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ON A CLASS OF CERTAIN ANALYTIC FUNCTIONS OF COMPLEX ORDER

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We introduce a class, namely, $H_n(b, M)$ of certain analytic functions. For this class we determine sufficient condition in terms of coefficients, coefficient estimate, maximization theorem concerning the coefficients, and radius problem.

Key Words : Analytic; Salagean Operator; Complex Order

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad \text{... (1.1)}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. We use Ω to denote the class of functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. For a function $f(z)$ in A, we define

$$D^0 f(z) = f(z), \qquad \text{... (1.2)}$$

$$D^1 f(z) = Df(z) = zf'(z), \qquad \text{... (1.3)}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \; (n \in N = \{1, 2, ...\}). \qquad \text{... (1.4)}$$

The differential operator D^n was introduced by Salagean[10]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to A is in the class $H_n(b, M)$ if and only if

$$\left| \frac{b - 1 + \dfrac{D^{n+1} f(z)}{D^n f(z)}}{b} - M \right| < M, \; z \in U, \qquad \text{... (1.5)}$$

where $M > \frac{1}{2}$ and $b \neq 0$, complex.

It follows by Kulshrestha[14] that $g(z) \in H_0(1, M) = F(1, M)$ if and only if for $z \in U$

$$\frac{zg'(z)}{g(z)} = \frac{1 + w(z)}{1 - m\,w(z)}, \qquad \dots (1.6)$$

where $m = 1 - \frac{1}{M}\left(M > \frac{1}{2}\right)$ and $w(z) \in \Omega$.

One can easily show that $f(z) \in H_n(b, M)$ if and only if there is a function $g(z) \in H_0(1, M) = F(1, M)$ such that

$$D^n f(z) = z\left[\frac{g(z)}{z}\right]^b. \qquad \dots (1.7)$$

Thus from (1.6) and (1.7) it follows that $f(z) \in H_n(b, M)$ if and only if for $z \in U$

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + [b(1+m) - m]\,w(z)}{1 - m\,w(z)}, \qquad \dots (1.8)$$

where $m = 1 - \frac{1}{M}\left(M > \frac{1}{2}\right)$ and $w(z) \in \Omega$.

By giving specific values to n, b and M, we obtain the following important subclass studied by various authors in earlier works :

(1) $H_0(b, M) = F(b, M)$ (Nasr and Aouf[7]) and $H_1(b, M) = G(b, M)$ (Nasr and Aouf[8]).

(2) $H_0(\cos\lambda e^{-i\lambda}, M) = F_{\lambda, M}$ and $H_1(\cos\lambda e^{-i\lambda}, M) = G_{\lambda, M}\left(|\lambda| < \frac{\pi}{2}\right)$ (Kulshrestha[4]).

(3) $H_0((1-\alpha)\cos\lambda e^{-i\lambda}, \infty) = S^{\lambda}(\alpha)\left(|\lambda| < \frac{\pi}{2}, 0 \le \alpha < 1\right)$ (Libera[6] and $H_1((1-\alpha)\cos\lambda e^{-i\lambda}, \infty) = C^{\lambda}(\alpha)\left(|\lambda| < \frac{\pi}{2}, 0 \le \alpha < 1\right)$ (Chichra[3] and Sizuk[13]).

(4) $H_0(b, M) = S(1-b)$ (Nasr and Aouf[9]) and $H_1(b, M) = C(b)$ (Wiatrowski[14] and Nasr and Aouf[8]).

(5) $H_0((1-\alpha)\cos\lambda e^{-i\lambda}, M) = F_M(\lambda, \alpha)$ and $H_1((1-\alpha)\cos\lambda e^{-i\lambda}, M) = G_M(\lambda, \alpha)$ $\left(|\lambda| < \frac{\pi}{2}, 0 \le \alpha < 1\right)$ Aouf[1 & 2].

(6) $H_0(1, 1) = F(1, 1)$ Singh[11] and $H_0(1, M) = F(1, M)$ Singh and Singh[12].

From the definitions of the classes $F(b, M)$ and $H_n(b, M)$, we observe that

$$f(z) \in H_n(b, M) \text{ if and only if } D^n f(z) \in F(b, M). \quad \dots (1.9)$$

The purpose of the present paper is to determine sufficient condition in terms of coefficients for function belong to $H_n(b, M)$, coefficient estimate, and maximization of $|a_3 - \mu a_2^2|$ on the class $H_n(b, M)$ for complex value of μ. Further we obtain the radius of disc in which $\text{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > 0$, wherever $f(z)$ belongs to $H_n(b, M)$.

2. A Sufficient Condition for a Function to be in $H_n(b, M)$

***Theorem* 1** — *Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{k=2}^{\infty} \{(k+1+|b(1+m)+m(k-1)|\} k^n |a_k| \le |b(1+m)|, \quad \dots (2.1)$$

hold, then $f(z)$ belongs to $H_n(b, M)$, where $m = 1 - \frac{1}{M}\left(M > \frac{1}{2}\right)$.

PROOF : Suppose that the inequality (2.1) holds. Then we have for $z \in U$.

$$|D^{n+1} f(z) - D^n f(z)| - |b(1+m) D^n f(z) + m(D^{n+1} f(z) - D^n f(z))|$$

$$= \left|\sum_{k=2}^{\infty} k^n (k-1) a_k z^k\right| - \left|b(1+m)\left\{Z + \sum_{k=2}^{\infty} k^n a_k z^k\right\} + m \sum_{k=2}^{\infty} k^n (k-1) a_k z^k\right|$$

$$\le \sum_{k=2}^{\infty} k^n (k-1) |a_k| r^k - \left\{|b(1+m)| r - \sum_{k=2}^{\infty} |b(1+m) + m(k-1)| k^n |a_k| r^k\right\}$$

$$= \sum_{k=2}^{\infty} \left\{(k-1) + |b(1+m) + m(k-1)\right\} k^n |a_k| r^K - |b(1+m)| r.$$

Letting $r \to -1$, then we have

$$|D^{n+1} f(z) - D^n f(z)| - |b(1+m) D^n f(z) + m(D^{n+1} f(z) - D^n f(z)|$$

$$= \sum_{k=2}^{\infty} \left\{(k-1) + |b(1+m) + m(k-1)\right\} k^n |a_k| r^K - |b(1+m)|$$

$$\le 0, \text{ by (2.1)}.$$

Hence, it follows that

$$\left| \frac{\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1}{b(1+m) + m\left\{\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1\right\}} \right| < 1,\ z \in U.$$

Letting

$$W(z) = \frac{\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1}{b(1+m) + m\left\{\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1\right\}},$$

then $w(0) = 0$, $w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Hence, we have

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + [b(1+m) - m]\, w(z)}{1 - mw(z)}.$$

Which shows that $f(z)$ belongs to $H_n(b, M)$.

3. Coefficient Estimate

***Theorem* 2** — *Let the function $f(z)$ defined by (1.1) be in the class $H_n(b, M), z \in U$.*

(a) If $2m(k-1)\, Re\ \{b\} > (k-1)^2 (1-m) - |b|^2 (1+m)$, let

$$N = \left[\frac{2m(k-1) \operatorname{Re}(b)}{(k-1)^2 (1-m) - |b|^2 (1+m)} \right],\ k = 1, 3, \ldots, j-1.$$

Then

$$|a_j| \le \frac{1}{j^n (j-1)!} \prod_{k=2}^{j} |b(1+m) + (k-2)m|, \qquad \ldots (3.1)$$

for $j = 2, 3, \ldots, N+2$; *and*

$$|a_j| \le \frac{1}{j^n (j-1)(N+1)!} \prod_{k=2}^{N+3} |b(1+m) + (k-2)m|, j > N+2. \qquad \ldots (3.2)$$

(b) If $2m(k-1)\ Re\ \{b\} \le (k-1)^2 (1-m) - |b|^2 (1+m)$, then

$$|a_j| \le \frac{(1+m)|b|}{j^n (j-1)},\ \text{for } j \ge 2, \qquad \ldots (3.3)$$

where $m = 1 - \dfrac{1}{M}\left(M > \dfrac{1}{2}\right)$ *and $b \ne 0$, complex.*

The inequalities (3.1) and (3.3) are sharp.

PROOF : Since $f(z) \in H_n(b, M)$, so from (1.8) we have that

$$\sum_{k=2}^{\infty} k^n (k-1) a_k z^k = \left\{ b(1+m) z + \sum_{k=2}^{\infty} k^n [b(1+m) + m(k-1)] a_k z^k \right\} w(z) \quad \text{... (3.4)}$$

which is equivalent to

$$\sum_{k=2}^{j} k^n (k-1) a_k z^k + \sum_{\kappa=2}^{\infty} d_k z^k$$

$$= \left\{ b(1+m) z + \sum_{k=2}^{j-1} k^n [b(1+m) + m(k-1)] a_k z^k \right\} w(z),$$

where d_j's are some complex numbers.

Then since $|w(z)| < 1$, we have

$$\left| \sum_{k=2}^{j} k^n (k-1) a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k \right|$$

$$\leq \left| b(1+m) z + \sum_{k=2}^{j-1} k^n [b(1+m) + m(k-1)] a_k z^k \right| \quad \text{... (3.5)}$$

Squaring both sides of (3.5) and integrating round $|z| = r < 1$, we get, after taking the limit with $r \to 1$

$$j^{2n} (j-1)^2 |a_j|^2 \leq (1+m)^2 |b|^2$$

$$+ \sum_{k=2}^{j-1} k^{2n} \left\{ |b(1+m) + m(k-1)|^2 - (k-1)^2 \right\} |a_k|^2. \quad \text{...(3.6)}$$

Now there may be following two cases :

(a) Let $2m(k-1) Re\{b\} > (k-1)^2 (1-m) - (1+m)|b|^2$. Suppose that $j \leq n+2$; then for $j = 2$, (3.6) gives

$$|a_2| \leq \frac{(1+m)|b|}{2^n}$$

which gives (3.1) for $j = 2$. We establish (3.1), by mathematical induction.

Suppose (3.1) is valid for $k = 2, 3, ..., j-1$. Then it follows from (3.6)

$$j^{2n}(j-1)^2|a_j|^2 \le (1+m)^2|b|^2$$

$$+\sum_{k=2}^{j-1} k^{2n} \quad \{|b(1+m)+m(k-1)^2-(k-1)^2\}.$$

$$\times \frac{1}{k^{2n}((k-1)!)^2}\prod_{p=2}^{k} |b(1+m)+(p+2)m|^2$$

$$= \frac{1}{((j-1)!)^2}\prod_{k=2}^{j} |b(1+m)+(k-2)m|^2.$$

Thus, we get

$$|a_j| \le \frac{1}{j^n(j-1)!}\prod_{k=2}^{j} |b(1+m)+(k-2)m|,$$

which completes the proof of (3.1).

Next, we suppose $j > N+2$. Then (3.6) gives

$$j^{2n}(j-1)^2|a_j|^2 \le (1+m)^2|b|^2$$

$$+\sum_{k=2}^{N+2} k^{2n} \quad \{|b(1+m)+m(k-1)|^2-(k-1)^2\}\ |a_k|^2$$

$$+\sum_{k=N+3}^{j-1} k^{2n} \quad \{|b(1+m)+m(k-1)|^2-(k-1)^2\}\ |a_k|^2$$

$$\le (1=m)^2|b|^2+\sum_{k=2}^{N+2} k^{2n} \quad \{|b(1+m)+m(k-1)|^2-(k-1)^2\}\ |a_k|^2.$$

On substituting upper estimates for $a_2, a_3, \ldots, a_{N+2}$ obtained above, and simplifying, we obtain (3.2).

(b) Let $2m(k-1)\ Re\ \{b\} \le (k-1)^2(1-m)-(1+m)|b|^2$, then it follows from (2.7)

$$j^{2n}(j-1)^2|a_j|^2 \le (1+m)^2|b|^2, (j \ge 2)$$

which proves (3.3)

The bounds in (3.1) are sharp for the function $f(z)$ given by

$$D^n f(z) = \begin{cases} \dfrac{z}{(1-mz)^{\frac{b(1+m)}{m}}}, & m \neq 0, \\ z \exp(bz), & m = 0. \end{cases} \quad \ldots (3.7)$$

The bounds in (3.3) are sharp for the function $f_k(z)$ given by

$$D^n f_k(z) = \begin{cases} \dfrac{z}{(1-mz^{k-1})^{\frac{b(1+m)}{m(k-1)}}}, & m \neq 0, \\ z \exp\left(\dfrac{b}{k-1} z^{k-1}\right), & m = 0. \end{cases} \quad \ldots (3.8)$$

4. Maximization of $|a_3 - \mu a_2^2|$

We shall need in our discussion the following lemma :

Lemma 1[5] — Let $w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$, if μ is any complex number, then

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}. \quad \ldots (4.1)$$

for any complex μ. Equality in (4.1) may be attained with the functions $w(z) = z^2$ and $w(z) = z$ for $|\mu| < 1$ and $|\mu| \geq 1$, respectively.

Theorem 3 — *If a function $f(z)$ defined by (1.1) is in the class $H_n(b, M)$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} \max\{1, |d|\}, \quad \ldots (4.2)$$

where
$$d = \frac{2.3^n \mu b(1+m)}{2^{2n}} - [b(1+m) + m]. \quad \ldots (4.3)$$

The result is sharp.

Proof : Since $f(z) \in H_n(b, m)$, we have

$$w(z) = \frac{D^{n+1} f(z) - D^n f(z)}{[b(1+m) - m] D^n f(z) + m D^{n+1} f(z)}$$

$$= \frac{\sum_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{b(1+m) + \sum_{k=2}^{\infty} k^n [b(1+m) + m(k-1)] a_k z^{k-1}}$$

$$= \frac{\sum_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{b(1+m)}$$

$$\times \left[1 + \frac{\sum_{k=2}^{\infty} k^n [b(1+m) + m(k-1)] a_k z^{k-1}}{b(1+m)} \right]^{-1} \qquad \dots (4.4)$$

Now compare the coefficients of z and z^2 on both sides of (4.4). We thus obtain

$$a_2 = \frac{b(1+m)}{2^n} c_1, \qquad \dots (4.5)$$

and

$$a_3 = \frac{b(1+m)}{2 \cdot 3^n} \{c_2 + [b(1+m) + m] c_1^2\}. \qquad \dots (4.6)$$

Hence,

$$a_3 - \mu a_2^2 = \frac{b(1+m)}{2 \cdot 3^n} [c_2 - d c_1^2], \qquad \dots (4.7)$$

where

$$d = \frac{2 \cdot 3^n \mu b(1+m)}{2^{2n}} - [b(1+m) + m].$$

Taking modulus both sides in (4.7), we have

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} |c_2 - d c_1^2|. \qquad \dots (4.8)$$

Using Lemma 1 in (4.8), we have

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m|}{2 \cdot 3^n} \max \{1, |d|\}.$$

Finally, the assertion (4.2) of Theorem 3 is sharp in view of the fact that the assertion (4.1) of Lemma 1 is sharp.

5. Radius Theorem

The following theorem may be obtained with the help of (1.9) and Theorem 3 of Nasr and Aouf[7].

Theorem* 4 — *Let the function $f(z)$ defined by (1.1) be in the class $H_n(b, M)$. Then

$$Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > 0 \text{ for } |z| < r_n,$$

where

$$r_n 2\left\{|b|(1+m) + \left[|b|^2(1+m)^2 - 4\left\{Re(b)\left(\frac{1+m}{m}\right) - 1\right\}\right]^{\frac{1}{2}}\right\}^{-1}. \quad \dots (5.1)$$

such that

$$|b|^2(1+m)^2 \geq 4\left\{Re(b)\left(\frac{1+m}{m}\right) - 1\right\}.$$

The result is sharp for the function $f_0(t)$, where

$$D^n f_0(z) = z(1 - m(z)^{-b\left(\frac{1+m}{m}\right)} \quad \dots (5.2)$$

and

$$t = \frac{r\left[r - m\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}\right]}{m\left[1 - m r\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}\right]}.$$

Remarks on Theorem 4 :

(i) Putting $n = 0$, we get the sharp radius of starlikeness of the class $F(b, M)$ studied by Nasr and Aouf[7]

(ii) Putting $n = 1$, we get the sharp radius of convexity of the class $G(b, M)$ which is investigated by Nasr and Aouf[8].

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