Indian J. pure appl. Math., 32(10) : 1479-1493, October 2001

HEDBERG'S THEOREM IN REAL LIPSCHITZ ALGEBRAS

DAVOOD ALIMOHAMMADI* AND ALI EBADIAN**

*Department of Mathematics, University of Arak, Arak 879, I. R. Iran
**Department of Mathematics, University of Urmia, Urmia, I. R. Iran

(*Received* 22 *December* 1999; *after revision* 4 *October* 2000; *accepted* 8 *January* 2001)

Let (X, d) be a compact metric space and τ be a topological involution on X. The real Banach function algebras $Lip^{\alpha}(X, \tau)$, for $\alpha \in (0, 1]$ and $lip^{\alpha}(X, \tau)$ for $\alpha \in (0, 1)$, were first defined by the authors. In this paper, we shall define a bigger class of these algebras. Also, we shall extend the Hedberg's Stone-Weierstrass theorem to the real Lipschitz algebra $lip^{\alpha}(X, \tau)$ and prove that the real Lipschitz algebra $Lip^{1}(X, \tau)$ is dense in $lip^{\alpha}(X, \tau)$ and prove that the real Lipschitz algebra $Lip^{1}(X, \tau)$ is dense in $lip^{\alpha}(X, \tau)$ for $\alpha \in (0, 1)$, without using the complexification techniques.

Key Words : Real Banach Function Algebras; Real Lipschitz Algebras; Lipschitz Involution; Hedberg's Theorem

INTRODUCTION

Let (X, d) be a compact metric space, and take α with $0 < \alpha \le 1$. Then $Lip^{\alpha}(X)$ is the complex algebra of bounded complex-valued function f on X such that

$$p_{\alpha}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d^{\alpha}(x, y)} : x, y \in X, x \neq y \right\}$$

is finite, and, for $0 < \alpha < 1$, $lip^{\alpha}(X)$ is the subalgebra of functions f on X such that

$$\frac{|f(x) - f(y)|}{d^{\alpha}(x, y)} \to 0 \text{ as } d(x, y) \to 0.$$

For $f \in \mathrm{Lip}^{\alpha}(X)$, set $\|f\|_{\alpha} = \|f\|_X + P_{\alpha}(f)$, where $\|f\|_X$ is the uniform norm on X. Then $(\mathrm{Lip}^{\alpha}(X), \|\cdot\|_{\alpha})$ and $(\mathrm{lip}^{\alpha}(X), \|\cdot\|_{\alpha})$ are Banach function algebras on X with the Lipschitz norm $\|\cdot\|_{\alpha}$. These algebras were first studied by Sherbert[11]. We denote

$$\mathrm{Lip}^{\alpha}_{\mathbb{R}}(X) := \{f \in \mathrm{Lip}^{\alpha}(X) : f \text{ is real-valued}\},$$

$$\mathrm{Lip}^{\alpha}_{\mathbb{R}}(X) := \{f \in \mathrm{Lip}^{\alpha}(X) : f \text{ is real-valued}\}.$$

Then $\text{Lip}^{\alpha}_{\mathbb{R}}(X)$ ($\text{lip}^{\alpha}_{\mathbb{R}}(X)$, respectively) is a closed real subalgebras of $\text{Lip}^{\alpha}(X)$ ($\text{lip}^{\alpha}(X)$, respectively). For further general facts about Lipschitz algebras the reader is referred to [13].

The theory of real Banach algebras was first introduced by Ingelstam[6]. Later on the theory of real function algebras was developed by Kulkarni and Limaye and received attention of the others. Their excellent monograph[9] presented all aspects the theory. Let X be a compact Hasudorff space. By $C(X)$ (respectively, $C_{\mathbb{R}}(X)$), we denote the complex (real, respectively) Banach algebra of all continuous complex-valued (real-valued, respectively) functions on X with the point-wise operations and the uniform norm on X. The continuous map $\tau: X \to X$ is called a topological involution on X if $\tau^2(x) = x$ for all x in X. If τ is a topological involution on X then the map $\sigma: C(X) \to C(X)$ defined by $\sigma(f) = \bar{f} \circ \tau$ is an algebra involution on $C(X)$, which is an isometry and is called the algebra involution induced by τ on $C(X)$. Moreover, every algebra involuion on $C(X)$ arises from a topological involution on X in the manner described above.

Now, by the above notation, we define $C(X, \tau) = \{f \in C(X) : \sigma(f) = f\}$. Then $C(X, \tau)$ with the uniform norm on X is a real Banach algebra with unit 1 and $C(X) = C(X, \tau) \oplus iC(X, \tau)$, that is, every $h \in C(X)$ can be expressed uniquely as $f + ig$ with f, g in $C(X, \tau)$. Moreover, $C(X, \tau)$ separates the points of X.

A real Banach function algebra on (X, τ), is a real subalgebra A of $C(X, \tau)$ such that separates the points of X, contains 1 and complete under an algebra norm $\|\cdot\|$. If the norm on a real Banach function algebra A is the uniform norm on X then A is called a real uniform (function) algebra on (X, τ).

In this paper, we first prove some results about the real Banach function algebras on (X, τ) which are the generalizations of [9, Theorems 1.3.22, 1.3.20]. Next, we introduce a bigger class of the real Lipschitz algebras $\text{Lip}^{\alpha}(X, \tau)$ and $\text{lip}^{\alpha}(X, \tau)$ that were first defined in[1]. We also extend the Hedberg's Stone-Weierstrass theorem[4] for the density of a real subalgebra of $lip^{\alpha}(X, \tau)$ in it and prove that the $Lip^1(X, \tau)$ is dense in the $lip^{\alpha}(X, \tau)$ for $\alpha \in (0, 1)$ without using the complexification technique. Finally we extend the Weaver's theorem [12, Theorem 1.4] to the real Lipschitz algebra $\text{lip}^1(X, \tau)$.

1. Basic Theorems in Real Banach Function Algebras

Let A be a real (respectively, complex) algebra. The carrier space of A, denote by Car (A), is the set of all nonzero homomorphisms from A to $\mathbb{C}$, regarded as a real (respectively, complex) algebra.

The following theorems are generalizations of [9, Theorems 1.3.22, 1.3.20] respectively. We shall use these results in later section the prove some results about the a real Lipschitz algebras.

Theorem 1.1 — *Let X be a compact Hausdorff space, τ be a topological involution on X and σ be the algebra involution induced by τ on $C(X)$. If $(B, \|\cdot\|)$ is a complex Banach function algebra on X such that $\sigma(B) = B$ and $A := \{h \in B : \sigma(h) = h\}$. Then*

(i) A is a real subalgebra of B;

(ii) Every h in B can be expressed uniquely as $f + ig$ with f, g in A;

(iii) There exists a constant $C \geq 1$ such that $\|\sigma(h)\| \leq C\|h\|$ for every $h \in B$ and max $\{\|f\|, \|g\|\} \leq C\|f + ig\|$ for all f, g in A;

(iv) $(A, \|\cdot\|)$ is a real Banach function algebra on (X, τ); and

(v) For ϕ in Car (A), define $\alpha(\phi)(f + ig) := \phi(f) + i\phi(g)$ for $f, g \in A$. Then $\alpha(\phi) \in$ Car (B) and map α is bijection Car (A) and Car (B).

PROOF : (*i*) Obvious.

(*ii*) Since $\sigma(B)=B$, for h in B, $\sigma\left(\frac{h+\sigma(h)}{2}\right)=\frac{1}{2}(h+\sigma(h))$ and $\sigma\left(\frac{h-\sigma(h)}{2i}\right)=\frac{h-\sigma(h)}{2i}$.

Thus $h=\frac{(h+\sigma(h))}{2}+i\left(\frac{((h-\sigma(h))}{2i}\right)$ with $\frac{(h+\sigma(h))}{2}, \frac{(h-\sigma(h))}{2i}$ in A. Further, if $h=f+ig$ with f and g in A, then $\sigma(h)=f-ig$ and hence $f=\frac{(h+\sigma(h))}{2}$ and $g=\frac{(h-\sigma(h))}{2i}$. This proves the uniqueness of f and g.

(*iii*) It is easy to see that every Banach function algebra is semisimple. Since $\sigma(B)=B$, we conclude that σ is an algebra involution on B, as a semisimple commutative Banach algebra. Thus σ is continuous by [3, Theorem 36.2]. Hence, there exists a positive real number C such that $\|\sigma(h)\| \leq C\|h\|$ for every $h \in B$. So, $\|h\| \leq C\|\sigma(h)\|$ for every $h \in B$. So, $C \geq 1$. Now, for every $f, g \in A$, we have

$$\|f\| = \left\|\frac{1}{2}[(f+ig)+\sigma(f+ig)]\right\| \leq \frac{1}{2}(\|f+ig\|+C\|f+ig\|) \leq C\|f+ig\|.$$

Similarly, $\|g\| \leq C\|f+ig\|$. This proves (iii).

(*iv*) Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(A, \|\cdot\|)$. Hence $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(B, \|\cdot\|)$. Since B is complete, there exists $h \in B$ such that $\lim_{n\to\infty} \|f_n - h\| = 0$. By continuity of the real linear map σ, we have $\lim_{n\to\infty} \|f_n - \sigma(h)\| = 0$, $\sigma(h)=h$. Hence $h \in A$ and $(A, \|\cdot\|)$ is complete. Since $\sigma(1) = 1$, $1 \in A$. By (ii), we conclude that A separates the points of X. Hence $(A, \|\cdot\|)$ is a real Banach function algebra on (X, τ).

(*v*) Obvious.

Theorem 1.2 — *Let X be a compact Hausdorff space, τ be a topological involution on X and σ be the algebra involution induced by τ on $C(X)$. Let $(A, \|\cdot\|)$ be a real Banach function algebra on (X, τ) and define $B := \{f+ig : f, g \in A\}$ Then :-*

(i) $\sigma(B)=B$ and $A = \{h \in B : \sigma(h)=h\} = B \cap C(X, \tau)$.

(ii) B is a complex subalgebra of $C(X)$ and $B = A \oplus iA$.

(iii) There is an algebra norm $|||\cdot|||$ on B such that $\|f\| = |||f|||$ for all $f \in A$ and $\max\{\|f\|, \|g\|\} \leq |||f+ig||| \leq 2\max\{\|f\|, \|g\|\}$ $(f, g \in A)$.

(iv) $(B, |||\cdot|||)$ is a Banach function algebra on X.

PROOF : (*i*) and (*ii*) are obvious

(*iii*) B can be viewed as the complexification of A, by (*ii*). Therefore by [2, Theorem 13.3], there exists an algebra norm $|||\cdot|||$ on B such that $\|f\| = |||f+i0||| = |||f|||$ for every $f \in A$ and

$$\max\{\|f\|, \|g\|\} \leq |||f+ig||| \leq 2\max\{\|f\|, \|g\|\} \ (f, g \in A).$$

(*iv*) Since A separates the points of X and $1 \in A$, B also separates the points of X and $1 \in B$. Since $(A, \|\cdot\|)$ is Banach, $(B, |||\cdot|||)$ also is Banach by [2, Theorem 13.3]. Therefore $(B, |||\cdot|||)$ is a Banach function algebra on X and the proof is complete.

2. Some Results on Real Lipschitz Algebras

Definition 2.1 — Let (X, d) be a compact metric space and τ be a topological involution on X. Then τ is called a Lipschitz involution on X, if there exists a constant $C > 0$ such that $d(\tau(x), \tau(y)) \leq Cd(x, y)$, for all x, y in X.

If $d(\tau(x), \tau(y)) = d(x, y)$ for every $x, y \in X$, then we say that τ is an isometric involution on X.

Remark 2.2 : Since τ^2 is the identity map on X, it is easy to see that $C \geq 1$, and if $C = 1$ then τ is an isometric involution on X.

In the following example we show that, there exists a Lipschitz involution which is not an isometric involution.

Example 2.3 — Ler $R > 1$ and $X = \left\{ z \in \mathbb{C} : \frac{1}{R} \leq |z| \leq R \right\}$. Now define the map $\tau : X \to X$ by $\tau(z) = -\frac{1}{z}$. Then τ is a topological involution on compact metric space X. Since for every $z, w \in X$,

$$d(\tau(z), \tau(w)) = \left| -\frac{1}{z} + \frac{1}{w} \right| = \frac{|z-w|}{|z||w|} \leq R^2 |z-w| = R^2 d(z, w)$$

and

$$d\left(\tau\left(-\frac{1}{R} \right), \tau\left(\frac{1}{R} \right) \right) = d(R, -R) = 2R = R^2 d\left(-\frac{1}{R}, \frac{1}{R} \right),$$

therefore, τ is a Lipschitz involution on X which is not an isometric involution.

Lemma 2.4 — Let (X, d) be a compact metric space and τ be a Lipschitz involution on X and let σ be the algebra involution induced by τ on $C(X)$. Then :-

(i) for every $0 < \alpha \leq 1$ and $f \in \mathrm{Lip}^{\alpha}(X)$, $p_{\alpha}(\sigma(f)) \leq C^{\alpha} p_{\alpha}(f)$.

(ii) for every $0 < \alpha \leq 1$, $\sigma(\mathrm{Lip}^{\alpha}(X)) = \mathrm{Lip}^{\alpha}(X)$,

(iii) for every $0 < \alpha < 1$, $\sigma(\mathrm{Lip}^{\alpha}(X)) = \mathrm{Lip}^{\alpha}(X)$,

(iv) for every $0 < \alpha \leq 1$ and every $f \in \mathrm{Lip}^{\alpha}(X)$, $\| \sigma(f) \|_{\alpha} \leq C^{\alpha} \| f \|_{\alpha}$.

(v) If τ is an isometric involution then for every $0 < \alpha \leq 1$ and $f \in \mathrm{Lip}^{\alpha}(X)$, $p_{\alpha}(\sigma(f)) = p_{\alpha}(f)$ and $\| \sigma(f) \|_{\alpha} = \| f \|_{\alpha}$.

Proof : (*i*) Let $0 < \alpha \leq 1$ and $f \in \mathrm{Lip}^{\alpha}(X)$. Then

$$p_{\alpha}(\sigma(f)) = \sup \left\{ \frac{|\sigma(f)(x) - \sigma(f)(y)|}{d^{\alpha}(x, y)} : x, y \in X, x \neq y \right\}$$

$$= \sup \left\{ \frac{|f(\tau((x) - f(\tau(y))|}{d^{\alpha}(x, y)} : x, y \in X, x \neq y \right\}$$

$$\leq \sup\left\{\frac{|f(\tau(x))-f(\tau(y))|}{C^{-\alpha}d^{\alpha}(\tau(x),\tau(y))} : x, y \in X, x \neq y\right\}$$

$$= C^{\alpha} p_{\alpha}(f).$$

(*ii*) By (*i*), $\sigma(\mathrm{Lip}^{\alpha}(X)) \subseteq \mathrm{Lip}^{\alpha}(X)$. Since $\sigma(\sigma(f)) = f$ for every $f \in \mathrm{Lip}^{\alpha}(X)$, $\sigma(\mathrm{Lip}^{\alpha}(X)) = \mathrm{Lip}^{\alpha}(X)$.

(*iii*) Let $0 < \alpha < 1$ and $f \in \mathrm{lip}^{\alpha}(X)$. Let $\varepsilon > 0$ be given. Since $f \in \mathrm{lip}^{\alpha}(X)$, there is a $\delta_1 > 0$ such that $\frac{|f(x)-f(y)|}{d^{\alpha}(x,y)} < C^{-\alpha}\varepsilon$ whenever $0 < d(x, y) < \delta_1$. Set $\delta = C^{-1}\delta_1$. Suppose $x, y \in X$ such that $0 < d(x, y) < \delta$. Since τ is Lipschitz involution on X, therefore

$$0 < d(\tau(x), \tau(y)) \leq Cd(x, y) < C\delta = \delta_1.$$

So $\frac{|f(\tau(x))-f(\tau(y))|}{d^{\alpha}(\tau(x),\tau(y))} < C^{-\alpha}\varepsilon$. This implies that $\frac{|\sigma(f)(x)-\sigma(f)(y)|}{d^{\alpha}(x,y)} < \varepsilon$.

So $\sigma(f) \in \mathrm{lip}^{\alpha}(X)$. Therefore, $\sigma(\mathrm{lip}^{\alpha}(X)) \subseteq \mathrm{lip}^{\alpha}(X)$ and since σ is an algebra involution, so $\sigma(\mathrm{lip}^{\alpha}(X)) = \mathrm{lip}^{\alpha}(X)$.

(iv) Let $0 < \alpha \leq 1$ and $f \in \mathrm{Lip}^{\alpha}(X)$. By (*i*) and $C \geq 1$, we have

$$\|\sigma(f)\|_{\alpha} = \|\sigma(f)\|_X + p_{\alpha}(\sigma(f)) = \|f\|_X + p_{\alpha}(\sigma(f))$$

$$\leq C^{\alpha}\|f\|_X + C^{\alpha}p_{\alpha}(f) = C^{\alpha}\|f\|_{\alpha}.$$

(*v*) See [1, Theorem 7, (iii)].

Remark 2.5 : (*i*) Let (X, d) be a compact metric space, $\alpha \in (0, 1)$ and A be any real Lipschitz algebra $\mathrm{Lip}_{\mathbb{R}}^{\alpha}(X)$ or $\mathrm{lip}_{\mathbb{R}}^{\alpha}(X)$. Then every linear isometry of A into A is of the form $f \to f \circ \tau$ for some isometry τ of X. Thus every antilinear isometry of a Lipschitz space is the algebra involution σ induced by an isometry τ on X[10].

(*ii*) Let (X, d) be a compact metric space, $0 < \alpha \leq 1$. If $B = \mathrm{Lip}^{\alpha}(X)$ or $\mathrm{lip}^{\alpha}(X)$, then the mapping $f \mapsto \bar{f}$ is an isometric algebra involution on B.

(*iii*) Let (X, d) be a compact metric space and τ be an isometric involution topological involution on X and let σ be the algebra involution induced by τ on $C(X)$. If $B = \mathrm{Lip}^{\alpha}(X)$ or $\mathrm{lip}^{\alpha}(X)$ then by Lemma 2.4, $\sigma|_B$ is an isometric algebra involution on B.

In the following example, we show that, there exists an algebra involution on some Lipschitz algebras, that is not an isometry.

Example 2.6 — Regard R, X and τ as Example 2.3. Let σ be the algebra involution induced by τ on $C(X)$, $0 < \alpha < 1$ and $B = \mathrm{Lip}^{\alpha}(X)$ or $\mathrm{lip}^{\alpha}(X)$. By Lemma 2.4 we have $\sigma(B) = B$. If $\eta = \sigma|_B$ then η is an algebra involution on B. Now, we show that η is not an isometry on B. Consider the map $f: X \to \mathbb{C}$ which is defined by $f(z) = z$. Clearly, $f \in B$ and we have

$$p_\alpha(f) = \sup\left\{ \frac{|f(z)-f(w)|}{|z-w|^\alpha} : z, w \in X, z \neq w \right\}$$

$$= \sup\left\{ |z-w|^{1-\alpha} : z, w \in X, z \neq w \right\}$$

$$= 2^{1-\alpha} R^{1-\alpha}.$$

Also, we have

$$p_\alpha(\eta(f)) = p_\alpha(\sigma(f)) = \sup\left\{ \frac{|\sigma(f)(z) - \sigma(f)(w)|}{|z-w|^\alpha} : z, w \in X < z \neq w \right\}$$

$$= \sup\left\{ \frac{|z-w|^{1-\alpha}}{|z||w|} : z, w \in X, z \neq w \right\}$$

$$\geq \sup\left\{ \frac{|z-w|^{1-\alpha}}{|z||w|} : |z| = \frac{1}{R}, |w| = \frac{1}{R} \right\}$$

$$= R^2 \sup\left\{ |z-w|^{1-\alpha} : |z| = \frac{1}{R,} |w| = \frac{1}{R} \right\}$$

$$= R^2 \left(\frac{2}{R}\right)^{1-\alpha} = 2^{1-\alpha} R^{1+\alpha}.$$

Therefore, $p_\alpha(\eta(f)) = R^{2\alpha} p_\alpha(f) > p_\alpha(f)$ and

$$\| \eta(f) \|_\alpha = \| \eta(f) \|_X + p_\alpha(\eta(f)) = \| \sigma(f) \|_X + p_\alpha(\eta(f))$$

$$= \|f\|_X + p_\alpha(\eta(f)) > \|f\|_X + p_\alpha(f) = \|f\|_\alpha.$$

Consequently, η is not an isometry on B.

Note that by the above example and Remark 2.5 (*iii*), we conclude that the class of the Lipschitz algebra which is defined in this paper, contains the real Lipschitz algebra defined in [1].

Theorem 2.7 — *Let (X, d) be a compact metric space, τ be a Lipschitz involution on X and σ be the algebra involution induced by τ on $C(X)$. We define,*

$$\mathrm{Lip}^\alpha(X, \tau) := \{h \in \mathrm{Lip}^\alpha(X) : \sigma(h) = h\} \quad (0 < \alpha \leq 1)$$

and

$$\mathrm{lip}^\alpha(X, \tau) := \{h \in \mathrm{lip}^\alpha(X) : \sigma(h) = h\} \quad (0 < \alpha < 1).$$

If $B = lip^\alpha(X)$ and $A = Lip^\alpha(X, \tau)$ ($B = lip^\alpha(X)$ and $A = lip^\alpha(X, \tau)$ respectively) then

(i) $B = A \oplus iA$.

(ii) For every $f, g \in A$, $\max\{\|f\|_\alpha, \|g\|_\alpha\} \leq C^\alpha \|f + ig\|_\alpha$

(iii) $(A, \|\cdot\|_\alpha)$ is a real Banach function algebra on (X, τ).

(iv) $(A, \|\cdot\|_\alpha)$ *is self-adjoint and* $\overline{A} = C(X, \tau)$.

(v) $Car\ (A) = \{e_x : x \in X\}$, *where* e_x *is the evaluation homomorphism on* A *at* $x \in X$, *that is,* $e_x(f) = f(x)$ *for all* f *in* A.

PROOF : Since $(B, \|\cdot\|_\alpha)$ is a complex Banach function algebra on X and by Lemma 2.4 $\sigma(B) = B$ and $\|\sigma(h)\|_\alpha \leq C^\alpha \|h\|_\alpha$ for each $h \in B$, therefore (*i*), (*ii*) and (*iii*) hold by Theorem 1.1.

(*iv*) Since B is self-adjoint and $A = B \cap C(X, \tau)$, so that A is self-adjoint. Clearly that $\overline{A}$ is a real uniform algebra on (X, τ) and $f \in \overline{A}$ implies that $\overline{f} \in \overline{A}$. Therefore, by Stone-Weierstrass theorem for real subalgebra of $C(X, \tau)$ [6], we conclude that $\overline{A} = C(X, \tau)$.

(*v*) Since $Car\ (B) = \{e_x : x \in X\}$ [11], we conclude that $Car\ (A) = \{e_x : x \in X\}$ by Theorem 1.1 (*v*).

Now, we show that the class of real Lipschitz algebras $\text{Lip}^\alpha\ (X, \tau)$ (respectively, $\text{lip}^\alpha\ (X, \tau)$) is bigger the class of complex Lipschitz algebras $\text{Lip}^\alpha\ (X)$ (respectively, $\text{lip}^\alpha\ (X)$), regarded as real Banach algebras.

Theorem 2.8 — *Let* (X, d) *be a compact metric space and* $\alpha \in (0, 1]$. *Then there exist a compact metric space* (Y, ρ), *an isometric involution* τ *on* Y *such that the complex Lipschitz algebra* $Lip^\alpha\ (X)$ *(*$lip^\alpha\ (X)$ *respectively, regarded as a real Banach algebra, is isometrically isomorphic to the real Lipschitz algebra* $Lip^\alpha\ (X, \tau)$ *(*$lip^\alpha(Y, \tau)$ *respectively).*

PROOF : Let $Y = X \times \{0, 1\}$. We define the map $\rho : Y \times Y \to I\!R$ by

$\rho((x, j), (y, k)) = \max\{d(x, y), |j - k|\}\ (x, y \in X, j, k \in \{0, 1\})$.

Then ρ is a metric on Y and the induced topology by ρ on Y coincides to the product topology on it. We define the map $\tau : Y \to Y$ by

$$\tau(x, 0) = (x, 1),\ \tau(x, 1) = (x, 0)\ (x \in X).$$

It is easy to see that τ is an isometric involution on Y. Now we define the map $\psi : \text{Lip}^\alpha(X) \to \text{Lip}^\alpha(Y, \tau)$ by

$$\psi(f)(x, 0) = f(x),\ \ \psi(f)(x, 1) = \overline{f}(x)\ \ (f \in \text{Lip}^\alpha(X), x \in X).$$

Then we can easily show that ψ is well-defined and it is an isometrically homomorphism of $(\text{Lip}^\alpha(X), \|\cdot\|_\alpha)$, regarded as a real Banach algebra, into $(\text{Lip}^\alpha(Y, \tau), \|\cdot\|_\alpha)$. For $g \in \text{Lip}^\alpha(Y, \tau)$, we define the map $f : X \to \mathbb{C}$ by $f(x) = g(x, 0)\ (x \in X)$. It is easy to see that $f \in \text{Lip}\,(X, \alpha)$ and $\psi(f) = g$. Therefore, ψ is an isometrically isomorphism of $\text{Lip}^\alpha\ (X)$ onto $\text{Lip}^\alpha(Y, \tau)$. Moreover, $f \in \text{lip}^\alpha(X)$ implies that $\psi(f) \in \text{lip}^\alpha(Y, \tau)$ and $g \in \text{lip}^\alpha(X, \tau)$ implies that $\psi^{-1}(g) \in \text{lip}^\alpha(X)$. Therefore, $\psi|_{\text{lip}^\alpha(X)}$ is an isometrically isomorphism of $(\text{lip}^\alpha(X), \|\cdot\|_\alpha)$, regarded as a real Banach algebra, onto $(\text{lip}^\alpha(Y, \tau), \|\cdot\|_\alpha)$. This completes the proof.

If (X, d) is a metric space and $0 < \alpha < \beta \leq 1$, then $\text{Lip}^\beta(X) \subsetneq \text{Lip}^\alpha(X)$[11]. In addition, if X is infinite set and $0 < \alpha < 1$ then $\text{lip}^\alpha(X)$ is a proper subalgebra of $\text{Lip}^\alpha(X)$ [4] (Also, see [7, theorem 4.7] and [8, Corollary 3]). Hence, we give the following result by Theorem 2.7.

Theorem 2.9 — *Let* (X, d) *be a compact metric space and* τ *be a Lipschitz involution on* X. *Then :*

(i) If $0<\alpha<\beta\le 1$, *then* $\mathrm{Lip}^{\beta}(X,\tau)$ *is a subalgebra of* $\mathrm{lip}^{\alpha}(X,\tau)$.

(ii) If $0<\alpha<1$ *and X is infinite set then* $\mathrm{lip}^{\alpha}(X,\tau)$ *is a proper subalgebra of* $\mathrm{Lip}^{\alpha}(X,\tau)$.

The Stone-Weierstrass theorem in real Lipschitz algebras $\mathrm{lip}^{\alpha}_{\mathbb{R}}(X)$ was first given by Hedberg[4] [Theorem 1]. This result proved in complex Lipschitz algebras $\mathrm{lip}^{\alpha}(X)$[5] [Remark 1]. We now extend the Hedberg's theorem in real Lipschitz algebras $\mathrm{lip}^{\alpha}(X,\tau)$, without using the complexification technique.

Theorem 2.10 — *Let (X, d) be a compact metric space, τ be a Lipschitz involution on X, and take $\alpha\in(0,1)$. Let A be a self-adjoint real subalgebra of $\mathrm{lip}^{\alpha}(X,\tau)$ which separates the points of X and contains the real-valued constant functions on X. Then A is dense in $\mathrm{lip}^{\alpha}(X,\tau)$ if for every $a\in X$, there are positive numbers M_a and δ_a such that for every $\delta\le\delta_a$, there is a $f\in A$, with $f(a)=1$, $f(x)=0$ on $S_\delta(a)=\{x\in X: d(x,a)=\delta\}$, and*

$$sup\left\{\frac{|f(y)-f(z)|}{d^{\alpha}(y,z)}: y,z\in B_\delta(a), y\neq z\right\}<\frac{M_a}{\delta^{\alpha}},$$

where $$B_\delta(a)=\{x\in X: d(x,a)\le\delta\}.$$

PROOF : Let B be the closure of A in $(\mathrm{lip}^{\alpha}(X,\tau), \|\cdot\|_\alpha)$, and set

$$B_{R+I}=\{Re\, f+Imf : f\in B\}.$$

Then B_{R+I} is a real subalgebra of $\mathrm{lip}^{\alpha}_{\mathbb{R}}(X)$, since for $f, g\in B$ we have

$$(Ref+Imf)\,(Reg+Img)=Reh+Imh$$

with $$h=\frac{1}{2}(fg+f\bar{g}+\bar{f}g-\bar{f}\bar{g})\in B.$$

It is clear that for $f\in C(X,\tau)$ and $x\in X$,

$$Ref(\tau(x))=Ref(x),\ Imf(\tau(x))=-Imf(x). \qquad \text{... (1)}$$

Since τ is a bijection of X onto X, by (1) we have for $f\in C(X,\tau)$,

$$\|Ref+Imf\|_X=\mathrm{Sup}\{|Ref(x)+Imf(x)|: x\in X\}$$

$$=\mathrm{Sup}\ \{|Ref(\tau(x))+Imf(\tau(x))|: x\in X\}$$

$$=\mathrm{Sup}\ \{|Ref(x)-Imf(x)|: x\in X\}$$

$$=\|Ref-Imf\|_X,$$

and for $f\in \mathrm{Lip}^{\alpha}(X,\tau)$,

$$p_\alpha(Ref - Imf) = \sup\left\{ \frac{\|[Ref(x) - Imf(x)] - [Ref(y) - Imf(y)]\|}{d^\alpha(x, y)} : x, y \in X, x \neq y \right\}$$

$$= \sup\left\{ \frac{|[Ref(\tau(x)) - Imf(\tau(x))] - [Ref(\tau(y)) - Imf(\tau(y))]|}{d^\alpha(\tau(x), \tau(y))} : x, y \in X, x \neq y \right\}$$

$$\leq \sup\left\{ \frac{|[Ref(x) + Imf(x)] - [Ref(y) + Imf(y)]|}{C^{-\alpha} d^\alpha(\tau(\tau(x)), \tau(\tau(y)))} : x, y \in X, x \neq y \right\}$$

$$= C^\alpha p_\alpha(Ref + Imf).$$

Hence, we have

$$\| Ref - Imf \|_\alpha \leq 2C^\alpha \| Ref + Imf \|_\alpha \ (f \in \mathrm{Lip}^\alpha(X, \tau)). \qquad \text{... (2)}$$

We claim that $B_{R=I}$ is a closed subset of $\mathrm{lip}^\alpha_{\mathbb{R}}(X)$ under the Lipschitz norm $\|\cdot\|_\alpha$. Let $\left\{ Ref_n + Imf_n \right\}_{n=1}^\infty$ be a Cauchy sequence in $(B_{R+I}, \|\cdot\|_\alpha)$, where $f_n \in B$ for each $n \geq 1$. Then inequality (2) implies that $\left\{ Ref_n \right\}_{n=1}^\infty$ and $\left\{ Imf_n \right\}_{n=1}^\infty$ are also Cauchy sequences in $(\mathrm{lip}^\alpha_{\mathbb{R}}(X), \|\cdot\|_\alpha)$. It follows that there are $g, h \in \mathrm{lip}^\alpha_{\mathbb{R}}(X)$ such that

$$\lim_{n \to \infty} \| Ref_n - g \|_\alpha = 0, \ \lim_{n \to \infty} \| Imf_n - h \|_\alpha = 0. \qquad \text{... (3)}$$

So

$$\lim_{n \to \infty} \| i\, Imf_n - ih \|_\alpha = 0. \qquad \text{... (4)}$$

Since A is self-adjoint, we conclude B is also self-adjoint and so Ref_n, $iImf_n \in B$ for each $n \geq 1$. By (3), (4) and the closedness of B in $(\mathrm{lip}^\alpha(X), \|\cdot\|_\alpha)$, we have $g, ih \in B$ and so $g + ih \in B$. Now set $f = g + ih$. Since g, h are real-valued, $g = Ref$, $h = Im\, f$ and by (3),

$$\lim_{n \to \infty} \| Ref_n + Imf_n \|_\alpha = \| Ref + Imf \|_\alpha.$$

Hence, our claim holds.

Now, we claim that B_{R+I} separates the points of X. Let $x, y \in X$ with $x \neq y$. Since A separates the points of X, there is an $f \in A \subset B$ such that $f(x) \neq f(y)$. Since A is self-adjoint, we conclude that $Re\, f \in A$. Firstly, suppose $Re\, f(x) \neq Ref(y)$. If $g = Re\, f$ then $g \in B$ and we have

$$Reg\ (x) + Im\ g(x) \neq Reg\ (y) + Img\ (y).$$

Secondly, suppose $Ref(x) = Ref(y)$ and $Imf\ (x) \neq Im\ f\ (y)$. If $g = f - Ref(x)\,1 - Ref(y)\,1$ then $g \in B$ and we have

$$Reg\ (x) + Img\ (x) = -\ Ref\ (y) + Imf\ (x) \neq -\ Ref\ (y) + Imf\ (y)$$

$$= -\ Ref\ (x) + Imf\ (y) = Reg\ (y) + Img\ (y).$$

Hence, our claim holds.

Now, let $a \in X$. By hypothesis there are positive numbers M_a and δ_a such that for every δ with $0 < \delta \leq \delta_a$, there is an $f \in A$ with $f(a) = 1$, $f(x) = 0$ on $S_a(\delta)$, and

$$\sup \left\{ \frac{|f(y) - f(z)|}{d^{\alpha}(y, z)} : y, z \in B_{\delta}(a), y \neq z \right\} < \frac{M_a}{\delta^{\alpha}}.$$

Therefore, we have $Ref + Imf \in B$ with $Ref(a) + Imf(a) = 1$, $Ref(x) + Imf(x) = 0$ on $S_a(\delta)$ and

$$\sup \left\{ \frac{|[Ref(y) + Imf(y)] - [Ref(z) + Imf(z)]|}{d^{\alpha}(y, z)} : y, z \in B_{\delta}(a), y \neq z \right\}$$

$$\leq \sqrt{2} \sup \left\{ \frac{|f(y) - f(z)|}{d^{\alpha}(y, z)} : y, z \in B_{\delta}(a), y \neq z \right\}$$

$$\leq \sqrt{2} \frac{M_a}{\delta^{\alpha}}.$$

Hence, we conclude that $B_{R+I} = \lim_{\mathbb{R}}^{\alpha}(X)$, by the Hedberg's Theorem in $\mathrm{lip}_{\mathbb{R}}^{\alpha}(X)$.

Now let $f \in \mathrm{lip}^{\alpha}(X, \tau)$. Then $Re\, f + Im\, f \in \mathrm{lip}_{\mathbb{R}}^{\alpha}(X) = B_{R+I}$. Hence, there is $g \in B$ such that

$$Ref + Imf = Reg + Img. \qquad \ldots (5)$$

Since $f, g \in C(X, \tau)$, by (1) we have for each $x \in X$,

$$Re\, f(x) - Imf(x) = Ref(\tau(x)) + Imf(\tau(x))$$

$$= Reg(\tau(x)) + Img(\tau(x))$$

$$= Reg(x) - Img(x).$$

Therefore, $Ref - Imf = Reg - Img$.... (6)

Now, we conclude that $f = g$, by (5), (6). Hence, $f \in B$ and so $B = \mathrm{lip}^{\alpha}(X, \tau)$. This completes the proof.

Notice that if (X, d) is a compact metric space and τ is the identity map on X then $\mathrm{Lip}^{\alpha}(X, \tau) = \mathrm{Lip}_{\mathbb{R}}^{\alpha}(X)$ and $\mathrm{Lip}^{\alpha}(X, \tau) = \mathrm{Lip}_{\mathbb{R}}^{\alpha}(X)$. Hence, the Theorem 2.10 is a generalization of the Hedberg's theorem. Note that Honary and Mahyar proved the Hedberg's theorem in complex Lipschitz algebras $\mathrm{lip}^{\alpha}(X)$ with using the Hedberg's theorem in real Lipschitz algebra $\mathrm{lip}_{\mathbb{R}}^{\alpha}(X)$.

Now, as a consequence of Theorem 2.10, we obtain the Hedberg's theorem in complex Lipschitz algebra $\mathrm{lip}^{\alpha}(X)$.

Corollary 2.11 — Let (X, d) be a compact metric space and take $\alpha \in (0, 1)$. Let B be a self-adjoint complex subalgebra of $\mathrm{lip}^{\alpha}(X)$ which separates the points of X and contains the complex-valued constant functions on X. Then B is dense in $\mathrm{lip}^{\alpha}(X)$ if for $a \in X$, there are positive numbers M_a and δ_a such that for every $\delta \leq \delta_a$, there is a $f \in B$ such that $f(a) = 1$, $f(x) = 0$ on $S_{\delta}(a) = \{x \in X : d(x, a) = \delta\}$, and

$$\sup \left\{ \frac{|f(y) - f(z)|}{d^{\alpha}(y, z)} : y, z \in B_{\delta}(a), y \neq z \right\} < \frac{M_a}{\delta^{\alpha}},$$

where $B_{\delta}(a) = \{x \in X : d(x, a) \leq \delta\}$.

PROOF : Let $Y = X \times \{0, 1\}$. Suppose ρ, τ and ψ are the metric on Y, the isometric involution on the compact metric space (Y, ρ) and the isomerically isomorphism of $(\mathrm{Lip}^{\alpha}(X), \|\cdot\|_{\alpha})$, regarded as a real Banach algebra, onto $(\mathrm{Lip}^{\alpha}(Y, \tau), \|\cdot\|_{\alpha})$, which have introduced in the proof of Theorem 2.8, respectively. Let

$$A = \{\psi(f) : f \in B\}.$$

Since $\psi|_{\mathrm{lip}^{\alpha}(X)}$ is an isometrically isomorphism of $\mathrm{lip}^{\alpha}(X)$, regarded as a real Banach algebra, onto $\mathrm{lip}^{\alpha}(Y, \tau)$, we conclude that A is a real subalgebra of $\mathrm{lip}^{\alpha}(Y, \tau)$ and the density of B in $(\mathrm{lip}(X, \alpha), \|\cdot\|_{\alpha})$ is equivalent to the density of A in $\mathrm{lip}^{\alpha}(Y, \tau)$. For each $f \in \mathrm{lip}^{\alpha}(X)$, we have $\overline{\psi(f)} = \psi(\bar{f})$. Therefore A is self-adjoint. Moreover, A contains the real-valued constant functions on Y, since the complex-valued function 1 on Y is in B and $\psi(1)$ is the real-valued function 1 on Y. Now we claim that A separates the points of zY. Let $(x, k), (y, l) \in Y$ with $(x, k) \neq (y, l)$. Firstly, we assume $x \neq y$ and $k = l$. Since B separates the points of X, there exists $f \in B$ such that $f(x) \neq f(y)$ and so $\bar{f}(x) \neq \bar{f}(y)$. Therefore $\psi(f) \in A$ and we have

$$\psi(f)(x, k) \neq \psi(f)(y, l).$$

Secondly, we assume $k \neq l$. Since the complex-valued constant function i belongs to B, we conclude that $\psi(i) \in A$ and

$$\psi(i)(x, k) = \pm i \neq \mp i = \psi(i)(y, l).$$

Hence, our claim holds.

Now, let $b = (a, j) \in Y$ with $a \in X$ and $j \in \{0, 1\}$. By hypothesis, there are positive numbers M_a and δ_a such that for every $\delta \leq \delta$ there is an element f in B such that

$$f(a) = 1, \qquad \text{... (1)}$$

$$f \equiv 0 \text{ on } S_{\delta}(a) = \{x \in X : d(x, a) = \delta\}, \qquad \text{... (2)}$$

and

$$\sup \left\{ \frac{|f(y) - f(z)|}{d^{\alpha}(y, z)} : y, z \in B_{\delta}(a), y \neq z \right\} < \frac{M_a}{\delta^{\alpha}}, \qquad \text{... (3)}$$

where $B_{\delta}(a) = \{x \in X : d(x, a) \leq \delta\}$. We choose $M_b = M_a$, $\delta_b = \min\left\{\frac{1}{2}, \delta_a\right\}$ and let $\delta \leq \delta_b$. Then

$\delta \le \delta_a$. Let $f \in B$ satisfy (1), (2) and (3) and $g = \psi(f)$. Then $g \in A$ and $g(b) = 1$. Moreover, since $\delta < 1$, $(y, l) \in \tilde{S}_\delta(b) = \tilde{S}_\delta(a, j) = \{(x, k) \in Y : \rho((x, k), (a, j)) = \delta\}$ implies that $l = j$ and so $y \in S_\delta(a)$. Therefore,

$$g \equiv 0 \text{ on } \tilde{S}_\delta(b).$$

Now, let (y, l), $(z, m) \in \tilde{B}_\delta(b) = \tilde{B}_\delta(a, j) = \{(x, k) \in Y : \rho((x, k), (a, j)) \le \delta\}$ with $(y, l) \neq (z, m)$. Since $\delta < 1$, we conclude that $l = m = j$ and so $y \neq z$, and $y, z \in B_\delta(a)$. Therefore, by (3) we have

$$\frac{|g(y, l) - g(z, m)|}{\rho^\alpha((y, l), (z, m))} = \frac{|f(y) - f(z)|}{\rho^\alpha((y, l), (z, m))} = \frac{|f(y) - f(z)|}{d^\alpha(y, z)}$$

$$\le \frac{M_a}{\delta^\alpha} = \frac{M_b}{\delta^\alpha}.$$

Thus

$$\sup\left\{\frac{|g(y, l) - g(z, m)|}{\rho^\alpha((y, l), (z, m)|)} : (y, l), (z, m) \in B_\delta(b), (y, l) \neq (z, m)\right\} \le \frac{M_b}{\delta^\alpha}.$$

Now by Theorem 2.10, we conclude that A is dense in $\mathrm{lip}^\alpha(Y, \tau)$. It follows that B is dense in $\mathrm{lip}^\alpha(X)$.

Bade, Curtis and Dales have proved the complex Lipschitz algebra $\mathrm{Lip}^1(X)$ is dense in $\mathrm{lip}^\alpha(X)$ for $\alpha \in (0, 1)$, by using the measure theory and duality[2], [Corollary 3.7]. Later, this result has proved by using the Hedberg's theorem in complex Lipschitz algebra $\mathrm{lip}^\alpha(X)$ in [5].

As an application of Theorem 2.10, we prove that the real Lipschitz algebra $\mathrm{Lip}^1(X, \tau)$ is dense in $\mathrm{lip}^\alpha(X, \tau)$ for $\alpha \in (0, 1)$ without using the complexification technique.

Corollary 2.12 — Let (X, d) be a compact metric space and τ be a Lipschitz involution on X. If $0 < \alpha < 1$ then $\mathrm{Lip}^1(X, \tau)$ is dense in $(\mathrm{lip}^\alpha(X, \tau), \|\cdot\|_\alpha)$.

PROOF : Let $a \in X$. Then either $\tau(a) = a$ or $\tau(a) \neq a$.

Case (i) — $\tau(a) = a$:

In this case, suppose $\delta > 0$ and define $f : X \to \mathbb{C}$ by

$$f(x) = [1 - \delta^{-1} d(x, a)]\,[1 - \delta^{-1} d(\tau(x), a)]. \qquad \ldots (1)$$

It can easily be shown that $f \in \mathrm{Lip}^1(X) \cap C(X, \tau)$. So, $f \in \mathrm{Lip}^1(X, \tau)$. Also, $f(a) = 1$ and $f(x) = 0$ for every $x \in S_\delta(a) = \{y \in X : d(y, a) = \delta\}$. Now, if $x, y \in B_\delta(a) = \{z \in X : d(a, z) \le \delta\}$ and $x \neq y$, then we have

$$\frac{|f(x) - f(y)|}{d^\alpha(x, y)} = \frac{|[1 - \delta^{-1} d(x, a)]\,[1 - \delta^{-1} d(\tau(x), a)] - [1 - \delta^{-1} d(y, a)]\,[1 - \delta^{-1} d(\tau(y), a)]|}{d^\alpha(x, y)}$$

$$\leq \frac{|[1-\delta^{-1} d(\tau(x),a)]-[1-\delta^{-1} d(\tau(y),a)]||1-\delta^{-1} d(y,a)|}{d^{\alpha}(x,y)}$$

$$+\frac{|[1-\delta^{-1} d(x,a)]-[1-\delta^{-1} d(y,a)]||1-\delta^{-1} d(\tau(x),a)|}{d^{\alpha}(x,y)}$$

$$\leq \frac{\delta^{-1} d(x,y)|1-\delta^{-1} d(\tau(x),\tau(a)|+\delta^{-1} d(\tau(x),\tau(y))}{d^{\alpha}(x,y)}$$

$$\leq \frac{\delta^{-1} d(x,y)[1+\delta^{-1} C d(x,a)]+\delta^{-1} Cd(x,y)}{d^{\alpha}(x,y)}$$

$$\leq \frac{\delta^{-1}(1+C) d(x,y)+\delta^{-1} Cd(x,y)}{d^{\alpha}(x,y)}$$

$$= \delta^{-1}(1+2+C) d^{1-\alpha}(x,y)$$

$$\leq \delta^{-1}(1+2+C)[d(x,a)+d(a,y)]^{1-\alpha}$$

$$= \delta^{-1}(1+2+C)(2\delta)^{1-\alpha}=\frac{(1+2+C)\,2^{1-\alpha}}{\delta^{\alpha}}.$$

Hence, in this case it is enough to consider $M_a=(2+C)\,2^{1-\alpha}$, $0<\delta<\delta_a$ and f given by (1).

Case (ii) — $\tau(a)\neq a$.

Set $\delta_a=\frac{1}{2} d(a,\tau(a))$. Corresponding with every $0<\delta<\delta_a$, we define the complex function f on X by

$$f(x)=[1-\delta^{-1} d(x,a)]\,[1-\delta^{-1} d(\tau(x),a)]\,[1-\delta^{-1} d(\tau(a),a)]^{-1}. \qquad \dots (2)$$

It can easily be shown that $f\in \mathrm{Lip}^1(X)\cap C(X,\tau)$. Thus $f\in \mathrm{Lip}^1(X,\tau)$. Also, $f(a)=1$ and $f(x)=0$ for every $x\in S_\delta(a)=\{y\in X: d(y),a)=\delta\}$. Now, let $x,y\in B_\delta(a)=\{z\in X: d(z,a)\leq\delta\}$ and $x\neq y$, then we have

$$\frac{|f(x)-f(y)|}{d^{\alpha}(x,y)}=\frac{|1-\delta^{-1} d(\tau(a),a)|^{-1}}{d^{\alpha}(x,y)}$$

$$\times|[1-\delta^{-1} d(x,a)]\,[1-\delta^{-1} d(\tau(x),a)]-[1-\delta^{-1} d(y,a)]\;[1-\delta^{-1} d(\tau(y),a)]|$$

$$\leq \frac{|1-\delta^{-1} d(\tau(a),a)|^{-1}}{d^{\alpha}(x,y)}\,\{|[1-\delta^{-1} d(x,a)]+[1-\delta^{-1}(y,a)]||1-\delta^{-1} d(\tau(x),a)|$$

$$+|[1-\delta^{-1} d(\tau(x),a)]-[1-\delta^{-1} d(\tau(y),a)]||1-\delta^{-1} d(y,a)|\}$$

$$\leq \frac{|1-\delta^{-1} d(\tau(a), a)|^{-1}}{d^{\alpha}(x, y)} \{\delta^{-1} d(x, y)[1+\delta^{-1} d(\tau(x), a)]+\delta^{-1} d(\tau(x), \tau(y))\}$$

$$\leq \frac{\delta}{[d(\tau(a), a)-\delta] d^{\alpha}(x, y)} \{\delta^{-1} d(x, y)[1+\delta^{-1} d(\tau(x), a)]+\delta^{-1} C d(x, y)\}$$

$$= \delta^{-1} d^{1-\alpha}(x, y)\left[\frac{d+d(\tau(x), a)}{d(\tau(a), a)-\delta}+\frac{C \delta}{d(\tau(a), a)-\delta}\right]$$

$$\leq \delta^{-1} d^{1-\alpha}(x, y)\left[\frac{2 \operatorname{diam} X}{\frac{1}{2} d(\tau(a), a)}+C\right]$$

$$\leq \delta^{-1}[d(x, a)+d(a, y)]^{1-\alpha} \frac{4 \operatorname{diam} X+C d(\tau(a), a)}{d(\tau(a), a)}$$

$$\leq \delta^{-1}(2 \delta)^{1-\alpha} \frac{4 \operatorname{diam} X+C d(\tau(a), a)}{d(\tau(a), a)}$$

$$= \frac{2^{1-\alpha}[4 \operatorname{diam} X+C d(\tau(a), a)]}{d(\tau(a), a)} \cdot \frac{1}{\delta^{\alpha}}.$$

Hence, in this case we consider $M_a = \frac{2^{1-\alpha}[4 \operatorname{diam} X+C d(\tau(a), a)]}{d(\tau(a), a)}$, $\delta_a = \frac{1}{2} d(\tau(a), a)$ and for every $0<\delta<\delta_a$, f is defined by (2). Therefore, by Theorem 2.9, we conclude that $\operatorname{Lip}^1(X, \tau)$ is dense in $(\operatorname{lip}^{\alpha}(X, \tau), \|\cdot\|_{\alpha})$.

Let (X, d) be a compact metric space. We denote by $\operatorname{lip}^1(X)$ (respectively, $\operatorname{lip}^1_{\mathbb{R}}(X)$) the set of all complex-valued (respectively, real-valued) functions f in Lipschitz algebra $\operatorname{Lip}^1(X)$ for which

$$\frac{|f(x)-f(y)|}{d(x, y)} \rightarrow 0 \text{ as } d(x, y) \rightarrow 0.$$

Then $\operatorname{lip}^1(X)$ ($\operatorname{lip}^1_{\mathbb{R}}(X)$) respectively, is a complex (real, respectively) subalgebras of $\operatorname{Lip}^1(X)$ which contains the constant function 1. Note that $\operatorname{lip}^1(X)$ or $\operatorname{lip}^1_{\mathbb{R}}(X)$) may be only consists constant functions. Let τ be a Lipschitz involution on X and σ be the algebra involution induced by τ on $C(X)$. We define

$$\operatorname{lip}^1(X, \tau) = \{f \in \operatorname{lip}^1(X) : \sigma(f)=f\}.$$

We can prove that $\operatorname{lip}^1(X, \tau)$ is real subalgebra of $\operatorname{lip}^1(X)$ and

$$\operatorname{lip}^1(X) = \operatorname{lip}^1(X, \tau) \oplus i \operatorname{lip}^1(X, \tau),$$

the similar way of the proof of Theorem 2.8.

Definition 2.13 — Let (X, d) be a compact metric space, τ be a Lipschitz involution on X and $\alpha \in (0, 1]$. Let A be a closed real subalgebra of $\mathrm{Lip}^{\alpha}(X, \tau)$, which contains 1 and has the separation property then $A = \mathrm{Lip}^{1}(X)$. It is said to be A has separation property if satisfies the following condition : there exists $a > 1$ such that for any $x, y \in X$, some $f \in A$ satisfies $\|f\|_{\alpha} \leq a$ and $|f(x) - f(y)| = d(x, y)$.

Note that this condition implies τ is an isometric involution on X. Weaver has proved that if A is a complex (real, respectively) closed self adjoint subalgebra of $\mathrm{lip}^{1}(X)$ ($\mathrm{lip}^{1}_{\mathbb{R}}(X)$ respectively)[12] [Theorem 1.4]. We can extend the Weaver's theorem in the real Lipschitz algebra $\mathrm{lip}^{1}(X, \tau)$.

Theorem 2.14 — *Let (X, d) be a compact metric space and τ be an isometric involution on X. Suppose A is a closed real subalgebra of $\mathrm{lip}^{1}(X, \tau)$ which contains 1 and has the separation property. Then $A = \mathrm{lip}^{1}(X, \tau)$.*

PROOF : Let $B = \{f + ig : f, g \in A\}$. It is easy to see that B is a self-adjoint complex closed subalgebra of $\mathrm{lip}^{1}(X)$ which contains 1 and has the separation property as a subset of $\mathrm{lip}^{1}(X)$. Therefore, $B = \mathrm{lip}^{1}(X)$ by Weaver's theorem. We now claim $A = \mathrm{lip}^{1}(X, \tau)$. Suppose $f \in \mathrm{lip}^{1}(X, \tau)$. Then $f \in \mathrm{lip}^{1}(X) = B$. Hence, there exist $g, h \in A$ such that $f = g + ih$. Since $\mathrm{lip}^{1}(X) = \mathrm{lip}^{1}(X, \tau) \oplus i\, \mathrm{lip}^{1}(X, \tau)$ and $f \in \mathrm{lip}^{1}(X, \tau)$, it must to be $h = 0$ and $f = g$. Hence, $f \in A$ and the proof is complete.

REFERENCES

1. Davood Alimohammadi and Ali Ebadian, *Far East J. Math. Sci.* **2**(2), (2000) 181-86.
2. W. G. Bade, P. C. Curtis, Jr., and H. G. Dales, *Proc. Lond. Math. Soc.*, **55** (1987) 359-77.
3. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, 1973.
4. L. I. Hedberg, *Ark. Math.* **8** (1969) 63-72.
5. T. G. Honary and H. Mahyar, *Queastions Mathematicae,* **23** (2000) 13-19.
6. L. Ingelstam, *Ark. Math* **5** (1964) 239-70.
7. J. A. Johnson, *Trans. Amer. Math. Soc.* **148** (1970) 147-69.
8. J&A. Johnson, *Pacific J. Math.* **51** (1974) 177-86.
9. S. H. Kulkarni and B. V. limaye, *Real Function Algebras*, Marcel Dekker, Inc. 1992.
10. E. Mayer-Wolf, *Israel. J. Math.* **38** (1981) 58-74.
11. D. R. Sherbert, *Trans. Amer. math. Soc.* **111** (1964) 240-72.
12. N. Weaver, *Pacific J. Math.* **173** (1996) 293.
13. N. Weaver, *Lipschitz Algebras*, World Scientific, New Jersey, 1999.