Indian J. pure appl. Math., 32(10) : 1513-1518, October 2001

FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS IN D-METRIC SPACES

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(*Received* 1 *January* 2001; *accepted* 9 *April* 2001)

In this paper, we define expansive mappings in the setting of D-metric spaces analogous to expansive mappings (Iseki[8]) in complete metric spaces. We also extend the results of Hid kaneko[1] for two mappings in the setting of D-metric spaces.

Key Words : Fixed Point; Expansive Mappings; D-Metric Spaces

INTRODUCTION

In 1992, a new structure of a generalised metric space was introduced by Dhage[2] on the lines of ordinary metric space defined as under:

Let R denote the real line and X denote a nonempty set. Let $D : X \times X \times X \to R$ be a function satisfying the following properties :

(*i*) $D(x, y, z) \geq 0$ for all $x, y, z \in X$, equality holds if and only if $x = y = z$.

(*ii*) $D(x, y, z) = D(y, x, z) = \ldots$ (symmetry).

(*iii*) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$. (rectangle inequality).

Then the function D is called a D-metric (also called a generalised metric) on X. The set X together with a D-metric D is called a D-metric space denoted by (X, D). Generally, the usual ordinary metric is called the distance function. D-metric is called diameter function of the points of X (Dhage[2]).

In 1963, Ghaler[4] introduced the concept of a 2-metric space. Recall (Ghaler[4]) that a function $d : X \times X \times X \to R$ is called a 2-metric on X satisfying the properties (*ii*), (*iii*) above and (*i*)$'$:

(*i*)$'$ For each distinct pair $x, y \in X$, there exists $z \in X$ such that

> $d(x, y, z) \neq 0$ and $d(x, y, z) = 0$ if any two elements of the triplet x, y, z are equal.

A number of fixed point theorems have been proved for 2-metric spaces. However, Hsiao[5] showed that all such theorems are trivial in the sense that the iterations of f are all collinear. Fortunately, in case of D-metric spaces the situation is not alike.

In this paper, we define expansive mappings in the setting of D-metric spaces analogous to expansive mappings (Iseki[8]) in complete metric spaces. We also extend the results of Hideaki Kaneko[1] for two mappings in the setting of D-metric spaces.

PRELIMINARIES

Definition 2.1 (Dhage[2]) — A sequence $\{x_n\}$ of points of a D-metric space X is said to be D-convergent and converges to a point $x \in X$, if for every $\varepsilon > 0$ there exists an $n_0 \in N$ such that for all $m, n \geq n_0, D(x_m, x_n, x) < \varepsilon$.

Definition 2.2 (Dhage[2]) — A sequence $\{x_n\}$ of points of a D-metric space X is said to be D-Cauchy if for $\varepsilon > 0$ there exists an $n_0 \in N$ such that for all $m > n$, $p \geq n_0$, $D(x_m, x_n, x_p) < \varepsilon$.

A complete D-metric space X is one in which every D-Cauchy sequence $\{x_n\}$ in X converges to a point x in X. A mapping $T : X \to X$ is said to be continuous if and only if $Tx_n \to Tx$ whenever $x_n \to x$ (Dhage[2]).

Definition 2.3 (Dhage[2]) — Let $x_n \in X$ be fixed and $\varepsilon < 0$ be given. We define an open ball $B(x_0, \varepsilon)$ in X with centre at x_0 of radius ε by $B(x_0, \varepsilon) = \bigcap_{z \in X} \left\{ y, z \in X : D(x_0, y, z) < \varepsilon \right\}$

Then the collection of all open balls $\{B(x, \varepsilon) : x \in X\}$ defines the topology on X denoted by τ. A D-metric space X is said to be precompact if there exists a finite number of elements $x_1, \ldots, x_n$ in X such that $X \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. If X is a compact D-metric space, then every sequence has a convergent subsequence and every continuous real function on X attains its maximum or minimum on it. In the sequel, we assume that X is equipped with the topology τ which is the same topology of D-metric space.

The following lemma is useful in the sequel :-

Lemma 2.1 (Dhage[2]) — The D-metric 'D' is a continuous function on X^3 in the topology of D-metric convergence.

MAIN RESULTS

***Theorem* 3.1** — *Let X be a complete D-metric space. If there exists a constant $a > 1$ such that*

$$D(Tx, Ty, Tz) \geq a\, D(x, y, z) \quad \forall\, x, y, z \in X \qquad \ldots (1)$$

and T is surjective, then T has a unique fixed point.

PROOF : Under the assumption, it is clear that T is injective. Let G be the inverse mapping of T. Therefore

$$D(x, y, z) = D(TT^{-1}x, TT^{-1}y, TT^{-1}z)$$

$$\geq a\, D(T^{-1}x, T^{-1}y, T^{-1}z) = a\, D(Gx, Gy, Gz)$$

or

$$D(x, y, z) \geq a\, D(Gx, Gy, Gz)$$

which implies $D(Gx, Gy, Gz) \leq \frac{1}{a} D(x, y, z)$

or $D(Gx, Gy, Gz) \leq h\, D(x, y, z)$ for $h < 1$, $h = \frac{1}{a}$.

Therefore, applying Theorem 2 (Dhage and Rhoades[3]), we conclude that G has a unique fixed point x_0. Hence, $G(x_0) = x_0 = T(G(x_0)) = Tx_0$. This gives that x_0 is also a unique fixed point of T. This completes the proof.

Theorem 3.2 — *Let X be a compact D-metric space. If T is a continuous bijective selfmap on X and there exists a real number a > 1 satisfying*

$$D(Tx, T^2x, T^3x) \geq a\, D(x, Tx, T^2x), \text{ for all } x \in X, x \neq Tx \qquad \text{... (2)}$$

Then T has a fixed point.

PROOF : We define $x_n = T^n x_0$ as $Tx_n = x_{n+1}$. Since T is bijective, there exists T^{-1} such that $T^{-1} x_{n+1} = x_n$. Let $G = T^{-1}$. Clearly G is continuous. We assume that $x_n \neq x_{n+1}$ for all n, otherwise G has a fixed point.

From (2), we get

$$D(Gx_n, G^2 x_n, G^3 x_n) \geq a\, D(x_n, Gx_n, G^2 x_n) \qquad \text{... (3)}$$

Now we consider

$$D(Gx_n, G^2 x_n, G^3 x_n) = D(x_{n-1}, G(Gx_n), G(G^2 x_n))$$

$$= D(x_{n-1}, Gx_{n-1}, G(Gx_{n-1}))$$

$$= D(x_{n-1}, x_{n-2}, x_{n-3})$$

Also $D(x_n, Gx_n, G^2 x_n) = D(x_n, x_{n-1}, x_{n-2})$

Using these values, (3) becomes

$$D(x_{n-1}, x_{n-2}, x_{n-3}) \geq a\, D(x_n, x_{n-1}, x_{n-2})$$

or $D(x_n, x_{n-1}, x_{n-2}) \leq h\, D(x_{n-1}, x_{n-2}, x_{n-3})$, for $0 < h < 1, h = \frac{1}{a}$

or $D(x_{n-2}, x_{n-1}, x_n) \leq hD(x_{n-3}\, x_{n-2}, x_{n-1})$

or $D(x_n, x_{n+1}, x_{n+2}) \leq h\, D(x_{n-1}, x_n, x_{n+1})$

or $D(x_n, x_{n+1}, x_{n+2}) \leq h\, D(x_{n-1}, x_n, x_{n+1}) \leq h^2 D(x_{n-2}, x_{n-1}, x_n)$

$$\ldots \leq h^n D(x_0, x_1, x_2),$$

which shows that $\{D(x_n, x_{n+1}, x_{n+2})\}$ is a monotone decreasing positive sequence with limit zero. Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and let z denote the limit of x_{n_k}.

Therefore $$z = \lim_{k \to \infty} x_{n_k}$$

Also $$Gz = G\left(\lim_{k \to \infty} x_{n_k}\right) = \lim_{k \to \infty} G(x_{n_k}) = \lim_{k \to \infty} x_{n_{k-1}}$$

Since D is continuous, therefore we get

$$D(z, Gz, G^2 z) = D\left(\lim_{k \to \infty} x_{n_k}, \lim_{k \to \infty} x_{n_{k-1}}, \lim_{k \to \infty} x_{n_{k-2}}\right)$$

$$= \lim_{k \to \infty} D(x_{n_k}, x_{n_{k-1}}, x_{n_{k-2}}),$$

which is zero as it is a monotone decreasing positive sequence with limit zero. Hence $D(z, Gz, G^2 z) = 0$, which implies that $z = Gz = G^2 z \Rightarrow z = Gz$. This shows that z is fixed point of G. Also $G(z) = z = T(Gz) = Tz$. Hence, z is also a fixed point of T. This completes the proof.

Pair of Mappings in D-Metric Spaces

Kaneko[1] proved a theorem for a pair of mappings. We extend his result in D-metric spaces, thus defining an expansive condition for a pair of mappings in theorem 4.2 below.

Theorem 4.1 — *Let (X, D) be a complete D-metric space. Let S, T and G be three onto selfmaps of X. Let there exist $q > 1$ such that*

$$D(Sx, Ty, Gz) \geq q\, D(x, y, z) \text{ for all } x, y, z \in X \quad \ldots (4)$$

Then $S = T$ and $S = G$.

PROOF : Assume on the contrary that either $S \neq T$ or $S \neq G$.

(i) Let $S \neq T$. Then for some $x \in X$

$$Sx \neq Tx \Rightarrow x \notin T^{-1}(Sx).$$

Therefore, $y \in T^{-1}(Sx)$ gives $Sx = Ty$. ... ($4i$)

(ii) Let $S \neq G$. Then for some $x \in X$

$$Sx \neq Gx \Rightarrow x \notin G^{-1}(Sx).$$

Therefore, $z \in G^{-1}(Sx)$ gives $Gz = Sx$. ... ($4ii$)

Consequently, we get $Sx = Ty$ and $Sx = Gz$. Hence, $D(Sx, Ty, Gz) = 0$.

Hence inequality (4) becomes

$$0 \geq q\, D(x, y, z) \text{ where } q > 1,$$

which is not possible since x, y, z are distinct. Hence our assumption is wrong. This proves the theorem.

Theorem 4.2 — *Let (X, D) be a complete D-metric space. Let S be a surjective and T an injective selfmaps of X satisfying the following condition. There exists $q > 1$ such that*

$$D(Sx, Sy, Sz) \geq q\, D(Tx, Ty, Tz) \qquad \text{... (5)}$$

for all $x, y, z \in X$. If S and T commute each other, then there exists a unique common fixed point of S and T.

PROOF : Under the assumption it is clear that S is injective. For each $x, y, z \in X$

$$D(x, y, z) = D(S(S^{-1}x), S(S^{-1}y), S(S^{-1}z))$$

$$\geq q\, D(T(S^{-1}x), T(S^{-1}y), T(S^{-1}z))$$

Hence, $$D(x, y, z) \geq q\, D(T(S^{-1}x), T(S^{-1}y), T(S^{-1}z))$$

or $$D(x, y, z) \geq q\, D(ToS^{-1}x, To\,S^{-1}y, ToS^{-1}z)$$

or $$D(ToS^{-1}x, ToS^{-1}y\; ToS^{-1}z) \leq hD(x, y, z),\ 0 < h < 1, \text{ where } h = \frac{1}{q}.$$

Then $D(Gx, Gy, Gz) \leq h\, D(x, y, z)$, where $G = ToS^{-1}$. By Theorem 2 (Dhage and Rhoades[3]), there exists a unique element $x_0 \in X$ such that $Gx_0 = x_0$ or $ToS^{-1}x_0 = x_0$. It follows that $S(T(S^{-1}x_0)) = Sx_0$. Since S adn T commute, therefore

$$T(S(S^{-1}x_0) = Sx_0 \text{ which gives } Tx_0, = Sx_0.$$

Put $y = z = S^{-1}x_0$ and replace x by x_0 in (5), we get

$$D(Sx_0, S(S^{-1}x_0), S(S^{-1}x_0)) \geq q\, D(Tx_0, T(S^{-1}x_0), T(S^{-1}x_0))$$

or $$D(Sx_0, x_0, x_0) \geq q\, D(Tx_0, x_0, x_0) = q\, D(Sx_0, x_0, x_0).$$

Since $q > 1$, hence $D(Sx_0, x_0, x_0) = 0$. This gives $Sx_0 = x_0$. Hence, $Sx_0 = Tx_0 = x_0$. This completes the proof.

Remark 1 : If S is assumed to be bijective, then there is no need to assume that T be injective.

Remark 2 : If x is a fixed point of S, then it is also a fixed point of T without any assumption about the commutativity of S and T.

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