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FUZZY DEFORMATION RETRACT OF FUZZY HOROSPHERES

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In this paper, we will introduce the definition of fuzzy horosphere of the fuzzy Lobachevskian space. The fuzzy retractions and fuzzy deformation retract are discussed. The relation between the fuzzy horosphere and fuzzy R^{n-1} is obtained. Some theorems which describe these relations are achieved.

Key Words : Fuzzy Horospheres; Fuzzy Deformation Retract

INTRODUCTION AND DEFINITIONS

The Deformation retract of hypersphere S^n introduced by Afwat Abd El-Maged, El-Ghoul and El-Ahmady[2], also the deformation retract of covering space of hypertori discussed by El-Ghoul[5], and more studies on the deformation retract of V_4 space and its topological folding were introduced by El-Ghoul and El-Ahmady[6].

Definition 1 — Topological folding of a fuzzy spheres[7]. A map $F = \tilde{S} \to \tilde{S}$, where $\tilde{S}$ is a fuzzy sphere, is said to be an isometric folding of $\tilde{S}$ into itself, iff for any piecewise geodesic path $\gamma : J \to \tilde{S}$, the induced path $F \circ \gamma : J \to \tilde{S}$ is a piecewise fuzzy geodesic and of the same length as γ, where $J = [0, 1]$. If F does not preserve length, then F is a topological folding of fuzzy spheres.

Definition 2 — A fuzzy subset $(\underset{\sim}{A}, \mu)$ of a fuzzy manifold $(\underset{\sim}{M}, \mu)$ is called a fuzzy retraction of $(\underset{\sim}{M}, \mu)$ if there exist a continuous map $\tilde{r} : (\underset{\sim}{M}, \mu) \to (\underset{\sim}{A}, \mu)$ such that $\tilde{r}(a, \mu(a)) = (a, \mu(a))$, $\forall$ $a \in \underset{\sim}{A}$, $\mu \in [0, 1]$[7].

There are three cases —

(*i*) $\tilde{r}(a, \mu(a)) = (a, \max \mu)$

(*ii*) $\tilde{r}(a, \mu(a)) = (a, \min \mu)$

(*iii*) $\tilde{r}(a, \mu(a)) = (a, \mu \in (0, 1))$

Definition (3) — A fuzzy subset $(\underset{\sim}{\overline{M}}, \overline{\mu})$ of a fuzzy manifold $(\underset{\sim}{M}, \mu)$ is called a fuzzy deformation retract if there exist a fuzzy retraction $\tilde{r} : (\underset{\sim}{M}, \mu) \to (\underset{\sim}{\overline{M}}, \overline{\mu})$ and a fuzzy homotopy $\tilde{F}$:

$(\underset{\sim}{M}, \mu) \times I \to (\underset{\sim}{M}, \mu)$ such that

$$\left.\begin{array}{l}\tilde{F}((x,\mu),0)=(x,\mu)\\ \tilde{F}((x,\mu),1)=\tilde{r}(x,\mu)\end{array}\right\} x \in \underset{\sim}{M}$$

$$\tilde{F}((a,\mu),t)=(a,\mu),\ \forall\ (a,\mu)\in \overline{M}, t\in I, \mu\in[0,1]$$

where $\tilde{r}(x,\mu)$ is any retraction mentioned above[7].

MAIN RESULTS

To obtain the main results we will introduce the following definitions : -

Definition (1) — The fuzzy line in the fuzzy Lobachevskian plane is the fuzzy Lobachevskian line attached with it $\mu \in (-\infty, \infty)$ otherwise it has no meaning.

The difference between the fuzzy line and fuzzy Lobachevskian line is that the line in Euclidean space attached with $\mu \in [a, b], a \neq \infty, b \neq \infty$[4].

Definition 2 — A fuzzy differentiable manifold is a C^{∞}-n-dimesional which has a physical character represented by the density function $\mu, \mu \in [0, 1]$[8].

Definition 3 — The fuzzy horocycle is defined also as fuzzy Riemannian circles with infinite radii and centers at infinity[1 & 4].

Definition 4 — The fuzzy horosphere in fuzzy Lobachevskian space is generated by a fuzzy horocycle rotating around of its axes[1&4].

Theorem (1) — *The fuzzy horosphere $\tilde{H}^2$ in fuzzy Lobachevskian space is homeomorphic to the whole fuzzy Lobachevskian plane $\tilde{L}^2$.*

PROOF : Let us choose the frame of reference $\left\{0, \tilde{x}_1, \tilde{x}_2\right\}$ in $\tilde{L}^2$ such that for each fuzzy point $\tilde{M} \in \tilde{L}^2, \tilde{M}(\tilde{x}_1, \tilde{x}_2)$ where

$$\tilde{x}_1 = \tanh \frac{\tilde{\eta}_1}{\tilde{R}} \text{ and } \tilde{x}_2 = \tanh \frac{\tilde{\eta}_2}{\tilde{R}}, \text{ see Fig. (1)}$$

Let $\tilde{H}^2$ be a fuzzy horosphere touching $\tilde{L}^2$ at 0. Consider the Beltrami map $\beta : \tilde{H}^2 \to \tilde{L}^2$ such that

$$\forall\ (\tilde{x}_1', \tilde{x}_2') \in \tilde{H}^2,\ \beta(\tilde{x}_1', \tilde{x}_2') = (\tilde{R}\tilde{x}_1, \tilde{R}\tilde{x}_2).$$

Let the fuzzy points $(\breve{x}_1, \breve{x}_2) \in \tilde{H}^2$, see Fig. (2), where $\tilde{x}_1' + \tilde{x}_2' = \tilde{R}^2$, correspond under β to the fuzzy points at infinity in $\tilde{L}^2$, i.e.,

$$\beta : \tilde{x}_1'2 + \tilde{x}_2'2 = \tilde{R}^2 \to \infty.$$

Then under Beltrami map the part of $\tilde{H}^2$ bounded by $\tilde{x}_1'2 + \tilde{x}_2'2 = \tilde{R}^2$ is considered as a fuzzy disc with radius of curvature $\tilde{R}$ and center at 0. On the fuzzy disc the frame of reference

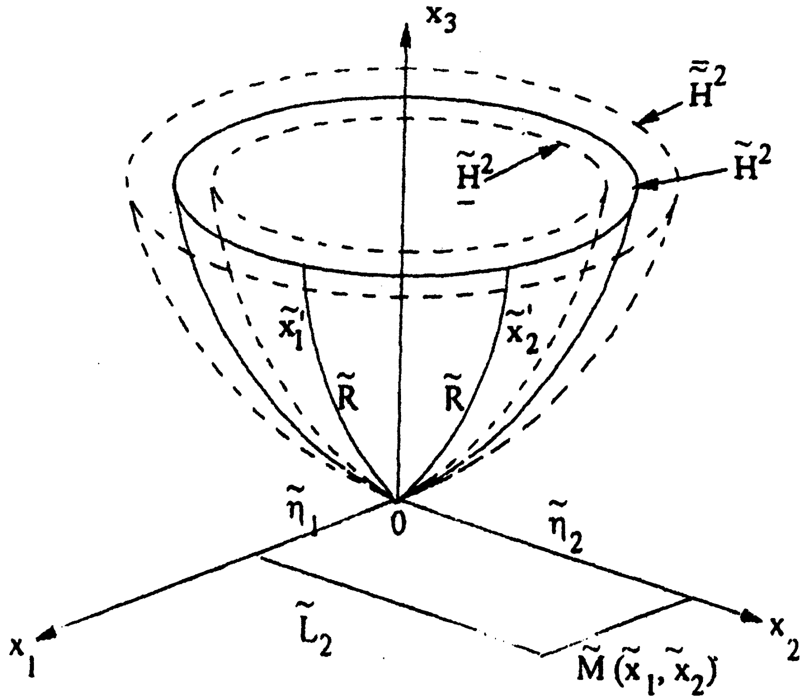

FIG. 1

$\left\{0, \tilde{x}_1, \tilde{x}_2\right\}$ corresponds to the frame $\left\{0, \tilde{x}_1, \tilde{x}_2\right\} \subset \tilde{L}^2$ where $0\,\tilde{x}_1'$ is an arc of fuzzy horocycle of length $\tilde{R}$. Each point $\tilde{M}(\tilde{x}_1', \tilde{x}_2')$ on the fuzzy disc satisfies the relation

$$\tilde{x}_1'\,2 + \tilde{x}_2'\,2 < \tilde{R}^2$$

It is clear from Beltrami map that the fuzzy disk $\tilde{K}$, For which

$$\tilde{K} = \left\{ (\tilde{x}_1' + \tilde{x}_2') : \tilde{x}_1'\,2 + \tilde{x}_2'\,2 \leqq \tilde{R}^2 \right\},$$

is homeomorphic to the whole fuzzy Lobacheviskian plane $\tilde{L}_2$.

Theorem 2 — *The fuzzy deformation retract of the fuzzy Lobacheviskian plane* $\tilde{L}_2$ *onto* $\tilde{B}_1 \subset \tilde{L}^2$ *induces the fuzzy deformation retract of the fuzzy disc* $\tilde{K}$ *in fuzzy horosphere* $\tilde{H}^2$.

PROOF : Consider the fuzzy disc $\tilde{K}_1$,

$$\tilde{K}_1 = \left\{ (\tilde{x}_1', \tilde{x}_2') : \tilde{x}_1'\,2 + \tilde{x}_2'\,2 \leqq \tilde{R}^2, \tilde{R}_1 < \tilde{R} \right\}$$

with the boundary $\tilde{B}$ such that

$$\tilde{B} = \left\{ (\tilde{x}_1', \tilde{x}_2') : \tilde{x}_1'\,2 + \tilde{x}_2'\,2 = \tilde{R}_1^2 \right\}.$$

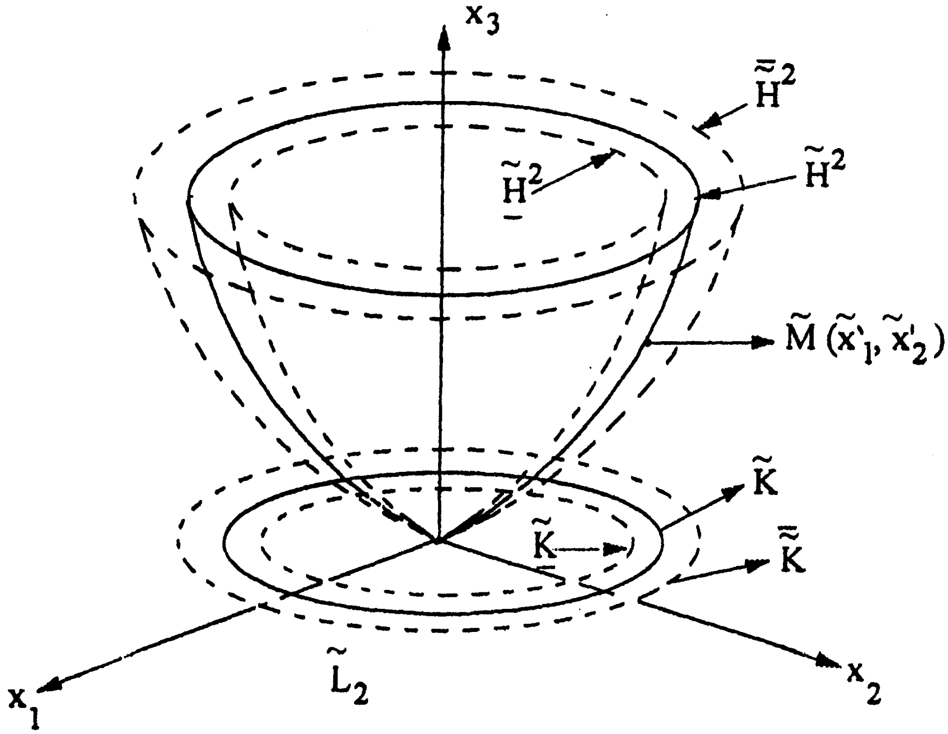

FIG. 2

Then we obtain

$$\beta(\tilde{K}_1) = \left\{ (\tilde{x}_1, \tilde{x}_2) : \tilde{x}_1^2 + \tilde{x}_2^2 \leqq \frac{\tilde{R}_1^2}{\tilde{R}^2} \subset \tilde{L}^2 \right.$$

with the boundary

$$(\tilde{B}_1) = \left\{ (\tilde{x}_1, \tilde{x}_2) : \tilde{x}_1^2 + \tilde{x}_2^2 = \frac{\tilde{R}_1^2}{\tilde{R}^2} \right\} \text{ and } \beta(\tilde{B}_1) = \tilde{B}.$$

We can define the fuzzy deformation retract as follows :

$$\tilde{\Phi} : \{\{\tilde{L}^2 - 0\},\ \mu\} \times I -. (\tilde{L}^2, \mu) \text{ such that}$$

$$\tilde{\Phi} : \{\{m, \mu\},\ t = (1-t)(m, \mu) + t\frac{(m, \mu)}{\alpha \| m, \mu \|},$$

where $(m, \mu) = (\tilde{x}_1, \tilde{x}_2),\ \alpha = \frac{\tilde{R}_1}{\tilde{R}},\ t \in I,\ \mu \in [0, 1]$

with $\tilde{\Phi}((m, \mu), 0) = (m, \mu)$ and $\tilde{\Phi}((m, \mu), 1) = \frac{(m, \mu)}{\alpha \| m, \mu \|}.$

Hence, we can induce the fuzzy deformation retract

$$\tilde{\Phi}' : \{\{\tilde{K} - 0\},\ \mu\} \times I \to \{\tilde{K}, \mu\}$$

such that

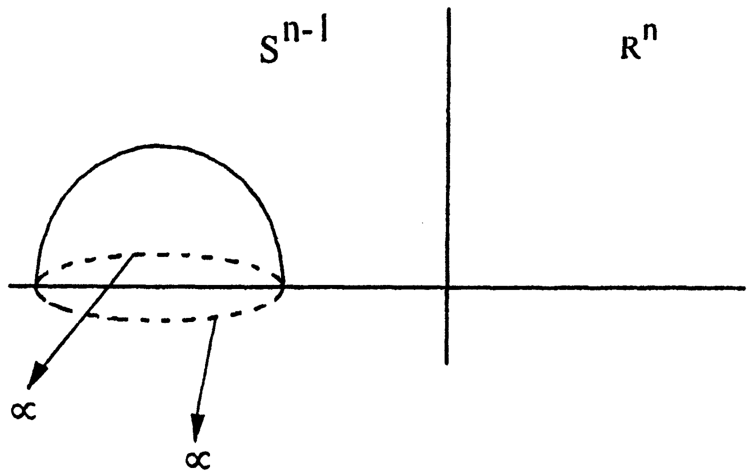

FIG. 3

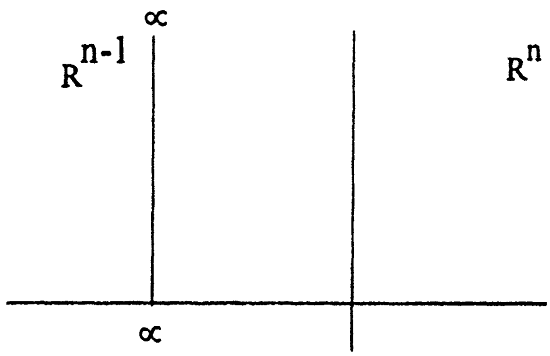

FIG. 4

$$\tilde{\Phi} : \{\{m', \mu\}, t\} = (1 - t)\ (m', \mu) + t \frac{(m', \mu)}{\alpha \| m', \mu \|},$$

where $\tilde{\beta}^{-1} \circ \tilde{\Phi}((m, \mu), 1) = \tilde{\Phi}' (\tilde{\beta}^{-1} (m, \mu), 1)$

and $\tilde{\beta}^{-1} \circ \tilde{\Phi}((m, \mu), 0) = \tilde{\Phi}' (\tilde{\beta}^{-1} (m, \mu), 0).$

Theorem 3 — *Every fuzzy retraction of fuzzy horosphere $\tilde{H}^2$ onto $\tilde{A}$ induces fuzzy retraction of fuzzy Lobacheviskian plane $\tilde{L}^2$ onto $\tilde{B}$ such that $\tilde{A}$ homeomorhic to $\tilde{B}$.*

PROOF : Let $\tilde{H}^2$ be a fuzzy horosphere with fuzzy retraction $\tilde{r}(\tilde{H}^2) = \tilde{A}$. Also let us consider the fuzzy retraction of fuzzy Lobacheviskian given by $\tilde{r}(\tilde{L}^2) = \tilde{B}$. Since the fuzzy horosphere is homeomorphic to the whole fuzzy Lobacheviskian plane, then $\tilde{A}$ homeomorphic to $\tilde{B}$.

Theorem 4 — *The fuzzy horospheres of a compact n-dimensional fuzzy manifold $\tilde{W}$ are diffeomorphic to fuzzy (n – 1) dimensional Euclidean space $\tilde{R}^{n-1}$ or diffeomorphic to a fuzzy semi hypersphere in $\tilde{R}^n$.*

PROOF : Let us consider the fuzzy horosphere, $\tilde{H}$. It is a paracompact fuzzy manifold. Let $\tilde{S}$ be a fuzzy compact subset of $\tilde{H}$, then $\tilde{\Phi}_t \tilde{S}$ is covered by fuzzy a disk $\tilde{D}$ of $\tilde{\Phi}\tilde{H}$. Since $\tilde{\Phi}^{-1} \tilde{D}$ is a disk which covers $\tilde{S}$ in $\tilde{H}$. Therefore, $\tilde{H}$ is diffeomorphic to $\tilde{R}^{n-1}$. See Fig. (3).

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