*Indian J. pure appl. Math.*, 32(10) : 1433-1441, October 2001
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# STABILITY IN VECTOR-VALUED AND SET-VALUED OPTIMIZATION*

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(*Received* 19 *July* 2000; *accepted* 8 *January* 2001)

In this paper, we discuss the stability of the sets of efficient points of vector-valued and set-valued optimization problems when the data $(E_n, f_n)$ (resp. $(E_n, F_n)$) of the approximate problems converge to the data $(E, f)$ (resp. $(E, F)$) of the original problem in the sense of Painleve-Kuratowski or Mosco. Our results improve and generalize those obtained by Attouch and Riahi in [1, Section 5].

**Key Words : Convergence of Set Sequence; Mosco Convergence; Painlevé- Kuratowski Convergence; Cone Extremization; Stability**

## 1. INTRODUCTION

The study of stability of optimization problems is an important topic in optimization theory and methodology. There have been a vast of works in the literature dealing with this problem under different assumptions. However, to our knowledge, Attouch and Riahi [1, Section 5] first studied the stability of a vector optimization problem under the pareto order in a finite dimensional space based on the concept of the convergence of the epigraphs of the corresponding vector-valued functions. In this paper, we consider the stability of the sets of efficient points of vector-valued and set-valued maps in a Banach space $Y$ under general cone order setting when the data $(E_n, f_n)$ (resp. $(E_n, F_n)$) of the approximate problems converge to the data $(E, f)$ (resp. $(E, F)$) of the original problem in the sense of Painleve-Kuratowski and Mosco (for details, see Section 2). Our results improve and generalize those in [1, Section 5] in the following ways: the objective space is generalized to an infinite space; the ordering cone is more general, which may be with empty interior; the objective vector-valued functions may be set-valued maps.

This paper is structured as follows. In Section 2, we present some concepts and notation. Section 3 is devoted to the stability results. Section 4 concludes the paper.

## 2. CONCEPTS AND NOTATIONS

In this section, we introduce some concepts and notations, which will be used in the sequel.

Throughout this paper, we assume $X$ and $Y$ are both Banach spaces. $Y$ is partially ordered by a nontrival, closed, pointed and convex cone $C$, i.e. $\forall\, y_1, y_2 \in Y, y_1 \leq_C y_2$ iff $y_2 - y_1 \in C$. Let $C^*$

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$= \{l \in Y^* : l(c) \geq 0, \ \forall\, c \in C\}$ denote the positive polar cone of $C$ and int $C$ denote the interior of $C$ if the interior of $C$ is nonempty.

We first recall the concepts of Painlevé-Kuratowski convergence and Mosco convergence of a sequence of sets.

*Definition* 2.1 — Let $Z$ be a first countable topological space. The Painlevé-Kuratowski convergence of a sequence of subsets $\{D_n : n \in N\}$ of $Z$ to a subset of $Z$ (i.e., $D_n \overset{P.K}{\rightarrow} D$) means

$$\limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n \quad \text{with}$$

$$\liminf_{n\to\infty} D_n = \{x = \lim_{n\to+\infty} x_n : x_n \in D_n,\ \forall\, n \in N\}$$

$$\limsup_{n\to\infty} D_n = \{x = \lim_{k\to+\infty} x_{n_k} : x_{n_k} \in D_{n_k},\ \forall\, k,\ \{n_k\} \text{ a subsequence of } N\}.$$

*Definition* 2.2 — Let $Z$ be a normed space. We say that a sequence of subsets $\{D_n\}$ of $Z$ Mosco converges to $D \subset Z$ if

$$w-\limsup_{n\to\infty} D_n \subset D \subset \liminf_{n\to\infty} D_n$$

with $$w-\limsup_{n\to\infty} = \{x = w-\lim_{k\to+\infty} x_{n_k} : x_{n_k} \in D_{n_k}\ \forall k,\ \{n_k\} \text{ a subsequence of } N\},$$

where $x = w-\lim\limits_{k\to+\infty} x_{n_k}$ stands for the weak convergence of $x_{n_k}$ to $x$.

*Definition* 2.3 — A vector-valued function $f: X \rightarrow Y$ is said to be lower semicontinuous (l.s.c) with respect to (w.r.t) $C$ if $\forall y \in Y$, $\{x \in X : f(x) \leq_C y\}$ is closed.

We use $ext_C A$ to denote the set of maximal (efficient) points of $A$, i.e., $z \in ext_C A$ iff $(z + C) \cap A = \{z\}$.

We introduce a virtual element $+\infty$ in $Y$ meaning that $+\infty - y \in C$, $\forall y \in Y$.

*Definition* 2.4 — We say a sequence of vector-valued functions $f_n$ (defined on $X$) Painlevé-Kuratowski (P.K. for short) resp. Mosco (M for short)) converges to a vector-valued function $f$ (defined on $X$) if

$$epi\,(f_n) = \{(x, y) : y \in f(x) + C\} \overset{P.K.}{\rightarrow} epi\,(f) = \{(x, y) : y \in f(x) + C\}$$

$$(epi\,(f_n) \overset{M}{\rightarrow} epi\,(f)).$$

*Definition* 2.5 — We say a sequence of nonempty set-valued maps $F_n$ (defined on $X$) P.K. (M) converges to a nonempty set-valued map $F$ (defined on $X$) if

$$epi\,(F_n) = \{(x, y) : y \in F_n(x) + C\} \overset{P.K.}{\rightarrow} epi\,(F) = \{(x, y) : y \in F(x) + C\}$$

$$(epi\,(F_n) \overset{M}{\rightarrow} epi\,(F)).$$

*Definition* 2.6 — Let $\{f_n : E_n \to Y, n = 1, 2, \ldots\}$ be a sequence of vector-valued functions and denote by $\{(E_n, f_n) : n = 1, 2, \ldots\}$ the corresponding sequence pairs. $f: E \to Y$. we say $(E_n, f_n)$ P.K. (M) converges to $(E, f)$ if $\bar{f}_n \overset{P.K.}{\to} \bar{f}(\bar{f}_n \overset{M}{\to} \bar{f})$, where

$$\bar{f}_n(x) = \begin{cases} f_n(x), & \text{if } x \in E_n, \\ +\infty, & \text{if } x \in X \setminus E_n; \end{cases}$$

$$\bar{f}_n(x) = \begin{cases} f(x), & \text{if } x \in E; \\ +\infty, & \text{if } x \in X \setminus E. \end{cases}$$

*Definition* 2.7 — Let $\{F_n : E_n \to 2^Y, n = 1, 2 \ldots\}$ be a sequence of nonempty set-valued maps and denote by $\{(E_n, F_n)\}$ the corresponding pairs. $F: E \to 2^Y$ is a nonempty set-valued map. We say $(E_n, F_n)$ *P. K.* (*M*) converges to $(E, F)$ if $\bar{F}_n \overset{P.K.}{\to} \bar{F}$, where

$$\bar{F}_n(x) = \begin{cases} F_n(x), & \text{if } x \in E_n, \\ +\infty, & \text{if } x \in X \setminus E_n; \end{cases}$$

and

$$\bar{F}(x) = \begin{cases} F(x), & \text{if } x \in E, \\ +\infty, & \text{if } x \in X \setminus E. \end{cases}$$

*Definition* 2.8 — Let $f: X \to Y$ be a vector-valued function. We say $f$ is bounded below if $\exists y_0 \in Y$ such that $f(X) - y_0 \subset C$.

Let $f_n : X \to Y$ be a sequence of vector-valued functions. We say $f_n$ are uniformly bounded below if $\exists y_0 \in Y$ such that $f_n(X) - y_0 \subset C$, $\forall n \in N$.

*Definition* 2.9 — Let $F: X \to 2^Y$ be a set-valued map. We say that $F$ is bounded below if $\exists y_0 \in Y$ such that $[F(X) - y_0] \subset C$, where $F(x) = \bigcup_{x \in X} F(x)$.

Let $F_n : X \to 2^Y$ be a sequence of set-valued maps. We say that $F_n$ are uniformly bounded below if there exists $y_0 \in Y$ such that $[F(X) - y_0] \subset C$ for all $n$.

## 3. Stability of the Set of Efficient (Minimal) Points

This section presents the main results, which generalize the corresponding results in [1, Section 5]. We shall first state the results and then prove them one by one.

***Theorem*** 3.1 — *Assume that* $intC \neq \theta, -C \subset \{y \in Y : l(y) + \varepsilon \| y \| \leq 0\}$ *for some* $l \in Y^*$ *and* $\varepsilon > 0$. $(E_n, f_n)$, $(E, f)$ *are as defined in Definition 2.6.* $\forall n \in N, E_n$ *is a nonempty closed subset of X,* $f_n$ *is l.s.c w.r.t. C.* $E \subset X$ *is nonempty closed.* $f$ *is l.s.c. w.r.t. C.*

***In addition,***

(a) $\inf_{n \in N} \inf_{x \in E_n} l(f_n(x)) > -\infty$;

(b) $(E_n, f_n) \overset{P.K.}{\rightarrow} (E, f)$;

(c) $\exists$ a *compact subset $K$ of $X$ such that $E_n \subset K$, $\forall n \in N$; and*

(d) $\forall\ \rho > 0, \exists$ *a compact subset $K_\rho$ of $Y$ such that $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, where $B$ is the unbit ball of $Y$.*

*Then $ext_{-C} f(E)$ is nonempty and*

$$ext_{-C} f(E) \subset \liminf_{n \to \infty} ext_{-C} f_n(E_n).$$

***Theorem*** 3.2 — *Assume that $C$ (without the assumption $intC \neq 0$), $f_n, E_n, E, f$ are as in Theorem 3.1. $\forall \lambda \in C^*, \lambda(f_n)$ is l.s.c. on $E_n$. $\lambda(f)$ is l.s.c. on $E$.*

*In addition,*

*(a)* $\inf_{n \in N} \inf_{x \in E_n} l(f_n(x)) > -\infty$;

*(b)* $(E_n, f_n) \overset{P.K.}{\rightarrow} (E, f)$;

*(c) $\exists$ a compact subset $K$ of $X$ such that $E_n \subset K$, $\forall n \in N$; and*

*(d) $\forall \rho > 0, \exists$ a compact subset $K_\rho$ of $Y$ such that $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, where $B$ is the unit ball of $Y$.*

*Then $ext_{-C} f(E)$ is nonempty and*

$$ext_{-C} f(E) \subset \liminf_{n \to \infty} ext_{-C} f_n(E_n).$$

***Theorem*** 3.3 — *Assume that $X$ and $Y$ are reflexive banach spaces. $C$, $E_n, f_n, E, f$ are as in Theorem 3.1. $f_n, f$ are l.s.c. (with respect to the weak topology of $X$) w.r.t. $C$.*

*In addition,*

*(a)* $\inf_{n \in N} \inf_{x \in E_n} l(f_n(x)) > -\infty$;

*(b)* $(E_n, f_n) \overset{M}{\rightarrow} (E, f)$;

*(c) $\exists$ a bounded closed subset $K$ of $X$ such that $E_n \subset K$, $\forall n \in N$; and*

*(d) $\forall \rho > 0, \exists$ a compact subset $K_\rho$ of $Y$ such that $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, where $B$ is the unit ball of $Y$.*

***Then** $ext_{-C} f(E)$ **is nonempty and***

$$ext_{-C} f(E) \subset \liminf_{n \to \infty} ext_{-C} f_n(E_n).$$

***Theorem*** 3.4 — *Let $X$, $Y$, $C$ (without the assumption int $C \neq 0$), $E_n, E, f_n, f$ be as in Theorem 3.3 suppose that $\forall \lambda \in C^*, \lambda(f_n), \lambda(f)$ are l.s.c. (with respect to the weak topology of $X$).*

*In addition, assume the following hold*

*(a)* $\inf_{n \in N} \inf_{x \in E_n} l(f_n(x)) > -\infty$;

*(b)* $(E_n, f_n) \overset{M}{\to} (E, f)$;

*(c)* $\exists$ *a bounded closed subset* $K$ *of* $X$ *such that* $E_n \subset K$, $\forall n \in N$;

*(d)* $\forall \rho > 0, \exists$ *a compact subset* $K_\rho$ *of* $Y$ *such that* $ext_{-C} f_n(E_n) \cap \rho B \subset K_\rho$, *where* $B$ *is the unit ball of* $Y$.

*Then* $ext_{-C} f(E)$ *is nonempty and*

$$ext_{-C} f(E) \subset \liminf_{n \to \infty} ext_{-C} f_n(E_n).$$

*Remark* 3.1 : (*i*) The condition $-C \subset \{y \in Y : l(y) + \varepsilon \| y \| \leq 0\}$ for some $l \in Y^*$ and $\varepsilon > 0$ is equivalent to the statement that $\exists l \in C^* \backslash \{0\}$ and $\varepsilon$ such that $l(c) \geq \varepsilon$ for any $c \in C$ with $\| c \| = 1$. The latter is fulfilled when $Y$ is a finite dimensional space and $C$ is a nontrivial, pointed, closed and convex cone. Note that Attouch and Riahi[1] considered only the special case when $Y = R^N$ and $C = R^N_+$. Thus, our Theorems 3.1 and 3.3 at least generalize Attouch and Riahi's result to the general ordering context in a finite dimensional space. Moreover, when $Y = R^N$, $C = R^N_+$, it is not hard to see that the conditions of Theorem 5.2 in [1] imply all the conditions in our Theorem 3.1. However, Definition5.1 in [1] is a stronger version of convergence than (*b*) in our Theorem 3.1. Therefore, our Theorem 3.1 improves Theorem 5.2 in [1]. On the other hand, our Theorems 3.2 and 3.4 generalize Attouch and Riahi's result [1, Theorem 5.1] to the context of finite dimensional spaces when our ordering cone $C$ is with nonempty interior. It is worth mentioning that our Theorems 3.2 and 3.4 apply to the case where the ordering cone $C$ is with empty interior.

(*ii*) It is clear that if $\bar{f}_n$ are uniformly bounded below (i.e., there exists $y_0 \in Y$ such that $f_n(x) - y_0 \in C$, $\forall x \in E_n$, $\forall n$), then condition (*a*) in the above theorems holds automatically.

We need the following lemmas to prove the theorems above.

*Lemma* 3.1 — Under the assumptions of Theorem 3.1 (or Theorem 3.2), we have $(E_n, f_n) \overset{P.K.}{\to} (E, f) \Rightarrow f_n(E_n) + C \overset{P.K.}{\to} f(E) + C$.

PROOF : Firstly, we prove

$$f(E) + C \subset \liminf_{n \to \infty} f_n(E_n) + C \quad \text{... (1)}$$

$\forall y \in f(E)$ with $y = f(x)$, $\forall c \in C$, then $(x, y + c) \in epi(\bar{f})$. Since $\bar{f}_n \overset{P.K.}{\to} \bar{f}$, we have $epi(\bar{f}) \subset \liminf_{n \to \infty} epi(\bar{f}_n)$.

So $\exists$ $(x_n, y_n) \in epi(\bar{f}_n)$ with $y_n = \bar{f}_n(x_n) + c_n$ such that $(x_n, y_n) \to (x, y + c)$.

Thus $\quad y_n \to y + C.$

Obviously, $x_n \in E_n$ when $n$ is sufficiently large. Hence, $y_n = \bar{f}_n(x_n) + x_n = f(x_n) + c_n \in f_n(E_n) + C$, when $c$ is sufficiently large, implying (1).

Secondly, We prove $\limsup_{n \to \infty} (f_n(E_n) + C) \subset f(E) + C$.

For any $x_{n_k} \in E_{n_k}, c_{n_k} \in C$ with $f_{n_k}(x_{n_k}) + c_{n_k} \in f_{n_k}(E_{n_k}) + C$ such that $f_{n_k}(x_{n_k}) + c_{n_k} \to y$.

Now that $\{x_{n_k}\} \subset K$ and $K$ is a compact subset of $X$, we deduce that there exist a subsequence $\{x_{n_{k_l}}\}$ and $x \in K$ such that $x_{n_{k_l}} \to x$. Noticing that $y \neq +\infty$, we obtain $x \in E$. So $(x, y) \in epi(f)$, i.e., $y \in f(E) + C$. The proof is complete.

Similarly, we can prove the following Lemma 3.2.

*Lemma* 3.2 — Under the assumptions of Theorem 3.3 (or Theorem 3.4),

$$(E_n, f_n) \overset{M}{\to} (E, f) \Rightarrow f_n(E_n) + C \overset{M}{\to} f(E) + C.$$

*Lemma* 3.3 — Let $E \subset X$ be nonempty compact. $f: E \to Y$ is l.s.c. w.r.t. $C$ on $E$. Then $f(E) + C$ is nonempty closed and $ext_{-C}[f(E) + C] = ext_{-C} f(E)$.

PROOF : For any $f(x_n) + c_n \in f(E) + C$ with $x_n \in E, c_n \in C$ such that

$$f(x_n) + c_n \to y. \qquad \text{... (2)}$$

By the compactness of $E$, we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that $x_{n_k} \to x$. This combined with (2) yields $f(x_{n_k}) + c_{n_k} \to y$.

Arbitrarily fix an $e \in$ int $C$, then $\forall \varepsilon > 0, \exists K_0$, when $k \geq K_0, f(x_{n_k}) + c_{n_k} \leq_C y + \varepsilon e$. So $f(x_{n_k}) \leq_C y + \varepsilon e$. By the l.s.c. of $f$ w.r.t. $C$, we know that $f(x) \leq_C y + \varepsilon e$, i.e., $f(x) - y - \varepsilon e \in -C$. Letting $\varepsilon \to 0$, we have $f(x) - y \in -C$ (since $-C$ is closed). Hence $y \in f(x) + C \subset f(E) + C$.

It is obvious that the relation $ext_{-C}[f(E) + C] = ext_{-C} f(E)$ holds.

*Lemma* 3.4 — Let $E \subset X$ be nonempty compact. Let $f: X \to Y$ be such that $\forall \lambda \in C^*$, $\lambda(f)$ is l.s.c. on $E$. Then $f(E) + C$ is nonempty closed and $ext_{-C}[f(E) + C] = ext_{-C}[f(E)]$.

PROOF : We only show that $f(E) + C$ is closed.

For any $f(x_n) + c_n \in f(E) + C$ with $x_n \in E, c_n \in C$ such that $f(x_n) + c_n \to y$. By the compactness of $E$, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that $x_{n_k} \to x$. Moreover, $f(x_{n_k}) + c_{n_k} \to y$. Hence $\lambda(f(x_{n_k}) + \lambda(c_{n_k}) \to \lambda(y)$, $\forall \lambda \in C^*$ ,implying $\lambda(f(x)) \leq \liminf_{n \to \infty} \lambda(f(x_{n_k}) \leq \lambda(y)$, $\forall \lambda \in C^*$. Thus $f(x) \leq_C y$, implying $y \in f(x) + C \subset f(E) + C$.

PROOF OF THEOREM 3.1 : We imply apply Theorem 3.3 in [1] with $C$ replaced by $-C$, $D_n = f_n(E_n) + C, D = f(E) + C$. By Lemma 3.1 and Lemma 3.3, we know that $D_n$, $D$ are nonempty closed and $D_n \overset{P.K.}{\to} D$. In addition, $\inf_{n \in N} \inf_{y \in D_n} l(y) \geq \inf_{n \in N} \inf_{x \in E_n} > -\infty$. Moreover, $\forall \rho > 0$,

$(ext_{-C}D_n) \cap \rho B = (ext_{-C}f_n(E_n)) \cap \rho B \subset K_\rho$. So all the conditions of Theorem 3.3 in [1] hold, hence, $ext_{-C}f(E) = ext_{-C}D \neq 0$ and $ext_{-C}f(E) = ext_{-C}\ D \subset \liminf_{n\to\infty} ext_{-C}D_n = \liminf_{n\to\infty} ext_{-C}f(E_n)$. The proof is completed.

Theorem 3.2 can be similarly proved.

*Lemma* 3.5 — Let $X$ be a reflexive Banach space and int $C \neq 0$. If $E \subset X$ is nonempty, closed and bounded. $f: E \to Y$ is l.s.c. (with respect to the weak topology of $X$) on $E$ w.r.t. $C$. Then $f(E) + C$ is a nonempty closed set and $ext_{-C}[f(E) + C] = ext_{-C}f(E)$.

PROOF : Since $X$ is reflexive, we know that $E$ is a weakly compact subset of $X$.

$$\forall f(x_n) + c_n \in f(E) + C \text{ with } x_n \in E, c_n \in C \text{ such that such that } f(x_n) + c_n \to y.$$

By the weak compactness of $E$, we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in E$ such that $x_{n_k} \xrightarrow{w} x$. Arbitrarily fix an $e \in$ int $C$, $\forall \varepsilon > 0, \exists K_0 > 0$, when $k \geq K_0$, we have $f(x_{n_k}) + c_{n_k} \leq_C y + \varepsilon e$. Thus $f(x_{n_k}) \leq_C y + \varepsilon e$ . By the l.s.c. of $f$ (w.r.t. $C$ and the weak topology of $X$), we have $f(x) \leq_C y + \varepsilon e$. Hence, $f(x) \leq_C y + \varepsilon e$, i.e., $y \in f(x) + C \subset f(E) + C$.

Similar to the Proof of Lemma 3.4 (applying the weak topology of $X$), we can prove the following Lemma 3.6.

*Lemma* 3.6 — Let $X$ be a reflexive Banach space. If $E \subset X$ is nonempty, closed and bounded, $f: E \to Y$ is such that " $\lambda \in C^*, \lambda(f)$ is l.s.c. (with respect to the weak topology of $X$) on $E$, then $f(E) + C$ is nonempty closed and $ext_{-C}[f(E) + C] = ext_{-C}f(E)$.

Applying our Lemmas 3.2, 3.5 and Theorem 3.5 in [1], we can easily prove Theorem 3.3

Applying our Lemmas 3.2, 3.6 and Theorem 3.5 in [1], we can also prove Theorem 3.4.

***Theorem*** 3.5 — *Let* $C \subset \{y \in Y: l(y) + \varepsilon \| y \| \leq 0\}$ *for some* $l \in Y^*, \varepsilon > 0$. *Assume that* $\forall\, n \in N, E_n$ is a nonempty closed subset of X, $F_n : X \to 2^Y$ *is u.s.c. nonempty compact-valued. E is a nonempty closed subset of* $X, F : X \to 2^Y$ *is u.s.c. nonempty compact-valued. In addition,*

*(a)* $\inf_{n \in N} \inf_{x \in E_n} \inf_{y \in F_n(x)} l(y) > -\infty$;

*(b)* $(E_n, F_n) \xrightarrow{M} (E, F)$;

*(c)* $\exists$ *a bounded closed subset K of X such that* $E_n \subset K$, $\forall n \in N$; *and*

*(d)* $\forall \rho > 0, \exists$ *a compact subset* $K_\rho$ *of Y such that* $ext_{-C}F_n(E_n) \cap \rho B \subset K_\rho$, *where B is the unit ball of Y.*

*Then* $ext_{-C}F(E)$ *is nonempty and* $ext_{-C}F(E) \subset \liminf_{n\to\infty} ext_{-C}F_n(E_n)$.

***Theorem*** 3.6 — *Let X, Y be reflexive Banach spaces,* $C, E_n, F_n, E, F$ *be as in Theorem 3.5. Assume that* $F_n, F$ *are u.s.c. (with respect to the weak topology of X). In addition,*

*(a)* $\inf_{n \in N} \inf_{x \in E_n} \inf_{y \in F_n(x)} l(y) > -\infty$;

*(b)* $(E_n, F_n) \xrightarrow{M} (E, F)$;

*(c)* $\exists$ *a bounded closed subset* $K$ *of* $X$ *such that* $E_n \subset K$, $\forall n \in N$;

*(d)* $\forall \rho > 0$, $\exists$ *a compact subset* $K_\rho$ *of* $Y$ *such that* $ext_{-C} F_n(E_n) \cap \rho B \subset K_\rho$, *where* $B$ *is the unit ball of* $Y$.

*Then* $ext_{-C} F(E)$ *is nonempty and* $ext_{-C} F(E) \subset \liminf_{n \to \infty} ext_{-C} F_n(E_n)$.

*Remark* 3.2 : If $\overline{F}_n$ are uniformly bounded below (i.e. there exists $y_0 \in Y$ such that $[F_n(E_n) - y_0] \subset C$, $\forall n$), then *(a)* in Theorems 3.5 and 3.6 holds automatically.

The following lemmas are needed to prove Theorems 3.5, 3.6.

*Lemma* 3.7 — Under the assumptions of Theorem 3.5, $(E_n, F_n) \xrightarrow{P.K.} (E, F)$ implies $F_n(E_n) + C \xrightarrow{P.K.} F(E) + C$.

*Lemma* 3.8 — Under the assumptions of Theorem 3.6, $(E_n, F_n) \xrightarrow{M} (E, F)$ implies $F_n(E_n) + C \xrightarrow{M} F(E) + C$.

The proofs of Lemma 3.7 and Lemma 3.8 are similar to those of Lemma 3.1, Lemma 3.2, respectively, we omit it.

*Lemma* 3.9 — Let $E \subset X$ be nonempty compact. Let $F: E \to 2^Y$ be u.s.c. nonempty compact-valued. Then $F(E) + C$ is nonempty closed and $ext_{-C} F(E) = ext_{-C} [F(E) + C]$.

PROOF : We only need to show that $F(E) + C$ is closed.

$\forall y_n + c_n \in F(x_n) + C$ with $y_n \in F(x_n)$ $(x_n \in E)$, $c_n \in C$ and $y_n + c_n \to y$. We prove that $y \in F(E) + C$ in the following two cases, respectively. (i) $x_n \equiv x$ for some $x \in X$ when $n$ is sufficiently large.

Then $y_n \in F(x)$, when $n$ is sufficiently large. By the compactness of $F(x)$, we get a subsequence $\{y_{k_k}\}$ of $\{y_n\}$ and $y' \in F(x)$ such that $y_{n_k} \to y'$. However, $y_{n_k} + c_{n_k} \to y$, so $c_{n_k} \to y - y' \in C$, i.e., $y \in F(x) + C \subset F(E) + C$.

(ii) $\exists$ a subsequence $\{x_{n_k}\}$ whose elements are different from one another such that $y_{n_k} \in F(x_{n_k})$.

By the compactness of $E$, we obtain a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ and $x \in E$ such that $x_{n_{k_l}} \to x$. By the u.s.c. of $F$ and the compactness of $F(x)$, we have a subsequence $\{y_{n_{k_{l_t}}}\}$ of $\{y_{n_{k_l}}\}$ and $y' \in F(x)$ such that $y_{n_{k_{l_t}}} \to y'$. So $c_{n_{k_{l_t}}} \to y - y' \in C$, i.e., $y \in F(x) + C \subset F(E) + C$. The proof is completed.

*Lemma* 3.10 — Let $X$ be a reflexive space, $E$ a nonempty closed bounded subset of $X$. Let $F: E \to 2^Y$ be an u.s.c. (w.r.t the weak topology of $X$) nonempty compact-valued map. Then $F(E) + C$ is nonempty closed and $ext_{-C} F(E) = ext_{-C} [F(E) + C]$.

The proof of Lemma 3.10 is almost the same as that of Lemma 3.9, the only difference being that the weak topology of $X$ should be applied.

The combination of our Lemmas 3.7, 3.9 and Theorem 3.3 in [1] completes the proof of our Theorem 3.5.

The combination of our Lemmas 3.8, 3.10 and Theorem 3.5 in [1] completes the proof of our Theorem 3.6.

## 4. CONCLUSIONS

This paper considered the stability of vector-valued and set-valued optimization problems based on the concepts of the Painleve-kuratowski and Mosco convergence of sets. The results generalized the corresponding results of Attouch and Riahi in [1, Section 5]. The generalization is threefold: the objective space $Y$ is extended from finite dimensional to finite dimensional; the dominating cone $C$ is extended from $R_+^N$ to a general order. The objective functions are extended to general vector-valued functions or set-valued maps.

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