometry*, Ellis Horwod Limited Publishers, 1982.
15. T. J. Willmore, In : *Topics in Differential Geometry*, Academic Press, New York 149-56, 1976.
16. J. Weiner, *Indian Univ. Math. J.*, **27** 1978 19-35.
17. C. E. Weatherburn, *Differential Geometry of Three Dimensions*, Cambridge Univ. Press, Vol. I, 1961.