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PERTURBATION THEORY FOR NONLINEAR SYSTEMS WITH LARGE DAMPING

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A second order nonlinear differential system modeling damped oscillatory process is considered. The new perturbation method based on the work of Krylov, Bogoliubov and Mitropolskii is developed to find approximate solutions of damped nonlinear systems (including some almost non-oscillatory nonlinear systems). The solution shows a good coincidence with the numerical solution.

Key Words : Damped Oscillation - Large Damping - Critical Damping

1. INTRODUCTION

Krylov and Bogoliubov[1] have used a perturbation method to discuss transients in the equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}) \qquad \text{... (1)}$$

where the overdots denote differentiation with respect to t and ε is a small parameter. This method has been amplified and justified by Bogoliubov and Mitropolskii[2] and later extended by Popov[3] to the following damped oscillatory system

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}), \qquad \text{... (2)}$$

where $k > 0$. Mendelson[4] has rediscovered the Popov's results. Murty, Deekshatulu and Krisna[5] have used Krylov-Bogoliubov-Mitropolskii (KBM) method to discuss transients in eq. (2) for the over-damped case, $k > \omega$. Murty[6] has presented a unified KBM method for solving eq. (2). Sattar[7] has found a solution of (2) characterized by critical damping, *i.e.*, $k = \omega$. Recently, the author[8] has extended the unified method of Murty[6] to critically damped nonlinear systems.

The aim of the present paper is to obtain a solution of (2) when its linear equation has two complex roots, $-k \pm i\,\omega_0$ where $\omega_0 = \sqrt{\omega^2 - k^2}$ and $\omega_0 \le k < \omega$. It is noted that Popov's[3] or Mendelson's[4] solution gives desired results when $k < \omega_0$ and as the limit $k \to 0$, it turns to the original solution obtained by KBM.

2. METHOD

When $\varepsilon = 0$, the solution of (2) is

$$x(t, 0) = e^{-kt}(a_0 \cos \omega_0 t + b_0 \sin \omega_0 t), \qquad \text{... (3)}$$

where a_0 and b_0 are arbitrary constants.

Now we seek a solution of (2) that reduces to (3) as the limit $\varepsilon \to 0$. We look for an asymptotic solution of (2) in the form

$$(t, \varepsilon) = e^{-kt} (a \cos \omega_0 t + b \sin \omega_0 t) + \varepsilon u_1 (a, b, t) + \varepsilon^2 \ldots, \qquad \ldots (4)$$

where a and b are functions of t, defined by the first order differential equations

$$\dot{a} = \varepsilon A_1 (a, b, t) + \varepsilon^2 \ldots$$

and

$$\dot{b} = \varepsilon B_1 (a, b, t) + \varepsilon^2 \ldots . \qquad \ldots (5)$$

Now differentiating (4) twice with respective to t, substituting for the derivatives $\dot{x}, \ddot{x}$ and x in (2), utilizing relations (5) and comparing the coefficients of various powers of ε, we get for the coefficient of ε :

$$e^{-kt} \left(\left(\frac{\partial A_1}{\partial t} + 2 \omega_0 B_1 \right) \cos \omega_0 t + \left(-2 \omega_0 A_1 + \frac{\partial B_1}{\partial t} \right) \sin \omega_0 t \right)$$

$$+ \frac{\partial^2 u_1}{\partial t^2} + 2k \frac{\partial u_1}{\partial t} + \omega^2 u_1 = -f^{(0)} (a, b, t), \qquad \ldots (6)$$

where $f^{(0)} = f(x_0, \dot{x}_0)$ and $x_0 = e^{-kt} (a \cos \omega_0 t + b \sin \omega_0 t)$.

Usually, eq. (6) is solved for the unknown functions A_1, B_1 and u_1 under the assumption that u_1 does not contain first harmonic terms. We shall follow this assumption (early imposed by KBM[1&12]) partially to obtain approximate solutions of nonlinear systems with large damping. We assume that u_1 does not contain first harmonic terms, which are contained in $(a_0 \cos \omega_0 t)^r$ and $(a_0 \cos \omega_0 t)^r b_0 \sin \omega_0 t, r > 1$ of $f^{(0)}$.

3. EXAMPLE

As an example of the above procedure we may consider the *Duffing*'s equation with a large linear damping

$$\ddot{x} + 2k \dot{x} + \omega^2 x = -\varepsilon x^3. \qquad \ldots (7)$$

Here

$$f^{(0)} = e^{-3kt} (a^3 \cos^3 \omega_0 t + 3a^2 b \cos^2 \omega_0 t \sin \omega_0 t + 3ab^2 \cos \omega_0 t \sin^2 \omega_0 t + b^3 \sin^3 \omega_0 t)$$

$$= e^{-3kt} \left(\frac{3}{4} a^3 \cos \omega_0 t + \frac{3}{4} a^2 b \sin \omega_0 t + \frac{1}{4} a^3 \cos 3 \omega_0 t \right.$$

$$+ \frac{3}{4} a^2 b \sin 3 \omega_0 t + 3ab^2 \cos \omega_0 t \sin^2 \omega_0 t + b^3 \sin^3 \omega_0 t \Bigg). \quad \text{... (8)}$$

Substituting $f^{(0)}$ from (8) into (6), we obtain following equations for A_1, B_1 and u_1 according to our new assumption

$$\frac{\partial A_1}{\partial t} + 2 \omega_0 B_1 = -\frac{3a^3 e^{-2kt}}{4}, \quad \text{... (9)}$$

$$-2 \omega_0 A_1 + \frac{\partial B_1}{\partial t} = -\frac{3a^2 be^{-2kt}}{4} \quad \text{... (10)}$$

and

$$\frac{\partial^2 u_1}{\partial t^2} + 2k \frac{\partial u_1}{\partial t} + \omega^2 u_1 = -e^{-3kt} \Bigg(\frac{1}{4} a^3 \cos 3 \omega_0 t + \frac{3}{4} a^2 b \sin 3 \omega_0 t$$

$$+ 3ab^2 \cos \omega_0 t \sin^2 \omega_0 t + b^3 \sin^3 \omega_0 t). \quad \text{... (11)}$$

The particular solutions of the above three differential equations give the three unknown functions A_1, B_1 and u_1. Solving eqs. (9)-(11) we obtain

$$A_1 = \frac{3a^2 (ka + \omega_0 b) e^{-2kt}}{8 (k^2 + \omega_0^2)},$$

$$B_1 = \frac{3a^2 (-\omega_0 a + kb) e^{-2kt}}{8 (k^2 + \omega_0^2)}, \quad \text{... (12)}$$

and

$$u_1 = -\frac{e^{-3kt}}{16 (k^2 + \omega_0^2)} \Bigg(\frac{3ab^2 (k \cos \omega_0 t - \omega_0 \sin \omega_0 t)}{k} + \frac{b^3 (\omega_0 \cos \omega_0 t + k \sin \omega_0 t)}{k}$$

$$+ (a^3 - 3ab^2) \left(\frac{(k^2 - 2 \omega_0^2) \cos 3 \omega_0 t - 3k \omega_0 \sin 3 \omega_0 t}{k^2 + 4 \omega_0^2} \right)$$

$$+ (3a^2 b - b^3) \left(\frac{3k \omega_0 \cos 3 \omega_0 t + (k^2 - 2 \omega_0^2) \sin 3 \omega_0 t}{k^2 + 4 \omega_0^2} \right) \Bigg). \quad \text{... (13)}$$

Substituting the values of A_1 and B_1 form (12) into (5), we integrate them with respect to t, by assuming that a and b are constants in the right hand sides of (5) and then obtain the following:

$$a = a_0 + \frac{3 \varepsilon a_0^2 (ka_0 + \omega_0 b_0) (1 - e^{-2kt})}{16k (k^2 + \omega_0^2)}$$

$$b = b_0 + \frac{3\,\varepsilon\, a_0^2 (-\,\omega_0\, a_0 + k b_0)\,(1 - e^{-2kt})}{16k\,(k^2 + \omega_0^2)}. \quad \ldots (14)$$

Therefore, the first approximate solution of (7) is

$$x = e^{-kt}\,(a \cos \omega_0\, t + b \sin \omega_0\, t) + \varepsilon\, u_1 \quad \ldots (15)$$

where a, b and u_1 are given by respectively (14) and (13). The method can be carried out to higher orders in the same way.

4. Popov's Solution

Solution (4) can be used as a Popov's[3] solution. According to Popov's method, u_1 does not contain first harmonic terms. Here $f^{(0)}$ of (6) can be expanded in a Fourier series as

$$f^{(0)} = \frac{e^{-3kt}}{4}\,(3a\,(a^2 + b^2) \cos \omega_0\, t + 3b\,(a^2 + b^2) \sin \omega_0\, t$$

$$+\ (a^3 - 3ab^2) \cos 3\,\omega_0\, t + (3a^2\, b - b^3) \sin 3\,\omega_0\, t). \quad \ldots (16)$$

Now substituting $f^{(0)}$ from (16) into (6), we obtain the equations for A_1, B_1 and u_1 according to Popov's assumption as

$$\frac{\partial A_1}{\partial t} + 2\,\omega_0\, B_1 = -\,\frac{3a\,(a^2 + b^2)\, e^{-2kt}}{4}, \quad \ldots (17)$$

$$-\,2\,\omega_0\, A_1 + \frac{\partial B_1}{\partial t} = -\,\frac{3b\,(a^2 + b^2)\, e^{2kt}}{4}, \quad \ldots (18)$$

and

$$\frac{\partial^2 u_1}{\partial t^2} + 2k\,\frac{\partial u_1}{\partial t} + \omega^2\, u_1 = -\,\frac{e^{-3kt}}{4}$$

$$\left((a^3 - 3ab^2) \cos 3\,\omega_0\, t + (3a^2\, b - b^3) \sin 3\,\omega_0\, t \right) \quad \ldots (19)$$

Solving eqs. (17)-(19) we obtain

$$A_1 = \frac{3(a^2 + b^2)\,(ka + \omega_0\, b)\, e^{-2kt}}{8\,(k^2 + \omega_0^2)},$$

$$B_1 = \frac{3(a^2 + b^2)\,(-\,\omega_0\, a + kb)\, e^{-2kt}}{8\,(k^2 + \omega_0^2)} \quad \ldots (20)$$

and
$$u_1 = -\frac{e^{-3kt}}{16(k^2+\omega_0^2)(k^2+4\omega_0^2)}((a^3-3ab^2)((k^2-2\omega_0^2)\cos 3\omega_0 t - 3k\omega_0 \sin 3\omega_0 t)$$

$$+ (3a^2 b - b^3)(3k\omega_0 \cos 3\omega_0 t + (k^2-2\omega_0^2)\sin 3\omega_0 t)). \qquad \text{... (21)}$$

Substituting the values of A_1 and B_1 form (20) into (5), we obtain

$$\left.\begin{aligned} \dot{a} &= \frac{3\varepsilon(a^2+b^2)(ka+\omega_0 b)e^{-2kt}}{8(k^2+\omega_0^2)} \\ \text{and} \quad \dot{b} &= \frac{3\varepsilon(a^2+b^2)(-\omega_0 a+kb)e^{-2kt}}{8(k^2+\omega_0^2)} \end{aligned}\right\} \qquad \text{... (22)}$$

Under transformations $a = \alpha\cos\varphi,\ b = -\alpha\sin\varphi$, (21) and (22) become

$$u_1 = -\frac{\alpha^3 e^{-3kt}}{16(k^2+\omega_0^2)(k^2+4\omega_0^2)}((k^2-2\omega_0^2)\cos 3(\omega_0 t+\varphi)$$

$$- 3k\omega_0 \sin 3(\omega_0 t+\varphi)), \qquad \text{... (21a)}$$

$$\left.\begin{aligned} \dot{\alpha} &= \frac{3\varepsilon k\alpha^3 e^{-2kt}}{8(k^2+\omega_0^2)} \\ \text{and} \quad \dot{\varphi} &= \frac{3\varepsilon\omega_0\alpha^2 e^{-2kt}}{8(k^2+\omega_0^2)} \end{aligned}\right\} \qquad \text{... (22a)}$$

$$\left.\begin{aligned} &\frac{\alpha_0}{\sqrt{1+\dfrac{3\varepsilon\alpha_0^2(e^{-2kt}-1)}{8(k^2+\omega_0^2)}}}, \\ \text{or} \quad &\varphi = \varphi_0 - \frac{\omega_0}{2k}\ln\left(1+\frac{3\varepsilon\alpha_0^2(e^{-2kt}-1)}{8(k^2+\omega_0^2)}\right) \end{aligned}\right\} \qquad \text{... (23)}$$

In this case, the first approximate solution of (7) can be transformed to

$$x - \alpha e^{-kt}\cos(\omega_0 t+\varphi) + \varepsilon u_1, \qquad \text{... (24)}$$

which is identical to Popov's[3] or Mendelson's[4] solution where α, φ and u_1 are given respectively by (23) and (21a).

5. Critically-Damped Solution

It is obvious that, when ω_0 is sufficiently small, *i.e.*, $\omega_0 \to 0$, one can replace $a_0 \cos \omega_0 t$ and $b_0 \sin \omega_0 t$ in linear solution (3) respectively by α_0 and $\beta_0 t$. So, when ω_0 is small, we may use transformations

$$\left.\begin{aligned} a \cos \omega_0 t &= \alpha(t) \\ b \sin \omega_0 t &= t\,\beta(t), \end{aligned}\right\} \qquad \dots (25)$$

where α and β satisfy the following differential equations

$$\left.\begin{aligned} \dot{\alpha} &= \varepsilon \bar{A}_1(\alpha, \beta, t) + \varepsilon^2 \dots \\ \dot{\beta} &= \varepsilon \bar{B}_1(\alpha, \beta, t) + \varepsilon^2 \dots. \end{aligned}\right\} \qquad \dots (26)$$

Differentiating (25) twice with respect to t and utilizing relations (5) and (26), we obtain

$$\left.\begin{aligned} \varepsilon\left(\frac{\partial A_1}{\partial t}\cos \omega_0 t - 2\,\omega_0 A_1 \sin \omega_0 t\right) + \varepsilon^2 \dots &= \omega_0^2 \alpha + \varepsilon \frac{\partial \bar{A}_1}{\partial t} + \varepsilon^2 \dots \\ \varepsilon\left(2\,\omega_0 B_1 \cos \omega_0 t + \frac{\partial B_1}{\partial t}\sin \omega_0 t\right) + \varepsilon^2 \dots &= \omega_0^2 t\,\beta + \varepsilon\left(2\bar{B}_1 + t\frac{\partial \bar{B}_1}{\partial t}\right) + \varepsilon^2 \dots \end{aligned}\right\} \qquad \dots (27)$$

When $\omega_0^2 = O(\varepsilon)$, we may equate the coefficients of ε on both sides of (27) as

$$\frac{\partial A_1}{\partial t}\cos \omega_0 t - 2\,\omega_0 A_1 \sin \omega_0 t = \varepsilon^{-1}\,\omega_0^2\,\alpha + \frac{\partial \bar{A}_1}{\partial t}$$

$$2\,\omega_0 B_1 \cos \omega_0 t + \frac{\partial B_1}{\partial t}\sin \omega_0 t = \varepsilon^{-1}\,\omega_0^2\,t\,\beta + 2B_1 + t\frac{\partial \bar{B}_1}{\partial t}. \qquad \dots (28)$$

Eliminating A_1 and B_1 form (6) by (28), we obtain

$$e^{-kt}\left(\frac{\partial \bar{A}_1}{\partial t} + 2\bar{B}_1 + t\frac{\partial \bar{B}_1}{\partial t}\right) + \frac{\partial^2 u_1}{\partial t^2} + 2k\frac{\partial u_1}{\partial t} + \omega^2 u_1 = -\varepsilon^{-1}\,\omega_0^2(\alpha + t\beta) - f^{(0)}. \qquad \dots (29)$$

The above relation is valied even $\omega_0 \to 0$. In the limiting case, (29) becomes

$$e^{-kt}\left(\frac{\partial \bar{A}_1}{\partial t} + 2\bar{B}_1 + t\frac{\partial \bar{B}_1}{\partial t}\right) + \frac{\partial^2 u_1}{\partial t^2} + 2k\frac{\partial u_1}{\partial t} + k^2 u_1 = -\bar{f}^{(0)}(\alpha, \beta, t), \qquad \dots (30)$$

where $\bar{f}^{(0)} = \lim_{\omega_0 \to 0} f^{(0)}$. Eq. (30) is similar to that obtained in [8] for the critical damping case, which has been found in [8] as a limiting case of an over-damped nonlinear system. In [8], it has been considered that $\bar{f}^{(0)}$ can be expanded in Maclaurin's series in t as :

$$\bar{f}^{(0)} = g_0(\alpha, \beta, t) + g_1(\alpha, \beta, t)\, t + g_2(\alpha, \beta, t)\, t^2 \dots \qquad \dots (31)$$

It is noted that $g_0, g_1, \dots$ contain only e^{-t} type terms.

Substituting (31) into (30) and equating the coefficients of t^0, t^1 and $t^r, r \geq 2$, it has been obtained that

$$e^{-kt}\left(\frac{\partial \bar{A}_1}{\partial t} + 2\bar{B}_1\right) = -g_0, \qquad \dots (32)$$

$$e^{-kt}\frac{\partial \bar{B}_1}{\partial t} = -g_1 \qquad \dots (33)$$

and

$$\frac{\partial^2 u_1}{\partial t^2} + 2k\frac{\partial u_1}{\partial t} + k^2 u_1 = -g_2 t^2 - \dots . \qquad \dots (34)$$

For (7), $\bar{f}^{(0)} = e^{-3kt}(\alpha^3 + 3\alpha^2\beta t + 3\alpha\beta^2 t^2 + \beta^3 t^3)$ and the non-vanishing terms of g are $g_0 = e^{-3kt}\alpha^3$, $g_1 = 3e^{-3kt}\alpha^2\beta$, $g_2 = 3e^{-3kt}\alpha\beta^2$ and $g_3 = e^{-3kt}\beta^3$.

Substituting the values of g_0 and g_1 into (32) and (33), and the values of g_2 and g_3 into (34) and then solving them, the following results have been obtained :

$$\bar{A}_1 = \frac{e^{-2kt}(k\alpha^3 + 3\alpha^2\beta)}{2k^2}, \qquad \dots (35)$$

$$\bar{B}_1 = \frac{3e^{-2kt}\alpha^2\beta}{2k},$$

and

$$u_1 = -\frac{e^{-3kt}}{4k^2}\left(\frac{9\alpha\beta^2}{2k^2} + \frac{3\beta^3}{k^3} + \left(\frac{6\alpha\beta^2}{k} + \frac{9\beta^3}{2k^2}\right)t + 3\left(\alpha\beta^2 + \frac{\beta^3}{k}\right)t^2 + \beta^3 t^3\right). \qquad \dots (36)$$

Substituting the values of $\bar{A}_1$ and $\bar{B}_1$ form (35) into (26), these can be integrated with respect to t, by assuming that α and β are constants in the right hand sides. The results are as follows :

and

$$\left.\begin{aligned} \alpha &= \alpha_0 + \frac{\varepsilon(k\alpha_0^3 + 3\alpha_0^2\beta_0)(1 - e^{-2kt})}{4k^3} \\ \beta &= \beta_0 + \frac{3\varepsilon\alpha_0^2\beta_0(1 - e^{-2kt})}{4k^2}. \end{aligned}\right\} \qquad \dots (37)$$

Thus the critically damped solution of (7) is

$$x = e^{-kt}(\alpha + \beta t) + \varepsilon u_1, \qquad \dots (38)$$

where α, β and u_1 are given by respectively (37) and (36). This solution shows a good agreement with the numerical solution[8].

6. Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this paper, we refer to works of Murty and Deekhatulu[9] and Mendelson[4]. In this paper, for different damping forces, $-2k\dot{x}$, i.e., for different values of k, analytical solutions (15) and (24) have been compared to those obtained by *Runge-Kutta* fourth-order procedure.

First of all, for a significant damping force $-2\dot{x}/\sqrt{3}$, i.e., for $k=1/\sqrt{3}$, $x(t)$ has been computed by author's solution (15) with initial conditions $[x(0) = 1,\ \dot{x}(0)=0]$. Then $x(t)$ has been computed by Popov's[3] solution (24). Finally, the numerical solution has been obtained and percentage error has been calculated. All the results are shown in Table I. From Table I it is seen that most of the time, t, errors of the results obtained by author's solution and Popov's solution are less than 1% and on an average percentage errors of author's solution are less than those computed by Popov's solution. Near a turning point, $t \approx 2.5$, percentage errors of (15) are more than 1% (it is noted that for $\varepsilon = 0.1$, percentage errors should be less than or equal to 1% [8]) but much smaller than those obtained by Popov's solution. At $t = 2.5$, percentage errors of (15) and (24) are respectively 1.1933 and 2.0901 (from Table I).

If the damping force is increased, author's solution (15) shows a good coincidence with the numerical solution. Contrary, errors of Popov's solution (24) increase. For verifying this, $x(t)$ has again been computed by (15), (24) and *Runge-Kutta* method with same initial conditions for a large damping force $-2\dot{x}/\sqrt{2}$ or $k=1/\sqrt{2}$ and are given (with percentage errors) in Table II. This table shows that percentage errors for (15) are less than 1%, while errors of Popov's solution are sometimes more than 5%.

For significant damping forces Popov's solution is a good analytical solution. But for a large damping force the new solution is more useful than Popov's[3] or Mendelson's[4] solution.

For various damping forces solutions (15), (24) and numerical solution of (7) with initial conditions $[x(0)=1, \dot{x}(0)=0]$ are given in the Tables I and II. In both the tables $x^{(1)}$ is computed by (15), $x^{(2)}$ is computed by *Runge-Kutta* fourth-order procedure and $x^{(3)}$ is computed by Popov's[3] solution (24). Percentage errors, $E^{(1)}$ are calculated for (15) and $E^{(3)}$ are calculated for (24).

Table I : When $k=\sqrt{\frac{1}{3}},\ \omega_0=\sqrt{\frac{2}{3}},\ \varepsilon=0.1$

t	$x^{(1)}$	$x^{(2)}$	$E^{(1)}$ (%)	$x^{(3)}$	$E^{(3)}$ (%)
0.0	1.000000	1.000000	0.0000	1.000000	0.0000
0.5	0.888461	0.888457	0.0005	0.888814	0.0402
1.0	0.648030	0.648069	− 0.0060	0.648672	0.0930
1.5	0.387460	0.387423	0.0096	0.388002	0.1494
2.0	0.167925	0.167802	0.0733	0.168241	0.2616
2.5	0.013314	0.013157	1.1933	0.013432	2.0901
3.0	− 0.075965	− 0.076109	− 0.1892	− 0.075977	− 0.1734
3.5	− 0.111949	− 0.112054	− 0.0937	− 0.112026	− 0.0250
4.0	− 0.111303	− 0.111363	− 0.0539	− 0.111399	0.0323
4.5	− 0.090212	−0.090234	− 0.0244	− 0.090300	0.0731
5.0	− 0.061578	− 0.061574	0.0065	− 0.061644	0.1137
5.5	− 0.034044	− 0.034026	0.0529	− 0.034085	0.1734
6.0	− 0.012253	− 0.012230	0.1880	− 0.012272	0.3434

TABLE II

When $k = \omega_0 = \sqrt{\frac{1}{2}}$, $\varepsilon = 0.1$

t	$x^{(1)}$	$x^{(2)}$	$E^{(1)}$ (%)	$x^{(3)}$	$E^{(3)}$ (%)
0.0	1.000000	1.000000	0.0000	1.000000	0.0000
0.5	0.892155	0.892678	− 0.0586	0.893243	0.0633
1.0	0.670497	0.671121	− 0.0930	0.672417	0.1931
1.5	0.438323	0.438773	− 0.1026	0.440211	0.3277
2.0	0.245383	0.245614	− 0.0941	0.246851	0.5036
2.5	0.106884	0.106951	− 0.0626	0.107866	0.8555
2.75	0.057495	0.057506	− 0.0191	0.058256	1.3042
3.0	0.019802	0.019773	0.1467	0.020368	3.0092
3.25	− 0.007703	− 0.007759	− 0.7217	− 0.007304	− 5.8642
3.5	− 0.026614	− 0.026684	− 0.2623	− 0.026352	− 1.2442
4.0	− 0.044827	− 0.044906	− 0.1759	− 0.044761	− 0.3229
4.5	− 0.045937	− 0.046005	− 0.1478	− 0.045978	− 0.0587
5.0	− 0.038418	− 0.038469	− 0.1326	− 0.038505	0.0936
5.5	− 0.027966	− 0.027999	− 0.1179	− 0.028060	0.2179
6.0	− 0.017902	− 0.017921	− 0.1060	− 0.017983	0.3460

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