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BEST PROXIMITY PAIR THEOREMS

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Let A be a non-empty approximately p-compact convex subset and B be a non-empty closed convex subset of a Hausdorff locally convex topological vector space E will a continuous semi-norm p. Given a Kakutani factorizable multifunction $T : A \to 2^B$ and a single valued function $g : A \to A$, best proximity pair theorems furnishing the sufficient conditions for the existence of an element $x_0 \in A$ such that

$$d_p(gx_0, Tx_0) = d_p(A, B),$$

are proved. Indeed, a generalization of Ky Fan's fixed point theorem for multifunctions is a consequence of a best proximity pair theorem. Also, a stochastic analogue of a best proximity pair theorem is proved.

Key Words : Kakutani Factorizable Multifunction; Best Proximity Pair Theorem; Random Operator and Random Fixed Point

INTRODUCTION

Consider the fixed point equation $Tx = x$ where T is a non-self operator. If this operator equation does not possess a solution, it is attempted to find an element x_∞ which is close to Tx_0. Indeed, a well known best approximation theorem due to Ky Fan[3] (see [8] for simple proof) states that if K is a non-empty compact convex subset of a hausdorff locally convex topological vector space E with a continuous semi norm p and $TK \to E$ is a single valued continuous map, then there exists an element x_0, called a best approximant, in K such that

$$p(x_0 - Tx_0) = d_p(Tx_0, K) := \text{Inf } \{p(Tx_0 - y) : y \in K\}$$

Later, Reich[7] has accomplished in weakening the compactness condition and thereby proving a generalized result that if K is a non-empty approximately p-compact, convex subset of a Hausdorff locally convex topological vector space E with a continuous semi norm p and $T : K \to E$ a single valued continuous map with $T(K)$ relatively compact, then there exists an element x_0 in K satisfying the conclusion of the aforesaid theorem.

Sehgal and Singh[10] have further extended this result to the one for multifunctions, and in the framework of normed linear spaces they have also proved the following generalization[11] of the result due to Prolla[6] :

If K is a non-empty approximately compact convex subset of a normed linear space X, $T: K \to X$ a single valued continuous map with $T(K)$ relatively compact and $g: K \to K$ an affine, continuous, surjective single valued map such that g^{-1} sends compact subsets of K onto compact sets, then there exists an element x_0 in K such that $d_p(gx_0, Tx_0) = d_p(Tx_0, K)$. Recently, in the setting of Hausdorff locally convex topological vector spaces, Vetrivel, Veeramani and Bhattacharyya[13] have proved a best approximation theorem for Kakutani factorizable multifunctions which generalizes the results due to Sehgal and Singh.

On the other hand, Beer and Pai[1] have investigated the conditions for the existence of proximal points of a pair of non-empty subsets of a Hausdorff locally convex topological vector space E, i.e., for the non-emptiness of the set

$$\text{Prox } (A, B) := \{(a, b) \in A \times B : d_p(a, b) = d_p(A, B)\}$$

where A and B are non-empty subsets of E and p is a continuous semi-norm on E.

Motivated by these interesting results of Beer, Pai, Sehgal, Singh and others, in this paper a best proximity pair theorem furnishing the sufficient conditions which ensure the existence of an element $x_0 \in A$ such that

$$d_p(gx_0, Tx_0) = d_p(A, B) := \text{Inf } \{p(a-b) : a \in A \text{ and } b \in B\}$$

is proved for the given non-empty subsets A and B of a Hausdorff locally convex topological vectoe space E with a continuous semi-norm p, a Kakutani factorizable multifunction $T: A \to 2^B$ and a single valued self-mapping g on A. The pair (x_0, Tx_0) is called a best proximity pair of T. Apart from the purpose of seeking an element x_0 close to Tx_0, these best proximity pair theorems also blend the results on best approximations and proximal points of a pair of sets. Indeed, if $T: A \to 2^B$ is a compact valued multi-function and $d_p(x_0, Tx_0) = d_p(A, B)$, then x_0 is a best approximant and Prox (A, B) is non-empty. Further, an anomaly about best proximity pair theorems proved in this paper is that they contain the fixed point theorems for upper semi-continuous multifunctions whereas the best approximation theorems for multifunctions do not contain so because of the continuity assumption on the underlying multifunctions. Indeed, a generalization of the following Ky Fan's fixed point theorem for multifunctions falls out from a best proximity pair theorem.

If K is a non-empty compact convex subset of a locally convex Hausdorff topological vector space and $T: K \to 2^K$ is an upper semicontinuous multifunction such that Tx is convex and closed for each x, in K, then there exists an element $x_0 \in K$ such that $x_0 \in Tx_0$.

Finally, invoking a measurable selection theorem due to Kuratowski and Ryll Nardzewski, a stochastic analogue of a best proximity pair theorem is proved. This theorem subsumes a classical result, due to Bharucha Reid [2], on random fixed points, which in turn is a stochastic analogue of Schauder's fixed point theorem.

2 PRELIMINARIES

A *multifunction* from a set A to another set B, denoted by $T: A \to 2^B$, is defined as a function assigning to each element in A a non-empty subset of B.

Let X and Y be any two topological spaces.

Let $T: X \to 2^Y$ be a multifunction. Then, T is said to be *upper semi-continuous* (resp. *lower semi-continuous*) if $T^{-1}(B) := \{x \in X : (Tx) \cap B \neq \Phi\}$ is closed (resp. open) in X whenever B is a closed (resp. open) subset of Y. Also, T is said to be continuous if it is both lower semi-continuous and upper semi-continuous.

A multifunction $T: X \to 2^Y$ is said to be a *compact* multifunction if $T(X)$ is contained in a compact subset of Y.

It is known that if $T: X \to 2^Y$ is an upper semicontinuous multifunction with compact values, then $T(K)$ is compact in Y whenever K is a compact subset of X.

A multifunction $T: X \to 2^Y$ is said to be a *Kakutani multifunction* [5] if the following two conditions are satisfied :

(i) T is upper semi-continuous

(ii) Either Tx is a singleton for each $x \in X$, Tx is a non-empty compact convex subset of Y (in which case Y is assumed to be a non-empty convex set in a Hausdirff topological vector space)

A *Kakutani factorizable multifunction* is defined as a multifunction which can be expressed as a composition of finitely many Kakutani multifunctions.

The class of all Kakutani factorizable multifunctions from the space X to the space Y is denoted by $\mathcal{K}_C(X, Y)$.

Let E be a Hausdorff locally convex topological vector space and p be a continuous semi-norm on it.

A non-empty subset A of E is said to be *approximately p-compact* if for each $y \in E$ and each net $\{x_\alpha\}$ in A satisfying the condition that $d_p(x_\alpha, y) \to d_p(y, A)$, there is a subnet of $\{x_\alpha\}$ converging to an element of A.

If A is a non-empty approximately p-compact convex subset of E, then the set $P_A(y)$ of all p-set approximations in A to any element $y \in E$ defined by

$$P_A(y) := \{x \in A : p(x-y) : p(x-y) = d_p(y, A)\}$$

is a non-empty convex compact subset of A and the mapping $y \to P_A(y)$ is an upper semicontinuous multifunction on E [7].

A non-empty subset A of E is said to be *boundedly p-compact* if its intersection with every closed p-ball $B[x, r] := \{y \in E : p(x-y) \leq r\}$ is relatively compact.

Given two non-empty subsets A and B of E, A is said to be *proximally p-compact* with respect to the set B if for every net $\{x_\alpha\}$ in A satisfying the condition that $d_p(x_\alpha, B) \to d_p(A, B)$, there is a subnet of $\{x_\alpha\}$ converging to an element of A.

A non-empty subset A of E is said to be *p-proximinal* if $P_A(y)$ is non-empty for every element $y \in E$.

A non-empty subset A of E is said to be *p-bounded* if Sup $\{p(x) : x \in A\}$ is finite.

Let C be a non-empty convex subset of E. A single valued function $g: C \to E$ is said to be *quasi p-affine* if for every real number $r \geq 0$ and $x \in E$, the set $\{u \in C : [g(u) - x] \leq r\}$ is convex.

A single valued function $g : A \to E$ is saod to be *p-continuous* if $p[g(x_\alpha) - g(x)] \to 0$ for each x in A and every net $\{x_\alpha\}$ is A converging to x.

3. Best Proximity Pair Theorems

The main theorems appearing in this section are obtained by invoking the following fixed point theorem, due to Lassonde[5], for Kakutani factorizable multifunctions:

Theorem A — *If S is a non-empty convex subset of a Hausdorff locally convex topological vector space, then any compact multifunction T in $\mathcal{K}_C(S, S)$ has a fixed point.*

For any two non-empty subsets A and B of a topological vector space with a continuous semi-norm p, the following notations are used in the sequel.

$$A_0 := \{a \in A : d_p(a, b) = d_p(A, B) \text{ for some } b \in B\}$$

$$B_0 := \{b \in B : d_p(a, b) = d_p(A, B) \text{ for some } a \in A\}$$

The main best proximity pair theorem is the following.

Theorem 3.1 — *Let E be Hausdorff locally convex topological vector space with a continuous semi-norm p. Let A be a non-empty approximately p-compact convex subset and B be a non-empty closed convex subset of E such that Prox (A, B) is non-empty. Suppose that*

(i) $T : A \to 2^B$ is a Kakutani factorizable multifunction satisfying the conditions that $T(A_0) \subseteq B_0$ and the convex hull of $T(A_0)$ is relatively compact.

(ii) $g : A \to A$ is a single valued map such that $g^{-1}(A_0) \subseteq A_0$

Then, there exists an element $x_0 \in A_0$ such that

$$d_p(gx_0, Tx_0) = d_p(A, B)$$

in each of the following cases.

(a) g is a bijective closed map

(b) g is a p-continuous, quasi p-affine, surjective map such that g^{-1} maps compact subsets of A onto compact sets.

Moreover, if $d_p(A, B) > 0$, then $x_0 \in g^{-1}(Bd(A))$ where $Bd(A)$ denotes the boundary of the set A.

Proof : Let $F := (S \circ T) : A_0 \to A_0$ where $S := (g^{-1} \circ P_A) : \overline{co}\, T(A_0) \to A_0$. Since $T(A_0)$ is contained in B_0 and the sets $P_A(B_0)$ and $g^{-1}(A_0)$ are contained in A_0, the set A_0 is invariant under F. Because A and B are convex, A_0 is also convex. As A is an approximately p-compact convex set, P_A is a compact convex valued upper semi-continuous multifunction.

It is claimed that F is a compact Kakutani factorizable multifunction in both cases (a) and (b).

If (a) is true, then g^{-1} is a single valued continuous function and hence F is a Kakutani factorizable compact multifunction.

Assume now that (b) is true. Since g is a quazi p-affine, S is a convex valued multi-function. In fact, if $x_1, x_2 \in Sx$, then $p[g(x_1) - x] = d_p(x, A) = p[g(x_2) - x]$. But, the quasi affinity of g implies that $\{u \in A : p[g(u) - x] \leq d_p(x, A)\}$ is convex. So, it follows that $p(g[\lambda x_1 + (1-\lambda) x_2] - x) = d_p(x, A)$. Therefore, Sx is convex. Moreover, Sx is compact as g^{-1} sends compact sets onto compact sets and $P_A(x)$ is compact. Also, F is a compact multifunction because both g^{-1} and P_A send compact sets onto compact sets.

It remains to show that S is an upper semi-continuous multifunction. Let C be any closed subset of A_0 and $\{x_\alpha\}$ be any net in $S^{-1}(C)$ such that $x_\alpha \to x$. Then, it follows that, for each α, $S(x_\alpha) \cap C$ is non-empty. Choose an element $y_\alpha \in S(x_\alpha) \cap C$ so that $d_p(g(y_\alpha), x_\alpha) = d_p(x_\alpha, A)$. Since $y_\alpha \in S(x_\alpha) \subseteq S(\overline{co}\, T(A_0)) = (g^{-1} \circ P_A)(\overline{co}\, T(A_0))$ which is a compact set, $\{y_\alpha\}$ has a convergent subnet. Without loss of generality it may be assumed that $\{y_\alpha\}$ has a convergent subnet. Without loss of generality it may be assumed that $\{y_\alpha\} \to y \in C$.

Now, $$p[g(y) - x] \leq p[g(y) - g(y_\alpha)] + p[g(y_\alpha) - x_\alpha] + p(x_\alpha - x)$$

$$= p[g(y) - g(y_\alpha)] + d_p(x_\alpha, A) + p(x_\alpha - x).$$

Since g is a p-continuous function, $p[g(y_\alpha) - g(y)] \to 0$. Also, $d_p(x_\alpha, A) \to d_p(x, A)$. So, $p[g(y) - x] = d(x, A)$ and hence $g(y) \in P_A(x)$. Therefore, $y \in S(x) \cap C$ which implies that $x \in S^{-1}(C)$. So, $S^{-1}(C)$ is closed and hence S is upper semi-continuous Thus, F is a compact Kakutani factorizable multifunction in both cases (a) and (b). So, Theorem A guarantees the existence of a point $x_0 \in A_0$ such that $x_0 \in Fx_0$ which means that $g(x_0) \in (P_A \circ T)(x_0)$. Consequently, there exists $y_0 \in Tx_0$ such that $g(x_0) \in P_A(y_0)$. Therefore, it follows that $d_p(g(x_0), y_0) = d_p(y_0, A)$. Further, as $Tx_0 \subseteq B_0$ and y_0 lies in Tx_0, $d_p(y_0, A) = d_p(A, B)$.

So, $$d_p(gx_0, Tx_0) = d_p(g(x_0), y_0)$$

$$= d_p(y_0, A)$$

$$= d_p(A, B)$$

and $$d_p(A, B) \leq d_p(gx_0, Tx_0).$$

Therefore, $d_p(gx_0, Tx_0) = d_p(A, B)$.

If $d_p(A, B) > 0$, then $Tx_0 \cap A$ is empty. Since Tx_0 is compact, there exists $u_0 \in Tx_0$ such that $d_p(gx_0, Tx_0) = p(gx_0 - u_0)$. Then, the set $D := \{\lambda g(x_0) + (1-\lambda) u_0 : \lambda \in [0, 1]\}$ intersects both A and $X - A$. Since D is connected, the boundary of A must intersect D.

If gx_0 does not lie in the boundary of A, then there exists $\lambda_0 \in (0, 1)$ such that $p(z_0 - u_0) < p(gx_0 - u_0)$ for $z_0 := \lambda_0 g(x_) + (1 - \lambda_0) u_0$ lying in $Bd(A)$.

This implies that $p(z_0 - u_0) < p(gx_0 - u_0)$

$= d_p(gx_0, Tx_0)$

$= d_p(A, B)$, which is a contradiction.

Hence, gx_0 must be in the boundary of A. This completes the proof of the theorem.

The following simple example illustrates the preceding theorem.

Let $E = R^2$ with the Euclidean norm.

Let $A := \{(1, y) : y \in [0, \infty)\}$ and $B = [0, \infty) \times [0, 1]$.

Let $C := \{(3, v) : v \in [0, 2]\}$ and $T := T_1 \circ T_2$ where $T_1 : A \to R^2$ and $T_2 : C \to R^2$ are defined as follows :

$$T_1(x, y) := \begin{cases} ((3, v) : v \in [0, y]) & \text{if } y \in [0, 2] \\ ((3, v) : v \in [0, 2]) & \textit{otherwise.} \end{cases}$$

$$T_2(x, y) := \begin{cases} ((2, v) : v \in [0, y]) & \text{if } y \in [0, 1] \\ ((u, 1) : u \in [2, y + 1]) & \textit{otherwise.} \end{cases}$$

Let $g : A \to A$ be defined by

$$g(x, y) := \begin{cases} (1/4)(4y + 1) & \text{if } 0 \le y \le (1/2) \\ (1/2)(y + 1) & \text{if } (1/2) \le y \le 1 \\ x^2 & \textit{otherwise} \end{cases}$$

Then, T and g satisfy the hypotheses of the theorem. The element satisfying the conclusion of the theorem is (1, 1).

The following result generalizes the Ky Fan's fixed point theorem for multifunctions.

Corollary 3.2 — Let A be a compact convex subet of a Hausdorff locally convex topological vector space E. If $T : A \to 2^A$ is a Kakutani factorizable multifunction and $g : A \to A$ is an affine, continuous, surjective single valued map, then there exists an element $x_0 \in A$ such that $gx_0 \in Tx_0$.

PROOF : Let $B = A$. Then, $A_\circ = B_\circ = A$. Let $\mathcal{F}$ be the collection of all continuous seminorms on E. From the preceding theorem, it follows that $K_q = \{x \in A : d_q(gx, Tx) = 0\}$ is non-empty for each $q \in \mathcal{F}$. Then, K_q is closed because of the upper semi-continuity of the multifunction T and the continuity of g and q. Consider an arbitrary finite subset $\{q_1, q_2, \ldots, q_n\}$ of $\mathcal{F}$. Let. $q_\circ = \sum_{i=1}^{n} q_i$

Then, $q_\circ$ is also a continuous semi-norm. As $K_{q_\circ} \subseteq \bigcap_{i=1}^{n} K_{q_i}$ and $K_{q_\circ}^{\cdot}$ is non-empty, the finite intersection $\bigcap_{i=1}^{n} K_{q_i}$ is non-empty. So, the family $\{K_q : q \in \mathcal{F}\}$ of closed subsets of A satisfies the finite intersection property. Since A is compact, there exists $x_0 \in A$ such that $d_q(gx_0, Tx_0) = 0$ for all $q \in \mathcal{F}$. This yields that $gx_0 \in Tx_0$.

Proof of the following corollary is immediate.

Corollary 3.3 — Let A be a non-empty compact convex subet of a Hausdorff locally convex topological vector space E. If $T: A \to A$ is a single valued continuous function and $g: A \to A$ is an affine, continuous, surjective single valued map, then there exists an element $x_0 \in A$ such that $gx_0 = Tx_0$.

Corollary 3.4 — Let E be a Hausdorff locally convex topological vector space with a continuous semi-norm p. Let A be a non-empty boundedly p-compact, closed, convex subset and B be a non-empty p-bounded, closed, convex, p-proximinal subset of. the space E. Suppose that

(i) $T: A \to 2^B$ is a Kakutani factorizable multifunction such that $T(A_0) \subseteq B_0$ and the convex hull of $T(A_0)$ is relatively compact.

(ii) g is as in Theorem 3.1

Then there exists an element $x_0 \in A$ such that

$$d_p(gx_0, Tx_0) = d_p(A, B).$$

PROOF : Because A is a boundedly p-compact closed subset, it is approximately p-compact[12] and proximally p-compact with respect to the p-bounded set B. But, the p-proximinality of B and the proximal p-compactness of A with respect to the set B ensures the non-voidness of Prox (A, B)[1]. So, the conclusion follows from Theorem 3.1.

Corollary 3.5 — Let E be a Hausdorff locally convex topological vector space with a continuous semi-norm p. Let A be a non-empty approximately p-compact convex subset and B a non-empty closed convex subset of E such that B is proximally p-compact with respect to A. If $T: A \to 2^B$ is a Kakutani factorizable multifunction such that $T(A_0) \subseteq B_0$ and g is as in Theorem 3.1, then there exists an element $x_0 \in A$ such that

$$d_p(gx_0, Tx_0) = d_p(A, B).$$

PROOF : Since B is a proximally p-compact with respect to the proximinal set A, the set Prox $(A, B$ is non-empty[1]. Also, it may be observed that B_0 is compact. Therefore, the closed convex hull of $T(A_0)$ is compact. Thus, all the hypotheses of Theorem 3.1 are satisfied and hence the conclusion holds good.

Because the compactness of A and the proximinality of B assures that Prox (A, B) is non-empty[1], Theorem 3.1 yields the following result.

Corollary 3.6 — Let A be a non-empty compact convex subset and B be a non-empty convex proximinal subset of a Banach space. If $T: A \to 2^B$ is a Kakutani factorizable multifunction such that $T(A_0) \subseteq B_0$ and g is as in Theorem 3.1, then there exists an element $x_0 \in A$ such that

$$d_p(gx_0, Tx_0) = d_p(A, B).$$

4. APPLICATIONS

A stochastic analogue of a best proximity pair theorem is proved as an application of a result derived in the previous section. The proof is based on the method followed in[9].

The following notions and results are needed in the sequel.

Let $(\Omega, \mathcal{M})$ be a measurable space where $\mathcal{M}$ is a σ-algebra of subsets of Ω, and Y be a topological space.

A multifunction $F: \Omega \to 2^Y$ is said to be measurable if for each open subset G of Y, $F^{-1}(G) \in \mathcal{M}$.

It is remarked that if $F(\omega)$ is compact for each $\omega \in \Omega$, then F is measurable if and only if $F^{-1}(C)$ is a measurable subset of Ω for each closed subset C of Y.

Given a non-empty subset S of the space Y, a multifunction $T: \Omega \times S \to 2^Y$ is said to be a random operator if for each fixed $x \in S$, the mapping $T(\cdot, x): \Omega \to 2^Y$ is measurable.

A random fixed point of the random operator $T: \Omega \times S \to 2^Y$ is defined to be a single valued measurable mapping $\Phi: \Omega \to Y$ such that $\Phi(\omega) \in \Phi(\omega))$ for all $\omega \in \Omega$.

The following measurable selection theorem, due to Kuratowski and Ryll Nardzewski[4], will be used subsequently.

Theorem **B** — *Let A be a separable closed subset of a Banach space X and $F: \Omega \to 2^A$ be a measurable function such that $F(\omega)$ is compact for all $\omega \in \Omega$. Then, there exists a single valued measurable function $\Phi: \Omega \to A$ such that $\Phi(\omega) \in F(\omega)$ for all $\omega \in \Omega$.*

The proof of the following lemma is similar to that of Lemma 2 given in[9].

Lemma C — Let A be a non-empty subset of a normed linear space X. If $T: A \to 2^E$ is a closed valued continuous multifunction such that $T(A)$ is relatively compact and $g: A \to A$ is a single valued continuous function, then the real valued function $x \to d_p(gx, Tx)$ is continuous on A.

The following theorem furnishes a stochastic analogue of a best proximity pair theorem.

Theorem 4.1 — *Let A be a non-empty compact convex subset and B be a non-empty convex proximinal subset of a Banach space X. Suppose that*

(a) $T: \Omega \times A \to 2^B$ is a random operator such that for each $\omega \in \Omega, T(\omega, A_0) := \bigcup \{T(\omega, x): x \in A_0\}$ is contained in B_0 and $T(\omega, \cdot): A \to 2^B$ is a continuous Kakutani factorizable multifunction.

(b) $g: A \to A$ is a continuous, quasi affine, surjective map such that $g^{-1}(A_0) \subseteq A_0$

Then, there exists a single valued measurable mapping $\Phi: \Omega \to A$ such that

$$d(g\,\Phi(\omega), T[\omega, \Phi(\omega)]) = d(A, B) \text{ for all } \omega \in \Omega.$$

PROOF : Let $F: \Omega \to 2^A$ be defined as

$$F(\omega) = \{x \in A : d(gx, T[\omega, x]) = d(A, B)\} \text{ for all } \omega \in \Omega.$$

Then, $F(\omega)$ is non-empty by Theorem 3.6. Also, from Lemma C, it follows that $F(\omega)$ is closed and hence it is compact.

It may be shown that F is a measurable multifunction. Let C be a closed subset of A and D be a countable dense subset of A.

For every positive integer n,

let $$D_n := \{x \in D : d(x, C) < (1/n)\}$$

and $$C_n := \bigcup_{x \in D_n} \{\omega \in \Omega : d(gx, T(\omega, x)) < d(A, B) + (1/n)\}$$

Then, C_n is a measurable set for each n. It may be proved that $F^{-1}(C) = \bigcap_{n=1}^{\infty} C_n$. If $\omega \in F^{-1}(C)$, then $F(\omega) \cap C$ is non-empty. So, there exists x_0 in C such that $d(gx_0, T[\omega, x_0]) = d(A, B)$. As D is dense in A, for each n, there exists an element $x_n \in D$ such that $d(x_n C) < (1/n)$

and $$d(gx_n, T[\omega, x_n]) \leq d(gx_0, T[\omega x_0]) + (1/2n)$$

$$= d(A, B) + (1/2n)$$

$$< d(A, B) + (1/n)$$

Therefore, $\omega \in \bigcap_{n=1}^{\infty} C_n$.

On the other hand, if $\omega \in \bigcap_{n=1}^{\infty} C_n$, then for each n, there exists an element x_n in D_n such that

$$d(gx_n, T[\omega, x_n]) < d(A, B) + (1/n).$$

Since $\{x_n\}$ is a sequence in the compact set A, there exists a subsequence $\{x_{n_k}\}$ converging to an element x_0 in C. So, $d(g x_0, T[\omega, x_0]) = d(A, B)$. Thus, $x_0 \in F(\omega) \cap C$ and hence $\omega \in F^{-1}(C)$. Therefore, $F^{-1}(C) = \bigcap_{n=1}^{\infty} C_n$. Since each C_n is measurable, $F^{-1}(C)$ is a measurable subset. So, F is a measurable multifunction.

Now, it is guaranteed by Theorem B that there exists a single valued measurable function $\Phi : \Omega \to A$ with $\Phi(\omega) \in F(\omega)$ for all $\omega \in \Omega$. From the definition of the multifunction F, it follows that

$$d(g\,\Phi(\omega), T[\omega, \Phi(\omega)]) = d(A, B) \text{ for all } \omega \in \Omega.$$

This completes the proof of the theorem.◊

A random fixed point theorem, due to Bharucha Reid[2], for continuous random operator is a special case of the following consequence of the preceding theorem.

Corollary 4.2 — If A is a non-empty compact convex subset of a Banach space and $T : \Omega \times A \to 2^A$ is a random operator such that for each $\omega \in \Omega$, the multifunction $T(\omega, \cdot) : A \to 2^A$ is continuous, then T has a random fixed point.

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