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ON THE LOCAL BOUNDEDNESS OF THE SOLUTIONS OF CERTAIN NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN CARNOT-CARATHÉODORY SPACES

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We prove the local boundedness of the solutions of some nonlinear partial differential equations in Carnot-Carathéodory spaces; such equations satisfy Serrin-type structure conditions.

Key Words : Carnot-Carathéodory Spaces; Nonlinear Partial Differential Equations; Structure Conditions

1. INTRODUCTION

In recent years, there has been a lot of interest in the study of nonlinear partial differential equations; one of the first problems one can meet in such studies is that one of the regularity of the weak solutions of the differential equations we are talking about.

A road to reach this intent is to prove first of all a L^∞-local bound for the solutions. In this work we obtain an a priori local bound in the L^∞-norm of the weak solutions of certain non-linear partial differential equations satisfying Serrin-type structure conditions in the context of the Carnot-Carathéodory spaces.

We begin with some notations and definitions.

Let $X = \{X_1, \dots, X_m\}$ be a system of vector fields of the type

$$X_j = \sum_{k=1}^{n} b_{jk} \frac{\partial}{\partial x_k} \quad j = 1, \dots, m,$$

where the coefficients b_{jk} are real-valued Lipschitz-continuous functions on an open subset Ω of $\boldsymbol{R}^n$, then for a function $f \in L^1_{loc}(\Omega)$ we can define the distributional derivative along the vector fields X_j with the following identity

$$\langle X_j f, \varphi \rangle \overset{def}{=} \int_\Omega f X_j^* \varphi \, dx \text{ for every } \varphi \in C_0^\infty(\Omega)$$

where $$X_j^* = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(b_{jk}^*) \text{ is the formal adjoint of } X_j.$$

Let now $\gamma: [0, 1] \to \boldsymbol{R}^n$ be a C^1-piecewise curve, then it is called *horizontal* if

$$\gamma'(t) = \sum_{j=1}^{m} a_j(t) X_j(\gamma(t))$$

whenever $\gamma'(t)$ exists, while the *horizontal length* of γ is defined by

$$l_h(\gamma) \overset{def}{=} \int_0^1 \left(\sum_{j=1}^{m} a_j(t)^2 \right)^{1/2} dt.$$

With this tools we can now define a new metric in $\boldsymbol{R}^n$ related to the vector field X_j : let x and y be two generic points in $\boldsymbol{R}^n$ then we set

$$d(x, y) \overset{def}{=} \inf l_h(\gamma),$$

where the infimum is taken over all horizontal curves γ which satisfy $\gamma(0) = x$ and $\gamma(1) = y$.

From now on we will assume that for each $x, y \in \boldsymbol{R}^n$ one has $d(x, y) < \infty$, so that $(\boldsymbol{R}^n, d)$ becomes a metric space, called the *Carnot-Carathéodory space* generated by the system X. In $(\boldsymbol{R}^n, d)$ the metric balls will be denoted by

$$B(x_0, R) \overset{def}{=} \{x \in \boldsymbol{R}^n \mid d(x, x_0) < R\}$$

ad if $B := B(x, R)$ we will usually write αB to denote $B(x, \alpha R)$.

We are now ready to state our general assumptions :

Hypothesis : For any set $U \subset \boldsymbol{R}^n$ with *diam* $(U) < \infty$ there exist some constants $c_1, c_2 > 0$, $R_0 \in]0, \infty]$ and $\alpha \geq 1$ such that for $x_0 \in U$ and $R \in]0, R_0[$ one has :

(H.1) $|B(x_0, 2R)| \leq c_1 |B(x_0, R)|$; and

(H.2) for every Lipschitz function u in $B(x_0, \alpha R)$ we have for any $\lambda > 0$,

$$|\{x \in B \,||\, u(x) - u_B| > \lambda\}| \leq \frac{c_2}{\lambda} R \int_{\partial B} |Xu(y)|\, dy.$$

We now introduce the geometrical and metrical properties of the open set Ω we need to state our result.

Definition 1.1 — An open set $\boldsymbol{\Omega}$ is called a $X - (PS)$ domain if there exist a covering $\{B\}_{B \in \mathcal{F}}$ of $\boldsymbol{\Omega}$ by metric balls, and numbers $N > 0$, $\alpha \geq 1$, $\nu \geq 1$ such that (*i*) $\sum_{B \in \mathcal{F}} \chi_{(\alpha+1)B}(x) \leq N_{\chi\Omega}(x)$ for every $x \in U$;

(*ii*) there exists a (central ball $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ one can find a chain $B_0, B_1, \ldots, B_{s(B)} = B$, with $B_i \cap B_{i+1} \neq 0$ and $|B_i \cap B_{i+1}| \geq N^{-1} \max(|B_i|, B_{i+1}|)$; and

(*iii*) for any $i = 0, \ldots, s(B)$, one has

$$B \subseteq \nu B_i$$

Let now $\varphi : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be a C^1 function and φ_0 be a positive number such that

$$0 \leq \frac{t\,\varphi'(t)}{\varphi(t)} \leq \varphi_0 \quad \forall t > 0 \qquad \ldots (1.1)$$

then we can define

$$\Phi(t) \overset{def}{=} \int_0^t \varphi(s)\,ds. \qquad \ldots (1.2)$$

It is easy to see that from (1.1) and (1.2) one has the following inequalities :

$$\frac{t\,\varphi(t)}{1+\varphi_0} \leq \Phi(t) \leq t\,\varphi(t) \quad \forall t \geq 0$$

$$\varphi(t) \leq \varphi(C(t) \leq C^{\varphi_0}\,\varphi(t) \quad \forall C > 1 \qquad \ldots (1.3)$$

$$\Phi(t) \leq \Phi(Ct) \leq C^{1+\varphi_0}(1+\varphi_0)\,\Phi(t) \quad \forall C > 1$$

Finally we will use in the sequel the simple inequality

$$a\,\varphi(b) \leq a\,\varphi(a) + b\,\varphi(b), \quad \forall a,\ b > 0 \qquad \ldots (1.4)$$

which is true thanks to the monotonicity of φ.

Let now $\Omega \subseteq \boldsymbol{R}^n$ be an X – *PS*-domain, then the nonlinear partial differential equations we will study are of the type :

$$\sum_{j=1}^{m} X_j^* A_j(x, u, Xu) + B(x, u, Xu) = 0 \qquad \ldots (1.5)$$

where the coefficients A_j ($j = 1, \ldots, m$) and B satisfy the following Serrin-type structure conditions:

$$|A(x, t, w)| \leq a\,\varphi(|w|) + b\,\varphi(|t|), \qquad \ldots (1.6)$$

$$|B(x, t, w)| \leq c\,\varphi(|w|) + d\,\varphi(|t|). \qquad \ldots (1.7)$$

$$\langle w, A(x, t, w) \rangle \geq \varphi(|w|)\,|w| - d\,\varphi(|t|)\,|t|, \qquad \ldots (1.8)$$

where a, b, c and d are non negative constants.

It is important here to observe that all the examples given in Lieberman[3] satisfy with an appropriate choice of B our structure conditions; in fact (1.6) and (1.8) are almost the same of Lieberman's relative conditions, while (1.7) involves only the function B; further we observe that our structure conditions are a natural generalization of Serrin's ones, see Serrin[5].

Finally in Lieberman[3] the system of vector fields is the classic euclidean one $X = \{\partial/\partial x_1, ..., \partial/\partial x_n\}$, while our hypotheses admit systems like the ones satisfying Hörmander's condition :

$$rankLie\ [X_1, ..., X_n]\,(x) = n \text{ at every } x \in \mathbf{R}^n;$$

or the generalized Baouendi-Grushin systems; $X_j = \partial/\partial x_j$ if $j = 1, ..., k$ and $X_j = (x_1^2 + \ldots + x_k^2)$ exp $(\alpha/2)\ \partial/\ \partial x_j$ for $j = k + 1, ..., n$ where $1 \leq k < n$ and $\alpha > 0$; or finally the systems of vector fields related to the Kohn Laplacian on the Heisemberg group.

2. Main Result

The following theorem is the main result of this paper :

Theorem 2.1 — *Every weak solution u of (1.5) is locally bounded, in fact for each $x_0 \in \Omega$ there exists an $R > 0$ such that*

$$\| u \|_{L^\infty(B(x_0, R))} \leq C \,\| \Phi(u) \|_{L^1(\Omega)}$$

where $$C = C(a, b, c, d, \varphi_0, X, \Omega).$$

Proof : In the proof of this result we will use an imbedding theorem of Sobolev-type in Carnot-Carathéodory spaces in one of the simpler cases, namely $S_0^{1,1}(\Omega) \to L^{Q/(Q-1)}(\Omega)$, which is contained for example in Garofalo-Nhieu[1], and we will use a modified version of Moser iteration method (see Moser[4]).

First of all we take a function $\eta \in C^{0,\infty}(\Omega)$ such that $supp(\eta) \subseteq B_t \subseteq \Omega$, for each $x \in \Omega$ one has $0 \leq \eta(x) \leq 1$ and for each $s, t \in]0, 1]$ with $s < t$ one has $\eta|_{B_s} = 1$ and $|X_\eta| \leq K/t - s$, where K is a constant depending only on the system X of vector fields. On the existence of such functions we refer to the work Garofalo-Nhieu[2].

Let now $\alpha > 1 + \varphi_0$, $l > 0$ and $q \geq 1$ be three fixed real number, then we can define the following functions :

$$M(\eta) \overset{def}{=} \eta^\alpha.$$

$$N(u) \overset{def}{=} \begin{cases} \Phi(|u|)^{q-1} u & \text{if } |u| \leq l \\ [(q-1)\,\Phi(l)^{q-2}\,\varphi(l)\,l + \Phi(l)^{q-1}]\,u = c_l\, u & \text{if } |u| > l. \end{cases}$$

$$\theta \overset{def}{=} M(\eta)\, N(u).$$

$$f \overset{def}{=} M(\eta)\, N(u)\, \varphi(|u|).$$

We can observe some facts :

a) $\theta \in S_0^{1,\Phi}(\Omega)$ thus it can be used as test function;

b) $f \in S_0^{1,1}(\Omega)$ so that we can apply the above embedding theorem of Sobolev-type, that is there exists a positive constant s (independent on u) such that $\|f\|_{Q/(Q-1)} \leq s \|Xf\|_1$;

c) we can relate $N'(u)$ with $N(u)/u$, in fact one has :

$$N'(u) = \begin{cases} (q-1)\,\Phi(|u|)^{q-2}\,\varphi(|u|)\dfrac{u}{|u|}u + \Phi(|u|)^{q-1} & \text{if } |u| \leq l \\ c_l & \text{if } |u| > l \end{cases}$$

so that $N'(u) \geq 0$ (while $N(u) \geq 0$ if and only if $u \geq 0$). Further if $|u| \leq 0$ we have :

$$q\,\Phi(|u|)^{q-1} \leq N'(u) \leq [(q-1)(1+\varphi_0)+1]\,\Phi(|u|)^{q-1}$$

that is
$$q\left|\frac{N(u)}{u}\right| \leq N'(u) \leq [q + \varphi_0(q-1)]\left|\frac{N(u)}{u}\right|$$

(obviously $|u| > l$ implies $N'(u) = N(u)/u = |N(u)/u|$).

Let now $k := Q/(Q-1)$ then, using the embedding theorem, one has :

$$\left[\int_\Omega |f|^k\right]^{1/k} \leq s \int_\Omega |Xf|$$

that is

$$\left[\int_\Omega |M(\eta)\,N(u)\,\varphi(|u|)|^k\right]^{1/k} \leq s \int_\Omega |X[M(\eta)\,N(u)\,\varphi(|u|)]|$$

$$\leq s\left[\int_\Omega |M'(\eta)\,N(u)\,\varphi(|u|)\,X\eta| + \left| M(\eta)\left(N'(u)\,\varphi(|u|) + N(u)\,\varphi'(|u|)\frac{u}{|u|}\right)Xu\right|\right]$$

$$\leq s\left[\int_\Omega M'(\eta)\,|X\eta|\,|N(u)|\,\varphi(|u|) + \int_\Omega M(\eta)\,N'(u)\,\varphi(|u|)\,|u|\right.$$

$$\left. + \int_\Omega M(\eta)\,N'(u)\,\varphi(|Xu|)\,|Xu| + \int_\Omega M(\eta)\left|\frac{N(u)}{u}\right|\frac{\varphi'(|u|)\,|u|}{\varphi(|u|)}\varphi(|u|)\,|Xu|\right]$$

$$\leq s\left[\int_\Omega \left(M'(\eta)\,|X\eta|\left|\frac{N(u)}{u}\right| + M(\eta)\,N'(u)\right)\varphi(+|u|)\,|u|\right.$$

$$+ \int_{\Omega} M(\eta)\, N'(u)\, \varphi(|Xu|)\,|Xu|$$

$$\left. + \varphi_0 \int_0 M(\eta) \left| \frac{N(u)}{u} \right| (\varphi(|u|)\,|u| + \varphi(|Xu|)\,|Xu|) \right].$$

We want now an upper bound for the terms containing $\varphi(|Xu|)\,|Xu|$, that is we want a Caccioppoli-type inequality.

To have this we put the test function θ in the weak form of (1.5) :

$$\int_{\Omega} \langle A(x, u, Xu), X[M(\eta)\, N(u)] \rangle = \int_{\Omega} B(x, u, Xu)\, M(\eta)\, N(u)$$

that is
$$\int_{\Omega} \langle A, X\eta \rangle M'(\eta)\, N(u) + \int_{\Omega} \langle A, Xu \rangle\, M(\eta)\, N'(u) = \int_{\Omega} BM(\eta)\, N(u)$$

and
$$\int_{\Omega} M(\eta)\, N'(u) \langle A, Xu \rangle = - \int_{\Omega} M'(\eta)\, N(u) \langle A, X\eta \rangle + \int_{\Omega} BM(\eta)\, N(u).$$

We now use the structure conditions (1.6), (1.7) and (1.8)

$$\int_{\Omega} M(\eta)\, N'(u)\, [\varphi(|Xu|)\,|Xu| - d\,\varphi(|u|)\,|u|] \leq \int_{\Omega} M(\eta)\, N'(u) \langle A, Xu \rangle$$

$$= - \int_{\Omega} M'(\eta)\, N(u) \langle A, X\eta \rangle + \int_{\Omega} BM(\eta)\, N(u)$$

$$\leq \int_{\Omega} M'(\eta)\,|N(u)|\,|A|\,|X\eta| + \int_{\Omega} |B|\, M(\eta)\,|N(u)|$$

$$\leq \int_{\Omega} M'(\eta)\,|N(u)|\,|X\eta|\,[a\,\varphi(|Xu|) + b\,\varphi(|u|)]$$

$$+ \int_{\Omega} M(\eta)\,|N(u)|\,[c\,\varphi(|Xu|) + d\,\varphi(|u|)]$$

and thus
$$\int_{\Omega} M(\eta)\, N'(u)\, \varphi(|Xu|)\,|Xu| \leq d \int_{\Omega} M(\eta)\, N'(u)\, \varphi(|u|)\,|u|$$

$$+ b \int_{\Omega} M'(\eta)\,|X\eta|\,|N(u)|\,\varphi(|u|) + d \int_{\Omega} M(\eta)\,|N(u)|\,\varphi(|u|)$$

$$+ a \int_{\Omega} M'(\eta)\,|X\eta|\,|N(u)|\,\varphi|(|Xu|) + c \int_{\Omega} M(\eta)\,|N(u)|\,\varphi(|Xu|)$$

We also have $|X\eta| \le K/\sigma$ (for $\sigma := t-s$), thus for $\varepsilon, \delta > 0$ one has :

$$\int_\Omega M(\eta)\, N'(u)\, \varphi(|Xu|)\, |Xu|$$

$$\le \int_\Omega \left[dM(\eta)\, N'(u) + bM'(\eta) \frac{K}{\sigma} \left| \frac{N(u)}{u} \right| + dM(\eta) \left| \frac{N(u)}{u} \right| \right] \varphi(|u|)\, |u|$$

$$+ a\,\varepsilon \int_\Omega M'(\eta)\, \eta\, K \left| \frac{N(u)}{u} + \frac{|u|}{\varepsilon\eta\sigma} \varphi(|Xu|) + c\,\delta \int_\Omega M(\eta) \right| \frac{N(u)}{u} \left| \frac{|u|}{\delta} \varphi(|Xu|)\right.$$

$$= I_1 + I_2 + I_3.$$

We apply now (1.4) with $a = |u|/\varepsilon\eta\sigma$ and $b = |Xu|$ in I_2, and again (1.4) with $a = |u|/\delta$ and $b = |Xu|$ in I_3 obtaining :

$$\int_\Omega M(\eta)\, N'(u)\, \varphi(|Xu|)\, |Xu|$$

$$\le \int_\Omega \left[dM(\eta)\, N'(u) + bM'(\eta) \frac{K}{\sigma} \frac{N(u)}{u} + dM(\eta) \frac{N(u)}{u} \right] \varphi(|u|)\, |u|$$

$$+ aK\,\varepsilon \int_\Omega M'(\eta)\, \eta \frac{N(u)}{u} \varphi\left(\frac{|u|}{\varepsilon\eta\sigma} \right) \frac{|u|}{\varepsilon\eta\sigma} + aK\,\varepsilon \int_\Omega M'(\eta)\, \eta \frac{N(u)}{u} \varphi(|Xu|)\, |Xu|$$

$$+ c\delta \int_\Omega M(\eta) \frac{N(u)}{u} \varphi\left(\frac{|u|}{\delta} \right) \frac{|u|}{\delta} + c\delta \int_\Omega M(\eta) \frac{N(u)}{u} \varphi(|Xu|)\, |Xu|$$

Next step is to put all the integrals with $\varphi(|Xu|)\, |Xu|$ at the first member and the other at the second one, in fact, using property (1.3)

$$\int_\Omega \left[M(\eta)\, N'(u) - aK\varepsilon M'(\eta)\, \eta \frac{N(u)}{u} - c\delta M(\eta) \frac{N(u)}{u} \right] \varphi(|X\bar{u}|)\, |Xu|$$

$$\le \int_\Omega \left[dM(\eta) N'(u) + bM'(\eta) \frac{K}{\sigma} \frac{N(u)}{u} + dM(\eta) \frac{N(u)}{u} \right.$$

$$\left. + \frac{aK}{\varepsilon^{\varphi_0} \sigma^{1+\varphi_0}} M'(\eta)\, \eta^{-\varphi_0} \frac{N(u)}{u} + \frac{c}{\delta^{\varphi_0}} M(\eta) \frac{N(u)}{u} \right] \varphi(|u|)\, |u|$$

we now substitute η^α to $m(\eta)$ and we use the relations between $\Lambda(u)/u$ and $N'(u)$ so that :

$$\int_\Omega (\eta^\alpha q - aK\,\varepsilon\,\alpha\,\eta^\alpha - c\,\delta\,\eta^\alpha) \frac{N(u)}{u} \varphi(|Xu|)\, |Xu|$$

$$\leq \int_{\Omega} \left[d\,\eta^{\alpha}\,[q + \varphi_0\,(q-1)] + \frac{b\,\alpha\,K}{\sigma}\,\eta^{\alpha - 1} + d\,\eta^{\alpha} \right.$$

$$\left. + \frac{aK\,\alpha}{\varepsilon^{\varphi_0}\,\sigma^{1+\varphi_0}}\,\eta^{\alpha - 1 - \varphi_0} + \frac{c}{\delta\,\varphi_0}\,\eta^{\alpha} \right] \frac{N(u)}{u}\,\varphi(|\,u\,|)\,|\,u\,|$$

we now choice ε and δ such that, recalling that $t - s < 1$, one has :

$$\int_{\Omega} \eta^{\alpha}\,\frac{N(u)}{u}\,\varphi(|\,Xu\,|)\,|\,Xu\,| = \frac{C_q}{(t-s)^{1+\varphi_0}} \int_{\Omega} \eta^{\alpha - 1 - \varphi_0}\,\frac{N(u)}{u}\,\varphi(|\,u\,|)\,|\,u\,| \qquad \text{... (2.1)}$$

with $C = C\,(a, b, c, d, \varepsilon, \delta, \varphi_0, K, \alpha)$ a positive constant.

Putting (2.1) into the following Sobolev type inequality

$$\left[\int_{\Omega} [\eta^{\alpha}\,|\,N(u)\,|\,\varphi(|\,u\,|)]^{k} \right]^{1/k}$$

$$\leq s \left[\int_{\Omega} \left[+ \frac{\alpha\,K}{\sigma}\,\eta^{\alpha - 1} + [q + (q-1)\,\varphi_0]\,\eta^{\alpha} + \varphi_0\,\eta^{\alpha} \right] \frac{N(u)}{u}\,\varphi(|\,u\,|)\,|\,u\,| \right.$$

$$\left. + \int_{\Omega} \eta^{\alpha}\,[q + \varphi_0\,(q-1) + \varphi_0]\,\frac{N(u)}{u}\,\varphi(|\,Xu\,|)\,|\,Xu\,| \right]$$

we obtain

$$\left[\int_{\Omega} [\eta^{\alpha}\,|\,N(u)\,|\,\varphi(|\,u\,|)]^{k} \right] 1/k \leq \frac{Cq^2}{(t-s)^{1+\varphi_0}} \int_{\Omega} \eta^{\alpha - 1 - \varphi_0}\,|\,N(u)\,|\,\varphi(|\,u\,|)$$

and thus, writing υ instead of $|\,N(u)\,|\,\varphi(|\,u\,|)$ and remembering the properties of η, one has :

$$\left[\int_{B_s} \upsilon^{k} \right] 1/k \leq \frac{Cq^2}{(t-s)^{1+\varphi_0}} \int_{B_t} \upsilon$$

we let now $l \rightarrow +\infty$ so that for the monotone convergence theorem one has :

$$\left[\int_{B_s} \Phi(|\,u\,|)^{q^k} \right]^{1/k} \leq \frac{Cq^2}{(t-s)^{1+\varphi_0}} \int_{B_t} \cdot\,\Phi(|\,u\,|)^{q}$$

that is

$$\|\,\Phi(|\,u\,|)\,\|_{L^{qk}(B_s)} \leq \left[\frac{Cq^2}{(t-s)^{1+\varphi_0}} \right]^{1/q} \|\,\Phi(|\,u\,|)\,\|_{(L^q B_t)} \qquad \text{... (2.2)}$$

let us put $q_\nu := k^\nu$, $t_\nu := (1 + 1/2^\nu) R$ and $s_\nu := t_{\nu+1}$ (so that $t_\nu - s_\nu = R/2^{\nu+1}$) into (2.2)

$$\| \Phi(|u|) \|_{L^{q_{\nu+1}}(B_{t_{\nu+1}})} \leq \left[\frac{C q_\nu^2}{R^{1+\varphi_0}} \right]^{1/q_\nu} (2^{1+\varphi_0})^{(\nu+1)/q_\nu} \| \Phi(|u|) \|_{L^{q_\nu}(B_{t_\nu})}$$

so that iterating

$$\| \Phi(|u|) \|_{L^{q_{\nu+1}}(B_{t_{\nu+1}})}$$

$$\leq \left[\frac{C q_\nu^2}{R^{1+\varphi_0}} \right]^{\sum_{i=0}^{\nu} 1/k^i} (k^2)^{\sum_{i=0}^{\nu} (i+1)/k^i} (2^{1+\varphi_0})^{\sum_{i=0}^{\nu} (i+1)/k^i} \| \Phi(|u|) \|_{L^1(B_{2R})}$$

$$\leq C' \| \Phi(|u|) \|_{L^1(B_2 R)}$$

we finally let $\nu \to +\infty$ so that

$$\| u \|_{L^\infty(B_R)} \leq C'' \| \Phi(|u|) \|_{L^\infty(B_R)} \leq C \ \| \Phi(|u|)_{L^1(B_{2R})} \leq C \| \Phi(|u|) \|_{L^1(\Omega)} < +\infty$$

which is what we are looking for, that is the local boundedness of the weak solutions of (1.5) and thus the theorem is proved. ■

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