*Indian J. pure appl. Math.*, 32(10) : 1473-1478, October 2001


# KELVIN-HELMHOLTZ INSTABILITY IN THE NEIGHBOURHOOD OF THE MINIMUM OF THE NEUTRAL CURVE IN ELECTROHYDRODYNAMICS, II

R. K. SINGLA[1], R. K. CHHABRA[2] AND S. K. TREHAN[3]

*Panjab University, Chandigarh* 160 014, *India*

(*Received* 10 *April* 2000; *accepted* 14 *March* 2001)

The nonlinear instability for two streaming superposed fluids in the presence of a normal electric field is investigated taking into account the spatial as well as temporal effects in the neighbourhood of the critical point $(k_c, E_c)$. Stability criteria are derived from the non-linear Klein-Gordon equations obtaining the bell shaped soliton and the kink solutions. It is demonstrated that the streaming difference decreases the critical electric field at which the instability sets in.

**Key Words : Kelvin-Helmholtz Instability; Neutral Curve; Electrohydrodynamics; Klein-Gordon Equations**

## 1. INTRODUCTION

The character of equilibrium of superposed fluids in relative motion has been the subject matter of great interest ever since the pioneering work by Helmholtz and Kelvin[1]. If a normal electric field is applied to a Rayleigh-Taylor surface such that there are no surface charges on the interface, one finds that stability criteria is marked by (i) a critical velocity for a prescribed electric field or (ii) a critical electric field for a prescribed velocity difference. In paper I, the first case was examined[2]. It does not appear obvious if the results of the second case can be inferred from the first one. The purpose of this papers to investigate how a continuous band width of the normal modes affects the description of the post critical normal field instability on charge free surface separating two semi-infinite dielectric fluids.

We show that the evolution of the small but finite amplitude wave packet is governed by the nonlinear Klein-Gordon equation. Depending upon the value of the electric field strength $E$, the dielectric constants and the streaming of the media involved, this equation gives the various stability criteria.

## 2. FORMULATION OF THE PROBLEM

We follow the notation of paper [1]. The solutions of the first order problem are valid provided $\omega$ satisfies the dispersion relation [(I) eqs. (18)-(22)].

[1]*Department of Computer Science and Application*; [2]*Department of Chemical Engineering and technology*; [3]*Department of Mathematics.*
*Address for correspondence*: Professor S. K. Trehan, House No. 146, Sector 09-B, Chandigarh 160 017.

$$D(\omega, k) = k(V^{(1)} - \omega/k)^2 + k\rho(V^{(2)} - \omega/k)^2 - (1-\rho) - \delta_0 k - k^2 = 0 \qquad \text{... (1)}$$

where
$$\delta_0 = -We(1/2)e^- E^2 \qquad \text{... (2)}$$

$$e(i/j) = e^{(i)}/e^{(j)}, \; e^{\pm} = e^{(2)} \pm e^{(1)} \text{ and } W = e^-/e^+.$$

On the neutral curve, the complex conjugate pair coalesces and hence the dispersion relation (1) must yield a double root for $\omega$ (real) and this can happen if the first derivative of the disperion relation $D(\omega, k)$ vanishes. Thus on the marginally neutral curve, we have [I eqs. (22)-(25)]

$$\omega = \frac{k}{(1+\rho)}(V^{(1)} + \rho V^{(2)}) \qquad \text{... (3)}$$

and the corresponding electric field is given by

$$E^2 = E_m^2(k) = \frac{e(2/1)}{ke^- W}\{k^2 + \delta_1 k + 1 - \rho\} \qquad \text{... (4)}$$

where
$$\delta_1 = -\frac{\rho}{1+\rho}(V^{(2)} - V^{(1)})^2. \qquad \text{... (5)}$$

The neutral stability curve provides the critical value of the electric field for the onset of instability. Thus the critical electric field $E_c^2$ occurs at $k = k_c = (1-\rho)^{1/2}$ and is given by

$$E_c^2 = \frac{e(2/1)}{We^-}[2(1-\rho)^{1/2} + \delta_1]. \qquad \text{... (6)}$$

It is to be observed that the system is unstable for $E \geq E_c$, for $E < E_c$ all wave solutions are stable. We now proceed to find solutions in the neighbourhood of the minimum of the neutral curve, $(k_c, E_c)$ for instability. We shall study the motions when

$$E^2 = E_c^2(1 + \varepsilon^2 \mu), \; \mu = \pm 1, \; \varepsilon \ll 1. \qquad \text{... (7)}$$

The neutral curve for stability is parabolic in the close vicinity of $(k_c, E_c)$. For any arbitrary initial data, the waves inside the unstable region with large growth rate will draw energy from the electric field system, thus giving rise to instability.

## 3. Evolution Equations

We carry out the problem to the second order equations following the procedure outlined in paper I. The solutions of the second order problem yield the solvability condition for $\eta_2$, where $\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + .....$ is the surface elevation. We then substitute the first and second order solutions into the third order problem and solve the resulting equations. At the minimum of the neutral curve $D_\omega = 0$ and $D_k = 0$ and the solvability condition for $\eta_3$ reduces to

$$\frac{1}{2} D_{\omega\omega} \frac{\partial^2 A}{\partial t_1^2} - D_{\omega k} \frac{\partial^2 A}{\partial x_1 \partial t_1} + \frac{1}{2} D_{kk} \frac{\partial^2 A}{\partial x_1^2} = \mu D_{E^2} A + G A^2 \bar{A} \qquad \text{... (8)}$$

where
$$G = -2(\omega - kV^{(1)})^2 (\lambda + k) + 2\rho(\omega - kV^{(2)})^2 (\lambda - k) - 2k^2(\lambda \delta_2 + k\delta_0) + 1.5k^4, \qquad \text{... (9)}$$

$$\delta_2 = -W\delta_0, \qquad \text{... (10)}$$

$$(1 - \rho - 2k^2)\lambda = k^2\left(V^{(1)} - \frac{\omega}{k}\right)^2 - \rho k^2 \left(V^{(2)} - \frac{\omega}{k}\right)^2 + k^2 E^{(1)} E^{(2)} e^- W^2, \qquad \text{... (11)}$$

and
$$D_{E^2} = \frac{\partial D}{\partial E^2}. \qquad \text{... (12)}$$

The various coefficients in eq. (8) are evaluated at the critical point and $G$ is the nonlinear interaction parameter, which can be either positive or negative. We may write eq. (8) as

$$\left(\frac{\partial}{\partial t_1} + V\frac{\partial}{\partial x_1}\right)^2 A - \Omega^2 \frac{\partial^2 A}{\partial x_1^2} = \alpha A + \beta A^2 \bar{A}, \qquad \text{... (13)}$$

where
$$V = \frac{V^{(1)} + \rho V^{(2)}}{1 + \rho}, \qquad \text{... (14)}$$

$$\Omega^2 = \frac{(1-\rho)^{1/2}}{(1+\rho)}, \qquad \text{... (15)}$$

$$\alpha = \frac{1-\rho}{(1+\rho)\,\varepsilon^2}\,[2(1-\rho)^{1/2} + \delta_1]\,[E^2/E_c^2 - 1] \qquad \text{.. (16)}$$

and
$$\beta = \frac{1}{2}\frac{(1-\rho)^{5/2}}{1+\rho}\left[16\left[W + \frac{\delta_1}{2}\left\{\frac{(1-\rho)^{1/2}}{1+\rho} + \frac{W}{(1-\rho)^{1/2}}\right\}\right]^2 - 5\right]. \qquad \text{... (17)}$$

Eq. (13) can be expressed as a nonlinear Klein-Gordon equation

$$\frac{\partial^2 A}{\partial T^2} - \Omega^2 \frac{\partial^2 A}{\partial X^2} = \alpha A + \beta A^2 \bar{A}, \qquad \text{... (18)}$$

with the transformation $X = x_1 - V t_1$, $T = t_1$.

It is interesting to note that the nonlinear interaction parameter $\beta$ appearing in (18) exerts a destabilizing influence whenever

$$W + \frac{1}{2[1-\rho]^{1/2}[1+\rho]}(1 - \rho + (1+\rho)W)\,\delta_1 > \sqrt{\frac{5}{16}}. \qquad \text{... (19)}$$

For $V^{(2)} = V^{(1)} = 0$, the coefficient $\beta$ is negative when $0.28286 < e\ (2/1) < 3.5353$ and positive otherwise, leading to soft and hard excitations. It can be shown that for a given value of density ratio $\rho$, the range of $e$ (2/1) values increases drastically with increase in the value of $(V^{(2)} - V^{(1)})^2$.

We shall now examine the solution of (18).

*Case* (I) — We consider only the temporal evolution of amplitude $A$ ($A$ being real). Therefore, (18) is reduced to

$$\frac{d^2 A}{dT^2} = \alpha A + \beta A^3. \qquad \text{... (20)}$$

The general solution of (20) can be found in terms of Jacobi Elliptic function and is sensitive to the sign of $\alpha$ and $\beta$. The stability of solution is discussed by the phase plane analysis. The first integral of (20) is

$$\left(\frac{dA}{dT}\right)^2 + \Phi(A) = r,$$

where $r$ is a constant of integration and is related to the wave amplitude $A$ and can be determined from the initial conditions $A = A_0$, $dA/dT = 0$. Also $\Phi(A) = -\alpha A^2 - \beta A^4/2$ can be regarded as the potential of (20). The solutions are classified into four cases :-

(i) $\alpha < 0, \beta < 0$ : Since both the linear and nonlinear terms are stabilizing, therefore all solutions are bounded and take the form of the *cn* function $A(T) = A_0\, cn\,(\alpha_1 T \mid m)$, $\alpha_1 = \sqrt{1 + A_0^2)}$ and $2\,(1 + A_0^2)\, m = A_0^2$.

Thus nonlinearity exerts a stabilizing influence.

(ii) $\alpha > 0, \beta > 0$ : Since both the linear and nonlinear terms are destabilizing, therefore the amplitude $A(T)$ grows exponentially, leading to "Explosive Instability". The perturbation analysis in this case ceases to be valid. We note that there exists only one equilibrium state at the origin $A = dA/dT = 0$.

(iii) $\alpha < 0, \beta > 0$ : In this case, the linear and nonlinear terms compete with each other giving the equilibrium amplitude : $A_{eq} = (|\alpha|/\beta)^{1/2}$.

$A_{eq}$ is a threshold value in the sense that if the initial amplitude $|A_0| < A_{eq}$ then the solution is bounded as $T \to \infty$ and is given in erms of the *sn* function as

$$A(T) = A_0\, sn\,(\alpha_1 T \mid m),\ \alpha_1 = \sqrt{1 - A_0^2/2} \text{ and } m\,(2 - A_0^2) = A_0^2.$$

If $|A_0| > A_{eq}$ and $dA/dT = 0$ initially then the solution breaks down after a finite time. Such an instability is termed as 'sub-critical' or 'nonlinear' instability, since the instability sets in for the finite amplitude even when all the linear perturbations are stable.

(iv) $\alpha > 0, \beta < 0$ : Since the linear term is destabilizing and the nonlinear term is stabilizing, therefore the solutions are bounded and oscillating of following different types :

1. Inside the separatrix $\Phi(A) < 0$, near the stabie equilibrium points $A = \pm A_{eq} = \pm(\alpha/|\beta|)^{1/2}$, $dA/dT = 0$, the solutions do not change sign and are given in terms of the *dn* function

$$A(T) = A_0\, dn\,(\alpha_1 T \mid m),\ \alpha_1 = A_0/\sqrt{2} \text{ and } mA_0^2 = 2\,(A_0^2 - 1).$$

2. Outside the separatrix $\Phi(A) > 0$, on the trajectories where $A$ does change sign, the solutions have the form of the *cn* function as in case (i)

$$A(T) = A_0\, cn\,(\alpha_1 T \mid m),\ \alpha_1 = \sqrt{A_0^2 - 1} \text{ and } 2\,(A_0^2 - 1)\,m = A_0^2.$$

3. On the separatrix $\Phi(A) = 0$, the solution is sech $A(T) = \sqrt{2}$ sech $(T)$.

*Case* (II) — Derivation of Solitary Wave Solution of (18)

We now look for the progressive wave solutions of permanent form taking into account the effects of modulation in both space ($x$-direction only) and time. For such a solution, we introduce a new variable $\xi = x_1 - vt_1$, which represents the position in a coordinate system moving at a constant velocity $v$ for which the wave appears stationary.

Hence $A\,(x_1, t_1) = A\,(x_1 - vt_1) = A\,(\xi)$ and accordingly (18) becomes ODE

$$(v^2 - \Omega^2)\frac{d^2 A}{d\,\xi^2} = \alpha A + \beta A^3 \qquad \text{... (21)}$$

with $\alpha > 0, \beta < 0$; let us normalize (21) by introducing new variables

$$T = \sqrt{\alpha}\, t_1,\ X = (\sqrt{\alpha}/\Omega)\, X_{1,}\ V' = V/\Omega, \qquad \text{... (22)}$$

$$A' = (-\beta/\alpha)^{1/2} A,\ \xi' = (\sqrt{\alpha}/\Omega)\ \xi \text{ and } \xi' = X - VT.$$

Thus, (21) becomes $(V^2 - 1)\dfrac{d^2 A}{d\,\xi^2} = A - A^3$, where we have dropped primes for convenience.

Eq. (22) yields different solutions depending on the sign of $(V^2 - 1)$.

(*a*) $V^2 > 1$. When $V^2 > 1$ then speed of the wave is always greater than the group velocity of the wave packet and the solutions are found in terms of *dn*, *cn* and *sech* functions.

The *dn* solution is $A = A_0\, dn\,(\Delta^{-1}\,(x - VT) \mid m)$,... (23)

provided $\quad V^2 = 1 + A_0^2\, \Delta^2/2,\ mA_0^2 = 2\,(A_0^2 - 1).$

In this case, the speed of wave is always greater than the group velocity and larger the scale $\Delta$, the faster the modulation propagates.

As $A_0 \to \sqrt{2}$ then $m \to 1$ and the *dn* function becomes

the hyperbolic secant:

$$A = \sqrt{2}\, sech\,(\Delta^{-1}\,(X - VT)). \qquad \text{... (24)}$$

where $$V^2 = 1 + \Delta^2. \qquad \text{... (25)}$$

Thus, the development of large modulations leads to the bell shaped soliton whose amplitude $A$ is given by (24). From (24) and (25), we observe that the soliton width $\Delta$ is entirely determined by $V$ in such a manner that the greater the width $\Delta$, the faster the soliton propagates. Here, the amplitude must be $\sqrt{2}$ and the speed depends only on $\Delta$, the width of the packet. The wider the packet, the faster it goes, as the packet becomes narrow, the speed approaches the lower limit of 1.

For $A_0 > \sqrt{2}$, the 3rd type of solution is

$$A = A_0\, cn\, (\Delta^{-1} (x - VT) \mid m), \qquad \text{... (26)}$$

provided $$V^2 = 1 + (A_0^2 - 1)\, \Delta^2,\ 2(A_0^2 - 1)\, m = A_0^2.$$

(*b*) $V^2 < 1$. The bounded solutions are given n terms of sn and tanh functions.

The sn solution is $A = A_0\, sn\, (\delta^{-1} (x - VT) \mid m)$, ... (27)

provided $$V^2 = 1 - \frac{1}{2}(2 - A_0^2)\, \delta^2,\ (2 - A_0^2)\, m = A_0^2,\ A_0^2 < 1.$$

Thus the shape (A function of m) depends only on the amplitude $A_0$. The speed $V$ depends on $A_0$ and the scale of oscillation $\delta$. As the scale $\delta$ decreases, the speed $V$ increases but it is always less than 1. As $A_0 \to 1$ then $m \to 1$ and the *dn* function becomes

$$A = \pm \tanh (\delta^{-1} (X - VT)), \qquad \text{... (28)}$$

provided $V^2 = 1 - \delta^2/2$. The solution (28) goes from an equilibrium solution of $A = -1$ to an equilibrium of $A = +1$, representing a single change of phase of $180^\circ$ that propagates with speed $V$. As the sharpness of the change increases i.e., $\Delta \to 0$ then the speed $V$ increases to the limit of unity. This solution (28) is in the form of a kink, which has different boundary conditions at $X = \pm \infty$. The solution with a plus sign is called kink and that with minus sign is called anti-kink. The kink and anti-kink are solitary waves rather than solitons because of their inability to survive collisions.

## 4. CONCLUSIONS

From the linear problem, we have shown that both the streaming velocity and the perpendicular fields are strictly destabilizing. For the critical point $(k_c, E_c)$, a generalized formulation of the non-linear problem leads to the nonlinear Klein-Gordon equation. The stability conditions of the latter equation are obtained, and the general solution of that equation, which is Jacobi elliptic function, is discussed. We find that the nonlinear effects can be stabilizing or destabilizing depending on both the dielectric constants and the streaming velocities.

## ACKNOWLEDGEMENT

SKT is grateful to the Indian National Science Academy for financial support as INSA Senior Scientist.

## REFERENCES

1. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford at the Clarendon Press.
2. R. K. Singla, R. K. Chhabra and S. K. trehan, *IJPAM*, **31** (2000) 72.