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SOME COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS

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The concept of compatibility between a multi-valued mapping and a single-valued mapping due to Kaneko and Sessa [11] is used as a tool to produce some common fixed point theorems on complete metrically convex metric spaces. Generalizations of known results are thereby obtained. In particular, a theorem by Rhoades [15] is generalized.

Key Words : Common Fixed Points; Compatible Mappings; Metrically Convex Metric Spaces

1. INTRODUCTION

Fixed point theorems for single-valued and multi-valued mappings have been studied extensively and applied to diverse problems during the last few decades. The applications of fixed point theorems encompass diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in : approximation theory, potential theory, game theory, mathematical economics, theory of differential equations, etc. Using fixed point techniques it is possible to analyse several concrete problems from science and engineering, where one is concerned with a system of differential, integral and functional equations. This approach is specially useful in dealing with certain problems and control systems and theory of elasticity.

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. Following the Banach contraction principle, Nadler [14] introduced the concept of multi-valued contraction mappings and established that a multi-valued contraction mapping possesses a fixed point in a complete metric space. Subsequently, many authors have generalized Nadler's fixed point theorem in different ways. In [3], Assad and Kirk gave sufficient conditions for non-self mappings to ensure the fixed point by proving a result on multi-valued contractions in complete metrically convex metric spaces. Several authors proved some fixed point theorems for non-self mappings (see, e.g., [1-9, 12]).

Recently, Ciric [5] proved a fixed point theorem for single-valued non-self mappings. He also established a multivalued version of his first result by using the δ-distance, so that the result and proof are identical to the single-valued case. Rhoades [15] gave another multi-valued version of the first result of Ciric [5] by using the Hausdorff distance.

The aim of our paper is to prove some common fixed point theorems by using the concept of compatibility between a multi-valued mapping and a single-valued mapping of Kaneko and Sessa

[11]. Our results generalize and improve the result of Rhoades [15]. In the following section, we introduce basic preliminaries. The main results are given in the last section.

BASIC PRELIMINARIES

Following Nadler [14], let (X, d) be a metric space. We denote by $CB(X)$ the set of all nonempty closed bounded subsets of (X, d) by H the Hausdorff metric defined on $CB(X)$

$$H(A, B) = \max\left\{ \sup_{x \in A} d(x, B), \ \sup_{y \in B} d(y, A) \right\},$$

where $A, B \in CB(X)$. Also, we define the function $D : X \times CB(X) \to [0, \infty)$ as follows

$$D(x, K) = \inf \{d(x, y) : y \in K\}.$$

It is well known that if (X, d) is complete, then so is $(CB(X), H)$ (cf. Kuratowskii [13]).

Let K be a nonempty closed and convex subset of X. Let $F, G : K \to CB(X)$, $S, T : K \to X$ satisfying the following condition

$$H(Fx, Gy) \leq h \max\left(\frac{d(Sx, Ty)}{a}, D(Sx, Fx), D(Ty, Gy), \frac{D(Sx, Gy) + D(Ty, Fx)}{a+h} \right), \quad \ldots (2.1)$$

for all x, y in X with $x \neq y$, where $0 < h < \frac{-1+\sqrt{5}}{2}$, $a \geq 1 + \frac{2h^2}{1+h}$.

Definition 2.1[3] — A metric space (X, d) is said to be **metrically convex** if for any x, y in X (with $x \neq y$) there exists a point z in X $(x \neq z \neq y)$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.1[3] — Let K be any non-empty closed subset of a complete metrically convex metric space (X, d). Then for any $x \in K$ and $y \notin K$ there exists a point $z \in \partial K$ (the boundary of K) such that

$$d(x, z) + d(z, y) = d(x, y).$$

Our main theorems is prefaced with the following lemma :

Lemma 2.2[14] — Let A, B be in $CB(X)$. Then for all $\alpha > 0$ and $a \in A$ there exists a point $b \in B$ such that

$$d(a, b) \leq H(A, B) + \alpha.$$

From Nadler [14], it is clear that

$$d(a, b) \leq kH(A, B),$$

where $k > 1$.

For single-valued mappings the notion of the weak commutativity is introduced by Sessa [16].

Definition 2.2 — The mappings $S, I : X \to X$ are said to be **weakly commuting** if

$$d(SIx, ISx) \leq d(Ix, Sx)$$

for all x in X.

It is clear that **two commuting mappings** are **weakly commuting** but the converse is not true as is shown in [16].

Jungck [10] generalized the concept of weak commutativity in the following way :

Definition 2.3 — Two single-valued mappings S and I of a metric space (X, d) into itself are **compatible** if $\lim_{n \to \infty} d(SX_n, ISx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ix_n = t$$

for some $z \in X$.

It is obvious that **two weakly commuting mappings** are **compatible** but the converse is false. Examples supporting this fact can be found in [10].

Kaneko and Sessa [11] extended the concept of compatibility for single-valued mappings to a single-valued mapping and a multi-valued mapping as follows :

Definition 2.5 — The mappings $A : X \to CB(X)$ and $S : X \to X$ are compatible if $SAx \in CB(X)$ for all $x \in X$ and $\lim_{n \to \infty} H(SAx_n, ASx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to M \in CB(X)$ and $Sx_n \to t \in M$.

Definition 2.4 extends Definition 2.3 as well as the definition 2.2, where one may replace $I : X \to X$ with $A : X \to CB(X)$, that is $S : X \to X$ and $A : X \to CB(X)$ and $H(SAx, ASx) \leq D(Sx, Ax)$.

3. Main Results

Theorem 3.1 — *Let (X, d) be a complete metrically convex metric space and K be a non-empty closed convex subset of X. If the mappings F, G, S, T satisfy the condition (2.1) such that*

(i) $\partial K \subseteq SK \cap XK$; $Fk \cap K \subseteq TK$, $GK \cap K \subseteq SK$,

(ii) $Tx \in \partial K \Rightarrow Fx \subseteq K$; $Sx \in \partial K \Rightarrow Gx \subseteq K$,

(iii) (F, S) and (G, T) are compatible mappings,

(iv) F, G, S, T are continuous on K,

then there exists a point z in X such that

$$Sz = Tz \in Fz \cap Gz.$$

PROOF : We shall assign $\alpha = h(1 + h)$. Let $x \in K$. Since $\partial K \subseteq SK$, there exists a point $x_0 \in K$ such that $x = S x_0$. From the implication $Sx \in \partial K \Rightarrow Fx \subseteq K$, we conclude that $Fx_0 \subseteq K \cap FK \subseteq TK$. Let $x_1 \in K$ be such that $y_1 = Tx_1 \in Fx_0 \subseteq K$. Since $y_1 \in Fx_0$, there exists a point $y_2 \in Gx_1$ such that

$$d(y_1, y_2) \leq H(Fx_0, Gx_0) + \alpha.$$

Suppose $y_2 \in K$. Then $y_2 \in K \cap GK \subseteq SK$ which implies that there exists a point $x_2 \in K$ such that $y_2 = Sx_2$. Otherwise if $y_2 \notin K$, then there exists a point $u \in \partial K$ such that

$$d(Tx_1, u) + d(u, y_2) = d(Sx_1, y_2).$$

Since $u \in \partial K \subseteq SK$, there exists a point $x_2 \in K$ such that $u = Sx_2$ and so

$$d(Tx_1, Sx_2) + d(Sx_2, y_2) = d(Tx_1, y_2).$$

Let $y_3 \in Fx_2$ be such that

$$d(y_2, y_3) \leq H(Gx_1, Fx_2) + \alpha.$$

Thus repeating the above arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{2n} \in Gx_{2n-1}, y_{2n+1} \in Fx_{2n}$,

(ii) $y_{2n} \in K \Rightarrow y_{2n} = Sx_{2n}$ or

$y_{2n} \notin K \Rightarrow Sx_{2n} \in \partial K$ and

$d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, y_{2n}) = d(Tx_{2n-1}, y_{2n})$,

(iii) $y_{2n+1} \in K, y_{2n+1} = Tx_{2n+1}$ or $y_{2n+1} \notin K, Sx_{2n+1} \in \partial K$ and

$d(Sx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, y_{2n+1}) = d(Sx_{2n}, y_{2n+1})$,

(iv) $d(y_{2n-1}, y_{2n}) \leq H(Gx_{2n-1}, Fx_{2n-2}) + \alpha^{2n-1}$,

$d(y_{2n}, y_{2n+1}) \leq H(Fx_{2n}, Gx_{2n-1}) + \alpha^{2n}$.

We denote

$$P_0 = \{Sx_{2i} \in \{Sx_{2n}\} : Sx_{2i} = y_{2i}\}$$

$$P_1 = \{Sx_{2i} \in \{Sx_{2n}\} : Sx_{2i} \neq y_{2i}\}$$

$$Q_0 = \{Tx_{2i+1} \in \{Tx_{2n+1}\} : Tx_{2i+1} = y_{2i+1}\}$$

$$Q_1 = \{Tx_{2i+1} \in \{Tx_{2n+1}\} : Tx_{2i+1} \neq y_{2i+1}\}.$$

First we show that $(Sx_{2n}, Tx_{2n+1}) \notin P_1 \times Q_1$ and $(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$.

If $Sx_{2n} \in P_1$, then $y_{2n} \neq Sx_{2n}$ and we have $Sx_{2n} \in \partial K$ which implies that $y_{2n+1} \in Fx_{2n} \subseteq K$. Hence $y_{2n+1} = Tx_{2n+1} \in Q_0$. Similarly, one can argue that $(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$.

There are three possibilities :

Case 1 — If $(Sx_{2n}, Tx_{2n+1}) \in P_0 \times Q_0$, then

$$d(Sx_{2n}, Tx_{2n+1}) = d(y_{2n}, y_{2n+1}) \leq H(Fx_{2n}, Gx_{2n-1}) + \alpha^{2n}$$

$$\leq h \max \left\{ \frac{d(Sx_{2n}, Tx_{2n-1})}{a}, D(Sx_{2n}, Fx_{2n}), D(Tx_{2n-1}, Gx_{2n-1}), \frac{D(Sx_{2n}, Gx_{2n-1}) + D(Tx_{2n-1}, Fx_{2n})}{a+h} \right\} + \alpha^{2n}$$

$$\leq h \max \left(\frac{d(Sx_{2n}, Tx_{2n-1})}{a}, d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Sx_{2n}), \frac{d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n+1})}{a+h} \right) + \alpha^{2n}$$

$$\leq \max \left(hd(Sx_{2n}, Tx_{2n-1}) + \alpha^{2n}, \frac{\alpha^{2n}}{1-h}, \frac{hd(Tx_{2n-1}, Sx_{2n}) + \alpha^{2n}(a+h)}{a} \right)$$

$$\leq hd(Sx_{2n}, Tx_{2n-1}) + \max \left\{ \frac{1}{1-h}, \frac{a+h}{a} \right\} \alpha^{2n}$$

$$= hd(Sx_{2n}, Tx_{2n-1}) + \frac{\alpha^{2n}}{1-h}.$$

Case 2 — If $(Sx_{2n}, Tx_{2n+1}) \in P_0 \times Q_1$, then by (iii), we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq d(Sx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1})$$

and it follows from case 1 that

$$d(Sx_{2n}, Tx_{2n+1}) \leq hd(Tx_{2n-1}, Sx_{2n}) + \frac{\alpha^{2n}}{1-h}.$$

Similarly, if $(Tx_{2n-1}, Sx_{2n}) \in Q_1 \times P_0$, then we have

$$d(Tx_{2n-1}, Sx_{2n}) \leq hd(Sx_{2n-2}, Tx_{2n-1}) + \frac{\alpha^{2n-1}}{1-h}.$$

Case 3 — If $(Sx_{2n}, Tx_{2n+1}) \in P_1 \times Q_0$, then $Tx_{2n-1} = y_{2n-1}$. Hence proceeding as in case 1, we have

$$d(Sx_{2n}, Tx_{2n+1}) = d(Sx_{2n}, y_{2n+1})$$

$$\leq d(Sx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1})$$

$$\leq d(Sx_{2n}, y_{2n}) + H(Fx_{2n}, Gx_{2n-1}) + \alpha^{2n}$$

$$\leq d(Sx_{2n}, y_{2n}) + h \max\left(\frac{d(Sx_{2n}, Tx_{2n-1})}{a}, D(Sx_{2n}, Fx_{2n}) \right.$$

$$\left. D(Tx_{2n-1}, Gx_{2n-1}), \frac{D(Sx_{2n}, Gx_{2n-1}) + D(Tx_{2n-1}, Fx_{2n})}{a+h} \right) + \alpha^{2n}$$

$$\leq d(Sx_{2n}, y_{2n}) + h \max\left(\frac{d(Sx_{2n}, Tx_{2n-1})}{a}, d(Sx_{2n}, Tx_{2n+1}), \right.$$

$$\left. d(Tx_{2n-1}, y_{2n}), \frac{d(Sx_{2n}, y_{2n}) + d(Tx_{2n-1}, Tx_{2n+1})}{a+h} \right) + \alpha^{2n}.$$

As $d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, y_{2n}) = d(Tx_{2n-1}, y_{2n})$, we get

$$d(Sx_{2n}, Tx_{2n+1}) \leq d(Tx_{2n-1}, y_{2n}) + h \max\left(\frac{d(Tx_{2n-1}, y_{2n})}{a}, \right.$$

$$\left. d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, y_{2n}), \frac{d(Sx_{2n}, y_{2n}) + d(Tx_{2n-1}, Tx_{2n+1})}{a+h} \right) + \alpha^{2n}$$

$$\leq \max\left((1+h)\, d(Tx_{2n-1}, y_{2n}) + \alpha^{2n}, \frac{\alpha^{2n}}{1-h}, \frac{hd(Sx_{2n}, y_{2n}) + (a+h)\,\alpha^{2n}}{a} \right)$$

$$\leq \max\left((1+h)\, d(Tx_{2n-1}, y_{2n}) + \alpha^{2n}, \frac{\alpha^{2n}}{1-h} \frac{hd(Tx_{2n}, y_{2n-2}) + (a+h)\,\alpha^{2n}}{a} \right).$$

Therefore

$$d(Sx_{2n}, Tx_{2n+1}) \leq \max$$

$$\left((1+h)\, d(Tx_{2n-1}, y_{2n}) + \alpha^{2n} \frac{hd(Tx_{2n-1}, y_{2n})}{1-h} + \frac{\alpha^{2n}}{1-h} \frac{hd(Tx_{2n-1}, y_{2n}) + (a+h)\,\alpha^{2n}}{a} \right)$$

$$\leq (1+h)\, d(Tx_{2n-1}, y_{2n}) + \frac{\alpha^{2n}}{1-h}$$

$$\leq h(1+h)\, d(Sx_{2n-2}, Tx_{2n-1}) + \frac{h\,\alpha^{2n-1}}{1-h} + \frac{\alpha^{2n}}{1-h}.$$

Thus if we put $z_{2n} = Sx_{2n}, z_{2n+1} = Tx_{2n+1}$, we have

$$d(z_n, z_{n+1} \leq \begin{cases} hd(z_{n-1}, z_n) + \dfrac{\alpha^n}{1-h} & \text{or} \\ h(1+h)\, d(z_{n-2}, z_{n-1}) + \dfrac{h\,\alpha^{n-1}}{1-h} + \dfrac{\alpha^n}{1-h} \end{cases}.$$

Now along the lines of Itoh [9], it can be shown that the sequence $\{z_n\}$ is a Cauchy and hence converges to a point z. Consequently, the subsequences $\{z_{2n}\} = \{Sx_{2n}\}$ and $\{z_{2n+1}\} = \{Tx_{2n+1}\}$ also converge to the point z. Therefore

$$\lim_{n \to \infty} D(Gx_{2n-1}, Tx_{2n-1}) = 0 \Rightarrow \lim_{n \to \infty} D(TSx_{2n}, GTx_{2n-1}) = 0,$$

which leads directly to $D(Tz, Gz) = 0$. So $Tz \in Gz$ as Gz is closed. Similarly, the continuity of S and F lead to $Sz \in Fz$.

Again, we obtain

$$d(Sz, Tz) \leq \frac{1}{\sqrt{h}} H(Fz, Gz)$$

$$\leq \sqrt{h} \max \left\{ \frac{d(Sz, Tz)}{a}, D(Sz, Fz), D(Tz, Gz), \frac{D(Sz, Gz) + D(Tz, Fz)}{a+h} \right\}$$

$$\leq \sqrt{h} \max \left\{ \frac{d(Sz, Tz)}{a}, \frac{2d(Tz, Sz)}{a+h} \right\} = \max \left\{ \frac{\sqrt{h}}{a}, \frac{2\sqrt{h}}{a+h} \right\} d(Sz, Tz)$$

which implies that $Sz = Tz$. Thus we have proved that $Sz = Tz \in Fz \cap Gz$.

Remark : In Theorem 3.1, if we set $F = G$ and $S = T = i_K$ (i_K: the identity mapping n K), we obtain the result of Rhoades [15].

Similarly, we can prove the following theorem :

Theorem 3.2 — *Let (X, d) be a complete metrically convex metric space and K a non-empty closed subset of X. Let $F, G : K \to C(X)$ and $S, T : K \to K$ satisfy the condition (2.1). If the conditions (i)-(iv) of Theorem 3.1 hold then there exists a point* z *in* K *such that*

$$z = Sz = Tz \in Fz \cap Gz.$$

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