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STRONGLY NA-PRECONTINUOUS FUNCTIONS

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In this paper, we define a function $f: X \to Y$ to be strongly na-precontinuous if $f^{-1}(V)$ is δ-open in X for each preopen set V of Y. We obtain some characterizations of such functions and the relationship among strong na-precontinuity and other strong forms of continuity.

Key Words : δ-open; α-open; na-Continuous; Precontinuous, Strongly na-Precontinuous.

1. INTRODUCTION

In 1986, Chae *et al.*[3] defined a function $f: X \to Y$ to be na-continuous if $f^{-1}(V)$ is δ-open in X for every feebly open set V of Y. Recently, Rose and Mahmoud[18] have defined a function to be strongly M-precontinuous if the inverse image of each preopen set is an open set. Quite recently, Mahmoud *et al.*[10] have introduced the notion of strongly na-continuous functions. The purpose of the present paper is to introduce and study the notion of strongly na-precontinuous functions. It will turn out that strong na-precontinuity is stronger than both na-continuity and strong M-precontinuity. In Section 3, we obtain some characterizations and several properties of strongly na-precontinuous functions. In Section 4, we investigate the relationship among several strong forms of continuity. In the last section, it is shown that if $f: X \to Y$ is strongly na-precontinuous surjection and X is connected (resp. nearly compact) then Y is preconnected (resp. strongly compact).

2. PRELIMINARIES

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces on which no separation axiom is assumed unless explicitly stated. Let A be a subset of a topological space X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1 — A subset A of a space X is said to be

(*a*) *semi-open*[6] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,

(*b*) *preopen*[11] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, (*c*) *α-open*[14] if $A \subset \text{Int}(\text{Cl}(A)))$,

(*d*) *regular open* if $A = \text{Int}(\text{Cl}(A))$,

(*e*) *δ-open*[20] if for each $x \in A$, there exists a regular open set U of X such that $x \in U \subset A$

(*f*) *feebly open*[8] if there exists an open set U of X such that $U \subset A \subset s\text{Cl}(U)$, where $s\text{Cl}(U)$ denotes the semi-closure[4] of U.

The following lemma is shown in[16] [Lemma 3.2] and the terms *feeble open* and *α-open* will be similarly used in the sequel.

Lemma 2.1 (Noiri) — A subset A of a space X is α-open if and only if it is feebly open.

For a space (X, τ), the collection of all δ-open sets of (X, τ) forms a topology for X which is usually called the *semiregularization* of τ and is denoted by τ_s. The family of all regular open (resp. preopen, semi-open, δ-open, α-open) sets of a space (X, τ) is denoted by $RO(X)$ (resp. $PO(X)$, $SO(X)$, $\delta(X)$, $\alpha\ (X)$ or τ^{α}). The family of regular open (resp. preopen) sets of (X, τ) containing $x \in X$ is denoted by $RO(X, x)$ (resp. $PO(X, x)$). The complement of a regular open (resp. semi-open, preopen, δ-open, α-open) set is said to be *regular closed* (resp. *semi-closed, preclosed, δ-closed, α-closed*). The intersection of all preclosed (resp. δ-closed) sets containing a subset A of X is called the *preclosure* (resp. *δ-closure*) of A and is denoted by $\text{pCl}(A)$ (resp. $\text{Cl}_{\delta}(A)$)

3. CHARACTERIZATIONS

In this section, we obtain several characterizations and basic properties of strongly na-precontinuous functions.

Definition 3.1 — A function $f: X \to Y$ is said to be *strongly na-precontinuous* (briefly *st.n.p.c.*) at a point $x \in X$ if for each $V \in PO(Y, f(x))$, there exists $U \in RO\,(X, x)$ such that $f(U) \subset V$. A function is said to be strongly na-precontinuous if it has the property for each point of the domain.

Theorem 3.1 — *The following are equivalent for a function* $f: X \to Y$*:*

(a) f *is strongly na-precontinuous;*

(b) for each $x \in X$ *and each* $V \in PO\,(Y, f(x))$*, there exists* $U \in \delta(X)$ *containing* x *such that* $f(U) \subset V$*;*

(c) $f^{-1}(V) \in \delta(X)$ *for every* $V \in PO\,(Y)$*;*

(d) $f^{-1}(F)$ *is δ-closed in* X *for every preclosed set* F *of* Y*;*

(e) $f(Cl_{\delta}(A)) \subset pCl\,(f(A))$ *for every subset* A *of* X*;*

(f) $Cl_{\delta}(f^{-1}(B)) \subset f^{-1}(p\,Cl\,(B))$ *for every subset* B *of* Y.

PROOF : This is shown by the usual technique and the proof is thus omitted.

Lemma 3.1 (Chae and Noiri)[2] — Let A be either dense or open in a space (X, τ). If U is a regular open set of (X, τ), then $A \cap U$ is regular open in the subspace $(A, \tau/A)$.

Theorem 3.2 — *If a function* $f: (X, \tau) \to (Y, \sigma)$ *is st.n.p.c. and* A *is either dense or open in* (X, τ)*, then the restriction* $f/A : (A, \tau/A) \to (Y, \sigma)$ *is st.n.p.c.*

PROOF : Let V be any preopen set of (Y, σ). Since f is *st.n.p.c*, $f^{-1}(V)$ is δ-open in X. By Lemma 3.1, $f^{-1}(V) \cap A$ is δ-open in the subspace $(A, \tau/A)$ and $f^{-1}(V) \cap A = (f/A)^{-1}(V)$. This shows that f/A is *st.n.p.c.*

Let $\{X_\lambda : \lambda \in \Lambda\}$ and $\{Y_\lambda : \lambda \in \Lambda\}$ be two families of topological spaces with the same index set Λ. The product space of $\{X_\lambda : \lambda \in \Lambda\}$ is denoted by $\Pi\ \{X_\lambda : \lambda \in \Lambda\}$ (or simply ΠX_λ). Let $f_\lambda : X_\lambda \to Y_\lambda$ be a function for each $\lambda \in \Lambda$. The product function $f : \Pi X_\lambda \to \Pi Y_\lambda$ is defined by $f(\{x_\lambda\}) = \{f_\lambda(x_\lambda)\}$ for each $\{x_\lambda\} \in \Pi X_\lambda$.

Lemma 3.2 (El-Deeb *et al.*)[5] — $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces, U_{λ_i} a subset of a space X_{λ_i} for each $i = 1, 2, ..., n$ and

$$U = \prod_{i=1}^{n} U_{\lambda_i} \times \prod_{\lambda \neq \lambda_i} X_\lambda .$$

Then U is preopen in ΠX_λ if and only if $U_{\lambda_i} \in PO(X_{\lambda_i})$ for each $i = 1, 2, ..., n$.

Let $f : X \to Y$ be a function. A function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is called the *graph function* of f.

Theorem 3.3 — *A function $f : X \to Y$ is st.n.p.c. if the graph function $g : X \to X \times Y$ is st.n.p.c.*

PROOF : Let $x \in X$ and $V \in PO(Y, f(x))$. Then $X \times V$ is a preopen set of $X \times Y$ containing $g(x)$ by Lemma 3.2. Therefore, there exists $U \in RO(X, x)$ such that $g(U) \subset X \times V$. This implies that $f(U) \subset V$. This shows that f is *st.n.p.c.*

Theorem 3.4 — *If a function $f : X \to \Pi Y_\lambda$ is st.n.p.c., then $p_\lambda \circ f : X \to Y_\lambda$ is st.n.p.c. for each $\lambda \in \Lambda$, where P_λ is the projection of ΠY_λ onto Y_λ.*

PROOF : Let V_λ be any preopen set of Y_λ. By Lemma 3.2, $P_\lambda^{-1}(V_\lambda)$ is preopen in ΠY_λ and hence $(P_\lambda \circ f)^{-1}(V_\lambda) = f^{-1}(P_\lambda^{-1}(V_\lambda))$ is δ-open in X. Therefore, $P_\lambda \circ f$ is *st.n.p.c.*

Theorem 3.5 — *If a function $f : \Pi X_\lambda \to \Pi Y_\lambda$ is st.n.p.c., then $f_\lambda : X_\lambda \to Y_\lambda$ is st.n.p.c. for each $\lambda \in \Lambda$.*

PROOF : Let V_λ be any preopen set of Y_λ. Then, by Lemma 3.2, $P_\lambda^{-1}(V_\lambda)$ is preopen in ΠY_λ. Since f is *st.n.p.v.*, $f^{-1}(P_\lambda^{-1}(V_\lambda))$ is δ-open in ΠX_λ and $f^{-1}(P_\lambda^{-1}(V_\lambda)) = f_\lambda^{-1}(V_\lambda) \times \Pi\ \{X_\alpha : \alpha \in \Lambda - \{\lambda\}\}$. Therefore, it follows from [3, Lemma 3.1] $f_\lambda^{-1}(V_\lambda)$ is δ-open in X_λ and hence f_λ is *st.n.p.c.*

4. COMPARISONS

In this section, we investigate the relationship among strongly na-precontinuous functions and other related functions.

Definition 4.1 —.A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *strongly semi-continuous*[1] (resp. *strongly M-precontinuous*[18], *strongly α-irresolute*[7 & 9]) if $f^{-1}(V)$ is open in (X, τ) for every

$V \in SO(Y)$ (resp. $V \in PO(Y)$, $V \in \sigma^{\alpha}$).

Definition 4.2 — A function $f: X \to Y$ is said to be *na-continuous*[3] (resp. *strongly na-continuous*[10]) if $f^{-1}(V)$ is δ-open in X for every feebly open (resp. semi-open) set V of Y.

Remark 4.1 : The following relationship hold among the six strong forms of continuity defined above :

(a) both strong *na*-continuity and strong na-precontinuity imply na-continuity and the converses are not true as shown by Example 4.1 below;

(b) both strong semi-continuity and strong M-precontinuity imply strong α-irresoluteness and the converses are not true as shown by Example 4.2 below; and

(c) strong na-precontinuity (resp. strong *na*-continuity, *na*-continuity) implies strong M-precontinuity (resp. strong semi-continuity, strong α-irresoluteness) and the converses are not true as shown by example 4.3 below.

Example 4.1 — Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$. Let $f: (X, \tau) \to (X, \tau)$ be a function defined as follows : $f(a) = f(b) = a$, $f(c) = c$ and $f(d) = d$. Then f is na-continuous since $\delta(X, \tau) = \tau = \tau^{\alpha}$. It is neither strongly na-continuous nor *st.n.p.c.* Because there exist $\{c, d\} \in SO(X, \tau)$ and $\{b, c, d\} \in PO(X, \tau)$ such that $f^{-1}(\{c, d\})$ and $f^{-1}(\{b, c, d\})$ are not δ-open in (X, τ).

Example 4.2 — Let $X = \{a, b, c, d\}$, $\sigma = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\tau = \{\phi, \{a, b, c\}, X\}$. Define a function $f: (X, \tau) \to (X, \sigma)$ as follows $f(a) = f(b) = b$, $f(c) = c$ and $f(d) = d$. Then f is strongly α-irresolute [7, Example 2.2] and it is neither strongly semi-continuous nor strongly M-precontinuous. Because, there exist $\{a, d\} \in SO(X, \sigma)$ and $\{a, b\} \in PO(X, \sigma)$ such that $f^{-1}(\{a, d\})$ and $f^{-1}(\{a, b\})$ are not open in (X, τ).

Example 4.3 — Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, c\}, X\}$. The identity function $f: (X, \tau) \to (X, \sigma)$ is strongly semi-continuous, strongly M-precontinuous and hence strongly α-irresolute because $PO(X, \sigma) = SO(X, \sigma) = \sigma^{\alpha} = \{a, b\} \cup \sigma$. However, f is not na-continuous because there exists $\{a, b\} \in \sigma^{\alpha}$ such that $f^{-1}(\{a, b\})$ is not δ-open in (X, τ).

A space X is said to be *submaximal* if every dense subset of X is open in X. A space (X, τ) is said to be *extremally disconnected* (briefly E.D.) if $Cl(U) \in \tau$ for every $U \in \tau$.

Theorem 4.1 — *Let Y be a submaximal space. Then a function $f: X \to Y$ is st.n.p.c. if and only if it is strongly M-precontinuous.*

PROOF : This follows immediately from the fact that if a space (X, τ) is submaximal then $PO(X, \tau) = \tau$ [18, Lemma 3].

Definition 4.3 — A function $f: X \to Y$ is said to be *super continuous*[13] if for each $x \in X$ and each neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(Int(Cl(U))) \subset V$.

It is shown in [13, Theorem 2.1] that a function $f: X \to Y$ is super continuous if and only if $f^{-1}(V)$ is δ-open in X for every open set V of Y. Super continuity implies continuity and is implied by na-continuity. However, the converses are not always true as shown by [13, Example 2.1] and [3, Example 4.2].

Theorem 4.2 — *Let (Y, σ) be a submaximal and E.D. space. Then the following are equivalent for a function $f: (X, \tau) \to (Y, \sigma)$:*

(a) f is st.n.p.c.;

(b) f is strongly na-continuous;

(c) f is na-continuous;

(d) f is super continuous.

PROOF : Since (Y, σ) is submaximal, PO $(X, \sigma) = \sigma$ [18; Lemma 3]. Let $V \in SO(Y, \sigma)$. Since (Y, σ) is E.D., we have Cl(V) = Cl (Int (V) $\in \sigma$ and hence $V \subset Cl(V) =$ Int (Cl (V)). Therefore, we obtain $V \in PO(X, \sigma)$ and hence $V \in \sigma$. Consequently, we obtain σ = PO (X, σ) = SO (X, σ). It is also shown in [15, Lemma 3.1] that σ^α = PO $(X, \sigma) \cap$ SO (X, σ). Therefore, we have $\sigma = \sigma^\alpha$ and the four statements are equivalent.

For any space (X, τ), let τ_p be the smallest topology containing PO (X). Then, we have $\tau \subset PO(X, \tau) \subset \tau_p$. The following two theorems are easily obtained and the proofs are thus omitted.

Theorem 4.3 — *The following properties are equivalent for a function $f: (X, \tau) \to (Y, \sigma)$:*

(a) $f: (X, \tau) \to (Y, \sigma)$ is st.n.p.c.;

(b) $f: (X, \tau) \to (Y, \sigma_p)$ is super continuous;

(c) $f: (X, \tau_s) \to (Y, \sigma_p)$ is continuous;

(d) $f: (X, \tau_s) \to (Y, \sigma)$ is strongly M-precontinuous.

Theorem 4.4 — *Let $f: X \to Y$ and $g: Y \to Z$ be functions. The composition $g \circ f: X \to Z$ is st.n.p.c. if f and g satisfy on at the following conditions:*

(a) f and g are st.n.p.c.,

(b) f is super continuous and g is strongly M-precontinuous.

5. STRONG na-PRECONTINUITY AND SOME SPACES

Definition 5.1 — A space X is said to be

(*a*) *preconnected*[17] if it cannot be expressed as the union of two nonempty disjoint preopen sets,

(*b*) *strongly compact*[12] (resp. *nearly compact*[19]) if every preopen (resp. regular open) cover of X has a finite subcover.

Theorem 5.1 — *If $f: X \to Y$ is a st.n.p.c. surjection and X is connected, then Y is preconnected.*

PROOF : Suppose that Y is not preconnected. There exist nonempty pre-open sets V_1 and V_2 of Y such that $Y = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$. Since f is *st.n.p.c.*, $f^{-1}(V_1)$ and $f^{-2}(V_2)$ are nonempty δ-open sets of X such that $X = f^{-1}(V_1) \cup f^{-1}(V_2)$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Since every δ-open set is open, X is not connected.

Theorem 5.2 — *If $f: X \to Y$ is a st.n.p.c. surjection and X is nearly compact, then Y is strongly compact.*

PROOF : Let $\{V_\alpha : \alpha \in \mathcal{A}\}$ be a preopen cover of Y. Since f is *st.n.p.c.* $\{f^{-1}(V_\alpha) : \alpha \in \mathcal{A}\}$ is a δ-open cover of X. Since X is nearly compact and each $f^{-1}(V_\alpha)$ is the union of regular open sets, there exists a finite subset $\mathcal{A}_l$ of $\mathcal{A}$ such that X = $\{f^{-1}(V_\alpha) : \alpha \in \mathcal{A}_l\}$. Therefore, we obtain Y = $\bigcup\{V_\alpha : \alpha \in \mathcal{A}_l\}$ and hence Y is strongly compact.

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