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FIXED POINTS OF ASYMPTOTICALLY DEMICONTRACTIVE MAPPINGS IN CERTAIN BANACH SPACES

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Let E be a q-uniformly smooth Banach space, $1 < q < \infty$, and let K be a nonempty closed convex subset of E. Iterative methods for the approximation of fixed points of asymptotically demicontractive mappings $T: K \to K$ are constructed. Our results show that the boundedness requirement imposed on the subset K in a recent result of Osilike[3] (which is itself a generalization of a Theorem of Qihou[1]) can be dropped. Furthermore, our results extend the results of Osilike to more general iteration methods with errors.

Key Words : Asymptotically Demicontractive Mappings; p-Fixed Points; Modified Mann and Ishikawa Iteration Methods (with errors); q-uniformly smooth Banach Spaces

INTRODUCTION

Let H be a Hilbert space and let K be a nonempty subset of H. A mapping $T: K \to K$ is said to be k-srictly asymptotically pseudocontractive[1] if there exists a sequence $\{k_n\}$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\| T^n x - T^n y \|^2 \le k_n^2 \| x - y \|^2 + k \| (I - T^n) x - (I - T^n) y \|^2, \quad \text{... (1)}$$

for some $k \in [0, 1)$ and for all $x, y \in K$, and $n \in I\!N$. T is called asymptotically demi contractive[1] if $F(T) = \{x \in K : Tx = x\} \neq 0$ and there exists a sequence $\{k_n\}$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\| T^n x - x^* \|^2 \le k_n^2 \| x - x^* \|^2 + k \| x - T^m x \|^2, \quad \text{... (2)}$$

for some $k \in [0, 1)$ and for all $x \in K$, $x^* \in F(T)$, and $n \in I\!N$. These classes of mappings were introduced recently by Qihou[1]. If $k = 0$ in (1) then T is called asymptotically nonexpansive. It is clear that a k-strictly asymptotically demicontractive. T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\| T^n x - T^n y \| \le L \| x - y \|, \quad \text{... (3)}$$

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for all $x, y \in K$ and $n \in IN$. In [1], Qihou using the modified Mann iteration method[2] introduced by Schu[2], proved the following convergence theorems for the iterative approximation of fixed points of k-strictly asymptotically pseudocontractive mappings and asymptotically demicontractive mappings.

Theorem Q1 ([1], p. 1835) — *Let H be a Hilbert space and K a nonempty closed convex and bounded subset of H. Let $T: K \to K$ be a completely continuous and uniformly L-Lipschitzian demicontractive mapping with sequence $\{k_n\} \subseteq [1, \infty)$, $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the condition : $0 < \varepsilon \le \alpha_n \le 1 - k - \varepsilon$ for all $n \in IN$ and for some $\varepsilon > 0$. For arbitrary $x_0 \in K$ define the sequence $\{x_n\}$ by*

$$x_{n=1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, n \ge 0. \qquad \dots (4)$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

Theorem Q2 ([1], p. 1836) — *Let H be a Hilbert space and K a nonempty closed convex and bounded subset of H. Let $T: K \to K$ be a completely continuous and uniformly L-Lipschitzian and k-strictly asymptotically pseudocontractive mapping with sequence $\{k_n\} \subseteq [1, \infty)$, $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the condition: $0 < \varepsilon \le \alpha_n \le 1 - k - \varepsilon$ for all $n \in IN$, and for some $\varepsilon > 0$. Define the sequence $\{x_n\}$ from an arbitrary $x_0 \in K$ by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

The modified Mann iteration method[4] and the more general *modified Ishikawa iteration method*[2] have been exmployed for the iterative approximation of asymptotically nonexpansive mappings and their generalizations (see for example [1-4]).

In [3] one of the authors proved that if $T: K \to K$ is k-strictly asymptotically pseudo-contractive with sequence $\{k_n\}$, then

$$\| T^n x - T^n y \| \le \frac{k_n + \sqrt{k}}{1 - \sqrt{k}} \| x - y \| \le \frac{D + \sqrt{k}}{1 - \sqrt{k}} \| x - y \| = L \| x - y \|, \ \forall \ x, y \in K,$$

where $D > 0$ is such that $k_n \le D$, $\forall \ n \ge 0$. Thus every k-asymptotically pseudocontractive map is uniformly L-Lipschitzian so that the requirement that T be uniformly L-Lipschitzian imposed in Theorem Q2 is superflous. It is also shown in [3] that if T is asymptotically demicontractive, then

$$\| T^n x - x^* \| \le \frac{D + \sqrt{k}}{1 - \sqrt{k}} \| x - x^* \|, \ \forall \ x \in K, x^* \in F(T). \qquad \dots (5)$$

The following Lemma is proved in [3].

Lemma 1 (Osilike[3]) — Let H be a Hilbert space and K a nonempty subset of H. Then $T: K \to K$ is k-strictly asymptotically pseudocontractive with sequence $\{k_n\}$ if and only if for all $x, y \in K$

$$Re \langle (I-T^n)x-(I-T^n)y, x-y \rangle \geq \frac{1}{2}(1-k) \| (I-T^n)x$$

$$-(I-T^n)y\|^2 - \frac{1}{2}(k_n^2-1)\|x-y\|^2. \quad \ldots (6)$$

Furthermore, it is also proved in[3] that $T: K \to K$ is asymptotically demicontractive if and only if $F(T) \neq 0$ and

$$Re \langle x-T^n x, x-x^* \rangle \geq \frac{1}{2}(1-k)\|x-T^n x\|^2 - \frac{1}{2}(k_n^2-1)\|x-x^*\|^2 \quad \ldots (7)$$

for all $x \in K$ and $x^* \in F(T)$.

Using Lemma 1, Osilike[3] extended the definition of k-asymptotically pseudocontractive and asymptotically demicontractive maps to arbitrary Banach Spaces.

Let $q > 1$. We denote by J_q the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In particular $J = J_2$ is the normalized duality mapping, and $J_q(x) = \|x\|^{q-2} J(x), x \neq 0$. If E^* is srictly convex, then J_q is single-valued. In the sequel we shall denote single-valued generalized duality mapping by j_q.

It follows from (6) that if E is an arbitrary Banach space, and K is a nonempty subset of E, then $T: K \to K$ is k-strictly asymptotically pseudocontractive if for all $x, y \in K$ there exist $j(x-y) \in J(x-y)$ and a constant $k \in [0, 1)$ such that

$$Re \langle (I-T^n)x-(I-T^n)y, j(x-y) \rangle \geq \frac{1}{2}(1-K) \| (I-T^n)x-(I-T^n)y\|^2$$

$$-\frac{1}{2}(k_n^2-1)\|x-y\|^2. \quad \ldots (8)$$

Furthermore, it follows from (7) that T is asymptotically demicontractive with sequence $\{k_n\}$ if $F(T) \neq 0$ and for all $x \in K$ and $x^* \in F(T)$, there exist $k \in [0, 1)$ and $j(x-x^*) \in J(x-x^*)$ such that

$$Re \langle x-T^n x, j(x-x^*) \rangle \geq \frac{1}{2}(1-k)\|x-T^n x\|^2 - \frac{1}{2}(k_n^2-1)\|x-x^*\|^2. \quad \ldots (9)$$

Using the above definitions, Osilike[3] extended Theorems Q1 and Q2 to the more general q-uniformly smooth Banach spaces, $1 < q < \infty$ (see definition below) and to the more general modified Ishikawa iteration methods. More precisely he proved the following theorem :

Theorem 1 (Osilike[3]) — *Let $q > 1$ and let E be a real q-uniformly smooth Banach space. Let K be a closed convex and bounded subset of E and $T: K \to K$ a completely continuous uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subseteq [1, \infty)$ satisfying*

$\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions*

(i) $0 \le \alpha_n, \beta_n \le 1, n \ge 0,$

(ii) $0 < \varepsilon \le c_q \alpha_n^{q-1} (1 + L\beta_n)^q \le \frac{q}{2} (1-k)(1+L)^{-(q-2)} - \varepsilon$, *for all* $n \ge 0$ *and for some* $\varepsilon > 0$

and (iii) $\sum_{n=0}^{\infty} \beta_n < \infty.$

Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by

$$y_n = (1-\beta_n) x_n + \beta_n T^n x_n, n \ge 0$$

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n T^n y_n, n \ge 0$$

converges strongly to a fixed point of T.

It is our purpose in this paper to prove that Theorem 1 remains true without the boundedness condition imposed on the subset K. Furthermore, our theorem will extend Theorem 1 to the more general modified Mann and Ishikawa iteration methods with errors in the sense of a recent result of Xu[5].

In the sequel we shall need the following definitions and results :

The *modulus of smoothness* of E is the function

$$\rho_E : [0, \infty) \to [0, \infty)$$

defined by

$$\rho_E(\tau) = \sup \left\{ \frac{1}{2} (\| x+y \| + \| x-y \| - 1 : \| x \| \le 1, \| y \| \le \tau \right\}.$$

E is *uniformly smooth* if and only if $\lim_{\tau \to 0} \dfrac{\rho_E(\tau)}{\tau} = 0.$

Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \le c\,\tau^q$. Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$ and the Sobolev spaces, W_m^p, $1 < p < \infty$ are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p \text{ (or } l_p\text{) or } W_m^p \text{ is } \begin{cases} p-\text{uniformly smooth if } 1 < p \le 2 \\ 2-\text{uniformly smooth if } p \ge 2. \end{cases}$$

Theorem HKX ([6], p. 1130) — *Let $q > 1$ and let E be a real Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that*

$$\| x+y \|^q \le \| x \|^q + q \langle y, j_q(x) \rangle + c_q \| y \|^q. \qquad \text{... (10)}$$

Inequality (10) implies that

$$\| x - y \|^q \leq \| x \|^q - q \langle y, j_q(x) \rangle + c_q \| y \|^q, \quad \ldots (11)$$

so that

$$\| x + y \|^q + \| x - y \|^q \leq 2 (\| x \|^q + c_q \| y \|^q).$$

Hence

$$\| x + y \|^q \leq 2 (\| x \|^q + c_q \| y \|^q). \quad \ldots (12)$$

Lemma Q ([1], p. 1836) — Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq a_n + b_n, \; n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n < \infty$ and $\{a_n\}$ has a subsequence which converges to 0, then $\lim_{n \to \infty} a_n = 0$.

MAIN RESULTS

Lemma 2 — Let E be a normed space, and K a nonempty convex subset of E. Let $T : K \to K$ be a uniformly L-Lipschitzian mapping and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$, $\{c_n'\}$ be sequences in [0, 1] with $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1 \; \forall \; n \geq 0$. Let $\{u_n\}$, $\{\upsilon_n\}$ be bounded sequences in K. For arbitrary $x_0 \in K$, generate the sequence $\{x_n\}$ by

$$y_n = a_n x_n + b_n T^n x_n + c_n u_n, \; n \geq 0$$

$$x_{n+1} = a_n' x_n + b_n' T^n y_n + c_n' \upsilon_n, \; n \geq 0.$$

Then
$$\| x_n - Tx_n \| \leq \| x_n - T^n x_n \| + L (1 + L)^2 \| x_{n-1}$$

$$- T^{n-1} x_{n-1} \| + L (1 + L) c_{n-1}' \| \upsilon_{n-1} - x_{n-1} \|$$

$$+ L^2 (1 + L) c_{n-1} \| u_{n-1} - x_{n-1} \| + L c_{n-1}' \| x_{n-1} - T^{n-1} x_{n-1} \|.$$

PROOF : Set $\lambda_n = \| x_n - T^n x_n \|$. Then

$$\| x_n - Tx_n \| \leq \| x_n - T^n x_n \| + L \| T^{n-1} x_n - x_n \|$$

$$\leq \lambda_n + L^2 \| x_n - x_{n-1} \| + L \| T^{n-1} x_{n-1} - x_n \|$$

$$= \lambda_n + L^2 \| a'_{n-1} x_{n-1} + b'_{n-1} T^{n-1} y_{n-1} + c'_{n-1} v_{n-1} - x_{n-1} \|$$

$$+ L \| a'_{n-1} x_{n-1} + b'_{n-1} T^{n-1} y_{n-1} + c'_{n-1} v_{n-1} - T^{n-1} x_{n-1} \|$$

$$= \lambda_n + L^2 \| b'_{n-1} (T^{n-1} y_{n-1} - x_{n-1}) + c'_{n-1} (v_{n-1} - x_{n-1}) \|$$

$$+ L \| a'_{n-1} (x_{n-1} - T^{n-1} x_{n-1}) + b'_{n-1} (T^{n-1} y_{n-1} - T^{n-1} x_{n-1})$$

$$+ c'_{n-1} (v_{n-1} - T^{n-1} x_{n-1}) \|$$

$$\leq \lambda_n + L^2 \| T^{n-1} y_{n-1} - x_{n-1} \| + L^2 c'_{n-1} \| v_{n-1} - x_{n-1} \| + L \| x_{n-1} - T^{n-1} x_{n-1} \|$$

$$+ L^2 \| y_{n-1} - x_{n-1} \| + Lc'_{n-1} \| v_{n-1} - x_{n-1} \| + Lc'_{n-1} \| x_{n-1} - T^{n-1} x_{n-1} \|$$

$$= \lambda_n + L \lambda_{n-1} + L(1+L) c'_{n-1} \| v_{n-1} - x_{n-1} \| + L^2 \| T^{n-1} y_{n-1} - x_{n-1} \|$$

$$+ L^2 \| y_{n-1} - x_{n-1} \| + Lc_{n-1} \lambda_{n-1}$$

$$\leq \lambda_n + L \lambda_{n-1} + L(1+L) c'_{n-1} \| v_{n-1} - x_{n-1} \| + L^2 (1+L) \| y_{n-1} - x_{n-1} \|$$

$$+ L^2 \| T^{n-1} x_{n-1} - x_{n-1} \| + Lc'_{n-1} \lambda_{n-1}$$

$$= \lambda_n + L(1+L) \lambda_{n-1} + L(1+L) c'_{n-1} \| v_{n-1} - x_{n-1} \|$$

$$+ L^2 (1+L) \| b_{n-1} (T^{n-1} x_{n-1} - x_{n-1}) + c_{n-1} (u_{n-1} - x_{n-1}) \| + Lc'_{n-1} \lambda_{n-1}$$

$$\leq \lambda_n + L (1+L)^2 \lambda_{n-1} + L(1+L) c'_{n-1} \| v_{n-1} - x_{n-1} \|$$

$$+ L^2 (1+L) c_{n-1} \| u_{n-1} - x_{n-1} \| + Lc'_{n-1} + \lambda_{n-1},$$

completing the proof of Lemma 2. □

Remark 1 : If $c_n = c'_n = 0, b_n = \beta_n, a_n = 1 - \beta_n, b'_n = \alpha_n, a'_n = 1 - \alpha_n$ for all $n \geq 0$ and for some $\{\alpha_n\}$, $\{\beta_n\} \subseteq [0, 1]$, then the sequence $\{x_n\}$ generated in Lemma 2 reduces to the usual modified Ishikawa iteration method introduced by Schu[2]. If $\beta_n = 0$ for all $n \geq 0$, then $\{x_n\}$ reduces to the usual modified Mann iteration method. Lemma 1.2 of Schu[2] is a special case of Lemma 2 for which $c_n = c'_n = 0, b_n = \beta_n, a_n = 1 - \beta_n, b'_n = \alpha_n = 1 - \alpha_n$. In the sequel $k \in [0, 1)$ is the constant appearing in the definition of asymptotically demicontractive mapping, and c_q is the constant appearing in inequality (10).

Theorem 2 — *Let E be a q-uniformly smooth Banach space, $1 < q < \infty$. Let K be a closed convex subset of E and let $T : k \to K$ be a completely continuous uniformly L-Lipschitzian* *asymptotically demicontractive mapping with sequence* $\{k_n\} \subseteq [1, \infty)$ *satisfying* $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$. *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$, *and* $\{c_n'\}$ *be a real sequences in* $[0, 1]$ *satisfying the conditions :*

(*i*) $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$,

(*ii*) $0 < \varepsilon \leq c_q (b_n')^{q-1} (1 + Lb_n)^q \leq \frac{q}{2}(1 - k)(1 + L)^{-(q-2)} - \varepsilon$, $\forall\, n \geq 0$ *and for some* $\varepsilon > 0$ *and*

(iii) $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$, *and* $\sum_{n=0}^{\infty} c_n' < \infty$.

Let $\{u_n\}$ *and* $\{\upsilon_n\}$ *be bounded sequences in K. Then the sequence* $\{x_n\}$ *generated from an arbitrary* $x_0 \in K$ *by*

$$y_n = a_n x_n + b_n T^n x_n + c_n u_n, n \geq 0 \quad \dots (13)$$

and

$$x_{n+1} = a_n' x_n + b_n' T^n y_n + c_n' \upsilon_n, n \geq 0, \quad \dots (14)$$

converges strongly to a fixed point of T.

PROOF : Let $x^* \in F(T)$. Then using (10) we obtain

$$\| x_{n+1} - x^* \|^q = \| a_n' x_n + b_n' T^n y_n + c_n' \upsilon_n - x^* \|^q$$

$$= \| (1 - (b_n' + c_n')) x_n + b_n' T^n y_n + c_n' \upsilon_n - x^* \|^q$$

$$= \| (x_n - x^*) + b_n' (T^n y_n - x_n) + c_n' (\upsilon_n - x_n) \|^q$$

$$\leq \| (x_n - x^*) + b_n' (T^n y_n - x_n) \|^q + qc_n' \langle \upsilon_n - x_n, j_q\ (x_n - x^* + b_n' (T^n y_n - x_n)) \rangle$$

$$+\ c_q + (c_n')^q \| \upsilon_n - x_n \|^q. \quad \dots (15)$$

Since $\{u_n - x^*\}$ and $\{\upsilon_n - x^*\}$ are bounded, let $M > 0$ be such that $\| u_n - x^* \| \leq M$, $\| \upsilon_n - x^* \| \leq M\ \ \forall\, n \geq 0$. Using inequality (12) we obtain

$$\| \upsilon_n - x_n \|^q \leq 2 (\| \upsilon_n - x^* \|^q + c_q \| x_n - x^* \|^q) \leq 2M^q + 2c_q \| x_n - x^* \|^q. \quad \dots (16)$$

Observe that

$$\| \upsilon_n - x_n \| \leq \| \upsilon_n - x^* \| + \| x_n - x^* \|$$

$$\leq M + \| x_n - x^* \|$$

$$\leq (1 + L) M + [2 = L(1 + L)] \| x_n - x^* \|. \quad \dots (17)$$

Furthermore,

$$\|(x_n - x^*) + b_n'(T^n y_n - x_n)\|^{q-1} \leq [\|x_n - x^*\| + L\|y_n - x^*\| + \|x_n - x^*\|]^{q-1}$$

$$\leq [2\|x_n - x^*\| + L(1+L)\| x_n - x^*\| + ML]^{q-1}$$

$$\leq [(1+L)M + (2 + L(1+L))\|x_n - x^*\|]^{q-1}. \quad \ldots (18)$$

Hence, (17) and (18) yield

$$\langle v_n - x_n, j_q(x_n - x^* + b_n'(T^n y_n - x_n))\rangle \leq \|v_n - x_n\| \|x_n - x^* + b_n'(T^n y_n - x_n)\|^{q-1}$$

$$\leq \{(2 + L(1+L))\|x_n - x^*\| + (1+L)M\}^q$$

$$\leq [2 \max \{(2 + L(1+L))\|x_n - x^*\|, (1+L)M\}]^q$$

$$\leq [2(2 + L(1+L))]^q \|x_n - x^*\|^q + [2(1+L)M]^q. \quad \ldots (19)$$

Furthermore,

$$\|(x_n - x^*) + b_n'(T^n y_n - x_n)\|^q \leq \|x_n - x^*\|^q + qb_n'\langle T^n y_n - x_n, j_q(x_n - x^*)\rangle$$

$$+ c_q(b_n')\|T^n y_n - x_n\|^q$$

$$= \|x_n - x^*\|^q + q b_n'\langle T^n y_n - T^n x_n, j_q(x_n - x^*)\rangle$$

$$- qb_n'\langle x_n - T^n x_n, j_q(x_n - x^*)\rangle$$

$$+ c_q(b_n')^q \|T^n y_n - x_n\|^q. \quad \ldots (20)$$

Observe that

$$\langle T^n y_n - T^n x_n, j_q(x_n - x^*\rangle \leq L\|y_n - x_n\| \|x_n - x^*\|^{q-1}$$

$$\leq L\|b_n(T^n x_n - x_n) + c_n(u_n - x_n)\| \|x_n - x^*\|^{q-1}$$

$$\leq \Big\{L[(1+L)b_n + c_n][\|x_n - x^*\| + Lc_n\|u_n - x^*\|\Big\}\|x_n - x^*\|^{q-1}$$

$$\leq L[(1+L)b_n + c_n]\|x_n - x^*\|^q + Lc_n\|u_n - x^*\| \|x_n - x^*\|^{q-1}$$

$$\leq L[(1+L)b_n + c_n]\|x_n - x^*\|^q$$

$$+ Lc_n \max\{\|u_n - x^*\|, \|x_n - x^*\|\} \|x_n - x^*\|^{q-1}$$

$$\leq L\,[(1+L)\,b_n + 2c_n]\,\| x_n - x^* \|^q + LM^q\, c_n, \qquad \text{... (21)}$$

$$\langle x_n - T^n x_n, j_q(x_n - x^*) \rangle = \| x_n - x^* \|^{q-2} \langle x_n - T^n x_n, j(x_n - x^*) \rangle$$

$$\geq \| x_n - x^* \|^{q-2} \left\{ \frac{1}{2}(1 - k \| x_n - T^n x_n \|^2 - \frac{1}{2}(k_n^2 - 1) \| x_n - x^* \|^2 \right\}$$

$$= \frac{1}{2}(1-k) \| x_n - T^n x_n \|^2 \| x_n - x^* \|^{q-2} - \frac{1}{2}(k_n^2 - 1) \| x_n - x^* \|^q. \qquad \text{.. (22)}$$

Since $\| x_n - T^n x_n \| \leq (1+L) \| x_n - x^* \|$, we have $\| x_n - x^* \| \geq (1+L)^{-1} \| x_n - T^n x_n \|$, so that (22) reduces to

$$\langle x_n - T^n x_n, j_q(x_n - x^*) \rangle \geq \frac{1}{2}(1-k)\,(1+L)^{-(q-2)})\, \| x_n - T^n x_n \|^q$$

$$- \frac{1}{2}(k_n^2 - 1) \| x_n - x^* \|^q. \qquad \text{... (23)}$$

We also have the following estimates :

$$\| T^n y_n - x_n \|^q = \| T^n y_n - T^n x_n + T^n x_n - x_n \|^q$$

$$\leq \| T^n x_n - x_n \|^q + q \langle T^n y_n - T^n x_n, j_q(T^n x_n - x_n) \rangle$$

$$+ \ c_q \| T^n y_n - T^n x_n \|^q$$

$$\leq \| T^n x_n - x_n \|^q + q \| T^n y_n - T^n x_n \| \, \| T^n x_n - x_n \|^{q-1}$$

$$+ \ c_q L^q \| y_n - x_n \|^q$$

$$\leq \| T^n x_n - x_n \|^q + qL \| y_n - x_n \| \, \| T^n x_n - x_n \|^{q-1}$$

$$+ \ c_q L^q \| y_n - x_n \|^q, \qquad \text{... (24)}$$

$$\| y_n - x_n \| = \| b_n (T^n x_n - x_n) + c_n (u_n - x_n) \|$$

$$\leq b_n \| x_n - T^n x_n \| + Mc_n + c_n \| x_n - x^* \|, \qquad \text{... (25)}$$

$$\| y_n - x_n \|^q \leq 2\,(b_n^q \| x_n - T^n x_n \|^q + c_q c_n^q \| u_n - x_n \|^q)$$

$$\leq 2b_n^q \| x_n - T^n x_n \|^q + 2c_q c_n^q [2\,(\| u_n - x^* \|^q + c_q \| x_n - x^* \|^q)]$$

$$\leq 2b_n^q \| x_n - T^n x_n \|^q + 4c_q c_n^q M^q + 4c_q^2 c_n^q \| x_n - x^* \|^q. \qquad \text{... (26)}$$

Using (25) and (26) in (24) we obtain

$$\|T^n y_n - x_n\|^q \leq \|x_n - T^n x_n\|^q$$

$$+ \ qL\,[b_n \|x_n - T^n x_n\| + Mc_n + c_n \|x_n - x^*\|]\, \|T^n x_n - x_n\|^{q-1}$$

$$+ \ c_q L^q\,[2b_n^q \|x_n - T^n x_n\|^q + 4c_q c_n^q M^q + 4c_q^2 c_n^q \|x_n - x^*\|^q]$$

$$\leq [1 + qLb_n]\, \|x_n - T^n x_n\|^q + qMLc_n (1+L)^{q-1} \|x_n - x^*\|^{q-1}$$

$$+ \ qLc_n (1+L)^{q-1} \|x_n - x^*\|^q$$

$$+ \ 2c_q L^q b_n^q \|x_n - T^n x_n\|^q + 4c_q^2 L^q c_n^q M^q + 4c_q^3 L^q c_n^q \|x_n - x^*\|^q$$

$$\leq [1 + qLb_n]\, \|x_n - T^n x_n\|^q + qMLc_n (1+L)^{q-1}\, [1 + \|x_n - x^*\|^q]$$

$$+ \ qLc_n (1+L)^{q-1} \|x_n - x^*\|^q + 2c_q L^q b_n^q (1+L)^q \|x_n - x^*\|^q$$

$$+ \ 4c_q^2 L^q c_n^q M^q + 4c_q^3 L^q c_n^q \|x_n - x^*\|^q$$

$$\leq [1 + qLb_n]\, \|x_n - T^n x_n\|^q$$

$$+ \ [qMLc_n (1+L)^{q-1} + qLc_n (1+L)^{q-1} + 2c_q L^q b_n^q (1+L)^q + 4c_q^3 L^q c_n^q]\, \|x_n - x^*\|^q$$

$$+ \ qMLc_n (1+L)^{q-1} + 4c_q^2 L^q c_n^q M^q. \qquad \ldots (27)$$

Define $f: [0, \infty) \to \mathcal{R}$ by $f(x) = (1+x)^q - qx - 1$. It follows from elementary calculus that f attains its minimum at $x = 0$, so that $f(x) \geq f(0) = 0$ for all $x \in [0, \infty)$. Thus $(1 + qx) \leq (1+x)^q$, $\forall$ $x \in [0, \infty)$. Setting $x = Lb_n$ we now obtain $(1 + qLb_n) \leq (1 + Lb_n)^q$ and using this in (27) we obtain

$$\|T^n y_n - x_n\|^q \leq (1 + Lb_n)^q \|x_n - T^n x_n\|^q$$

$$+ \ [qMLc_n (1+L)^{q-1} + qLc_n (1+L)^{q-1} + 2c_q L^q b_n^q\, (1+L)^q + 4c_q^3 L^q c_n^q]\, \|x_n - x^*\|^q$$

$$+ \ qMLc_n (1+L)^{q-1} + 4c_q^2 L^q c_n^q M^q. \qquad \ldots (28)$$

Using inequalities (21), (23) and (28) in (20) we obtain

$$\|x_n - x^* + b_n' (T^n y_n - x_n\|^q \leq \|x_n - x^*\|^q$$

$$+ \ b_n' \ \{L(1+L)\, b_n + 2c_n]\, \|x_n - x^*\|^q + Lc_n M^q\}$$

$$- q b_n' \left\{ \frac{1}{2}(1-k)(1+L)^{-(q-2)} \| x_n - T^n x_n \|^q - \frac{1}{2}(k_n^2 - 1) \| x_n - x^* \|^q \right\}$$

$$+ (b_n')^q c_q \{(1 + Lb_n)^q \| x_n - T^n x_n \|^q$$

$$+ [qL(q+L)^{q-1}(1+M) c_n + 2c_q L^q b_n^q (1+L)^q + 4L^q c_n^q c_q^3] \| c_n - x^* \|^q$$

$$+ qLc_n M (1+L)^{q-1} + 4c_n^q c_q^2 M^q L^q\}. \qquad \ldots (29)$$

Using inequalities (16), (19) and (29) in (15) we obtain

$$\| x_{n+1} - x^* \|^q \le \left\{ 1 + qLb_n' [(1+L) b_n + 2c_n] + \frac{q}{2} b_n' (k_n^2 - 1) + 2c_q^2 (c_n')^q \right.$$

$$+ qc_n' [2(2 + L(1+L))]^q$$

$$+ c_q (b_n')^q [qL(1+L)^{q-1}(1+M) c_n + 2c_q L^q b_n^q (1+L)^q + 4c_q^3 c_n^q L^q]\} \| x_n - x^* \|^q$$

$$- b_n' \left\{ \frac{q}{2}(1-k)(1+L)^{-(q-2)} - (b_n')^{q-1} c_q (1 + Lb_n)^q \right\} \| x_n - T^n x_n \|^q$$

$$+ qb_n' Lc_n M^q + qLc_n M (1+L)^{q-1} c_q (b_n')^q$$

$$+ 4c_n^q (b_n') c_q^3 M^q L^q + 2c_q (c_n')^q M^q + qc_n' [2(1+L)M]^q$$

$$= [1 + \delta_n] \| x_n - x^* \|^q$$

$$- b_n' \left\{ \frac{q}{2}(1-k)(1+L)^{-(q-2)} - (b_n')^{q-1} c_q (1 + Lb_n)^q \right\} \| x_n - T^n x_n \|^q + \sigma_n, \qquad \ldots (30)$$

where
$$\delta_n = qLb_n' [(1+L)b_n + 2c_n] + \frac{q}{2} b_n' (k_n^2 - 1) + 2c_q^2 (c_n')^q + qc_n' [2(2 + L(1+L)))]^q$$

$$+ c_q (b_n') [qL(1+L)^{q-1}(1+M) c_n + 2c_q L^q b_n^q (1+L)^q + 4c_q^3 c_n^q L^q]$$

and
$$\sigma_n = qb_n' Lc_n M^q + qLc_n M (1+L)^{q-1} c_q (b_n')^q$$

$$+ 4c_n^q c_q^3 M^q L^q + 2c_q (c_n')^q M^q + qc_n' [2(1+L)M]^q.$$

Condition (*iii*) and $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$ imply that $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} \sigma_n < \infty$. From (30) we obtain

$$\| x_{n+1} - x^* \|^q \le [1 + \delta_n] \| x_n - x^* \|^q + \sigma_n,$$

from which it follows easily that $\{\| x_n - x^* \|\}$ is bounded. Suppose $\| x_n - x^* \|^q \le D, \ \forall\ n \ge 0.$

Then it follows from (30) that

$$\| x_{n+1} - x^* \|^q \le \| x_n - x^* \|^q - b_n'$$

$$\left\{ \frac{q}{2}(1-k)(1+L)^{-(q-2)} - (b_n')^{q-1} c_q (1+Lb_n)^q \right\} \| x_n - T^n x_n \|^q + \lambda_n, \qquad \text{... (31)}$$

where $\lambda_n = \sigma_n + D\,\delta_n$. Using condition (ii) in (31) we obtain

$$\| x_{n+1} - x^* \|^q \le \| x_n - x^* \|^q - b_n' \varepsilon \| x_n - T^n x_n \|^q + \lambda_n. \qquad \text{... (32)}$$

Furthermore, condition (ii) implies that

$$b_n' \ge \left[\frac{\varepsilon}{c_q (1+Lb_n)^q} \right]^{\frac{1}{q-1}},$$

and since $$\lim_{n\to\infty} \left[\frac{\varepsilon}{c_q (1+Lb_n)^q} \right]^{\frac{1}{q-1}} = \left(\frac{\varepsilon}{c_q} \right)^{\frac{1}{q-1}} > 0,$$

there exists a positive integer N_0 such that

$$b_n' \ge \left[\frac{\varepsilon}{c_q (1+Lb_n)^q} \right]^{\frac{1}{q-1}} \ge \frac{1}{2} \left(\frac{\varepsilon}{c_q} \right)^{\frac{1}{q-1}}, \ \forall\ n \ge N_0. \qquad \text{... (33)}$$

Using (33) in (32) we have

$$\| x_{n+1} - x^* \|^q \le \| x_n - x^* \|^q - \frac{\varepsilon}{2} \left(\frac{\varepsilon}{c_q} \right)^{\frac{1}{1-q}} \| x_n - T^n x_n \|^q + \lambda_n,$$

so that

$$\frac{\varepsilon}{2} \left(\frac{\varepsilon}{c_q} \right)^{\frac{1}{1-q}} \sum_{j=N_0}^{n} \| x_j - T^j x_j \|^q \le \| x_{N_0} - x^* \|^q + \sum_{j=N_0} \lambda_j \ .$$

Since $\sum_{n=0}^{\infty} \lambda_n < \infty$, we obtain that $\sum_{n=0}^{\infty} \| x_n - T^n x_n \|^q < \infty$, so that $\lim_{n\to\infty} \| x_n - T^n x_n \| = 0$. From Lemma 2 we obtain

$$0 \le \| x_n - Tx_n \| \le \| x_n - T^n x_n \| + L(1+L)^2 \| x_{n-1} - T^{n-1} x_{n-1} \|$$

$$+ L(1+L)c_n'(M+D^{\frac{1}{q}}) + L^2(1+L)c_{n-1}(M+D^{\frac{1}{q}})$$

$$+ Lc_{n-1}' \| x_{n-1} - T^{n-1}x_{n-1} \| \to 0 \text{ as } n \to \infty.$$

Hence, $$\lim_{n\to\infty} \| x_n - Tx_n \| = 0. \qquad ...(34)$$

Since T is completely continuous and $\{x_n\}$ is bounded, it follows that $\{Tx_n\}$ has a convergent subsequence $\left\{Tx_{n_j}\right\}_{j=0}^{\infty}$, so that (34) implies that $\{x_n\}$ has a convergent subsequence $\left\{x_{n_j}\right\}_{j=0}^{\infty}$. Let $\lim_{j\to\infty} x_{n_j} = p$. Then from (34) we obtain $p = Tp$ so that p is a fixed point of T. Hence, it follows from (32) that

$$\| x_{n+1} - p \|^q \le \| x_n - p \|^q + \lambda_n, \; n \ge 0.$$

SInce $\{\| x_n - p \|\}$ has a subsequence which converges to 0 and $\sum_{n=0}^{\infty} \lambda_n < \infty$, it follows from Lemma Q that $\lim_{n\to\infty} \| x_n - p \| = 0$, completing the proof of Theorem 2. □

Remark 2: If we set $\beta_n = b_n$, $\alpha_n = b_n'$, $a_n = 1 - \beta_n$, $a_n' = 1 - \alpha_n$, and $c_n = c_n' = 0$ in Theorem 2, conditions (ii) and (iii) reduce to conditions (ii) and (iii) of Theorem 1, so that we obtain Theorem 1 without the boundedness condition imposed on the subset K.

Corollary — Let E, K, T, $\{k_n\}$, and $\{\upsilon_n\}$ be as in Theorem 2. Let $\{a_n'\}$, $\{b_n'\}$ and $\{c_n'\}$ be sequences satisfying the conditions :

(i) $a_n' + b_n' + c_n' = 1, n \ge 0$ and

(ii) $0 < \varepsilon \le c_q (b_n')^{q-1} \le \frac{q}{2}(1-k)(1+L)^{-(q-2)} - \varepsilon$, for all $n \ge 0$ and for some $\varepsilon > 0$. Then sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = a_n' x_n + b_n' T^n x_n + c_n' \upsilon_n, \; n \ge 0$$

converges strongly to a fixed point of T.

PROOF : Set $a_n = 1, b_n = c_n = 0, \; \forall \; n \ge 0$ in Theorem 2 and the Corollary follows.

Remark 3 : Suitable choices of ε, and our real sequences in Theorem 2 are :

$$\varepsilon = \frac{1}{q}\left(1 - \frac{1}{q}\right)\left[\frac{q}{2}(1-k)(1+L)^{-(q-2)}\right],$$

$$b_n = \frac{1}{(n+2)^2}, \; c_n = \frac{1}{(n+1)(n+2)^2}, \; a_n = 1 - (b_n + c_n),$$

$$b_n' = \left[\frac{1}{qc_q}\left(1 - \frac{1}{q}\right)\frac{q}{2}(1-k)(1+L)^{-(q-2)} \right]^{\frac{1}{q-1}} \left[\frac{(n+2)^{2q}}{(L+(n+2)^2)^q} \right]^{\frac{1}{q-1}},$$

$$c_n' = \left[\frac{1}{qc_q}\left(1 - \frac{1}{q}\right)\frac{q}{2}(1-k)(1+L)^{-q-2)} \right]^{\frac{1}{q-1}} \left[\frac{L^q}{(L+(n+2)^2)^q} \right]^{\frac{1}{q-1}},$$

$$a_n' = 1 - (b_n' + c_n'), \ \forall\ n \geq 0.$$

Remark 4 : Since $\sum_{n=0}^{\infty} (k_n - 1) < \infty$ if and only if $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$, one can replace the condition $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$ with $\sum_{n=0}^{\infty} (k_n - 1) < \infty$ in our theorem.

Lemma 3 — Let E be a normed space and $T : E \to E$ a uniformly L-Lipschitzian mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in [0, 1], and let $\{\varepsilon_n\}$, $\{\delta_n\}$ be sequences in E such that $\lim_{n\to\infty} \| \varepsilon_n \| = \lim_{n\to\infty} \| \delta_n \| = 0$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in E$ by

$$y_n = (1-\beta_n)\, x_n + \beta_n\, T^n x_n + \varepsilon_n,\ n \geq 0$$

and

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, T^n y_n + \delta_n,\ n \geq 0.$$

Then

$$\| x_n - Tx_n \| \leq \| x_n - T^n x_n \| + L(1+L)^2 \| x_{n-1} - T^{n-1} x_{n-1} \| + L^2 (1+L)\, \varepsilon_{n-1} + L(1+L)\, \delta_{n-1}.$$

PROOF : Set $\lambda_n = \| x_n - T^n x_n \|$. Then

$$\| x_n - Tx_n \| \leq \| x_n - T^n x_n \| + L \| T^{n-1} x_n - x_n \|$$

$$\leq \lambda_n + L^2 \| x_n - x_{n-1} \| + L \| T^{n-1} x_{n-1} - x_n \|$$

$$= \lambda_n + L^2 \| (1-\alpha_{n-1})\ x_{n-1} + \alpha_{n-1} T^{n-1} y_{n-1} + \delta_{n-1} - x_{n-1} \|$$

$$+ L \| (1-\alpha_{n-1})\, x_{n-1} + \alpha_{n-1} T^{n-1} y_{n-1} + \delta_{n-1} - x_{n-1} \|$$

$$\leq \lambda_n + L^2 [\| T^{n-1} y_{n-1} - x_{n-1} \| + \delta_{n-1}]$$

$$+ L [\, \| x_{n-1} - T^{n-1} x_{n-1} \| + \| T^{n-1} y_{n-1} - T^{n-1} x_{n-1} \| + \delta_{n-1}]$$

$$\leq \lambda_n + L^2 \| T^{n-1} y_{n-1} - x_{n-1} \| + L^2 \sigma_{n-1} + L \| x_{n-1} - T^{n-1} x_{n-1} \|$$

$$+ L^2 \| y_{n-1} - x_{n-1} \| + L \delta_{n-1}$$

$$\leq \lambda_n + L \lambda_{n-1} + L^3 \| y_{n-1} - x_{n-1} \| + L^2 \| T^{n-1} x_{n-1} - x_{n-1} \|$$

$$+ L(1+L) \delta_{n-1} + L^2 \| y_{n-1} - x_{n-1} \|$$

$$\leq \lambda_n + L(1+L) \lambda_{n-1} + L^2 (1+L) \| (1-\beta_{n-1}) x_{n-1} + \beta_{n-1}$$

$$T^{n-1} x_{n-1} + \varepsilon_{n-1} - x_{n-1} \| + L(1+L) \delta_{n-1}$$

$$\leq \lambda_n + L(1+L)^2 \lambda_{n-1} + L^2 (1+L) \varepsilon_{n-1} + L(1+L) \delta_{n-1},$$

completing the proof of Lemma 3. □

***Theorem* 3** — *Let E be a q-uniformly smooth Banach space, $1 < q < \infty$. Let $T : E \to E$ be a completely continuous uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subseteq [1, \infty)$ satisfying $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$. Let $\{\delta_n\}$, $\{\varepsilon_n\}$ be sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$ real sequences in $[0, 1]$ satisfying the conditions :*

(i) $0 < \varepsilon \leq c_q \alpha_n^{q-1} (1 + L\beta_n)^q \leq \frac{q}{2} (1-k)(1+L)^{-(q-2)} - \varepsilon$, *for all $n \geq 0$ and for some $\varepsilon > 0$;*

and

(ii) $\sum_{n=0}^{\infty} \beta_n < \infty$, $\sum_{n=0}^{\infty} \| \varepsilon_n \| < \infty$, *and* $\sum_{n=0}^{\infty} \| \delta_n \| < \infty$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by

$$y_n = (1-\beta_n) x_n + \beta_n T^n x_n + \varepsilon_n, \; n \geq 0$$

and
$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n T^n y_n + \delta_n, \; n \geq 0$$

converges strongly to a fixed point of T.

PROOF : Let $x^* \in F(T)$. Then using (10) we obtain

$$\| x_{n+1} - x^* \|^q = \| (1-\alpha_n) x_n + \alpha_n T^n y_n + \delta_n - x^* \|^q$$

$$= \| x_n - x^* + \alpha_n (T^n y_n - x_n) + \delta_n \|^q$$

$$\leq \| x_n - x^* + \alpha_n (T^n y_n - x_n) + \delta_n \|^q + q \langle \delta_n, j_q (x_n - x^* + \alpha_n (T^n y_n - x_n)) \rangle + c_q \| \delta_n \|^q$$

$$\leq \| x_n - x^* + \alpha_n (T^n y_n - x_n) \|^q + q \| \delta_n \| \| x_n - x^* + \alpha_n (T^n y_n - x_n) \|^{q-1} + c_q \| \delta_n \|^q$$

$$\leq \| x_n - x^* + \alpha_n (T^n y_n - x_n) \|^q + q \| \delta_n \| \; [1 + \| x_n - x^* + \alpha_n (T^n y_n - x_n) \|^q] \; + \; c_q \| \delta_n \|^q$$

$$= [1 + q \| \delta_n \|] \| x_n - x^* + \alpha_n (T^n y_n - x_n) \|^q + q \| \delta_n \| + c_q \| \delta_n \|^q. \qquad \text{... (35)}$$

Using inequality (10) we obtain

$$\| x_n - x^* + \alpha_n (T^n y_n - x_n \|^q \leq \| x_n - x^* \|^q + q \, \alpha_n \langle T^n y_n - x_n, j_q (x_n - x^*) \rangle$$

$$+ \; c_q \, \alpha_n^q \| T^n y_n - x_n \|^q$$

$$= \| x_n - x^* \|^q + q \, \alpha_n \langle T^n y_n - T^n x_n, j_q (x_n - x^*) \rangle$$

$$- q \, \alpha_n \langle x_n - T^n x_n, j_q (x_n - x^*) \rangle$$

$$+ \; c_q \, \alpha_n^q \| T^n y_n - x_n \|^q. \qquad \text{... (36)}$$

Observe that

$$\langle T^n y_n - T^n x_n, j_q (x_n - x^*) \rangle \; \leq L \| y_n - x_n \| \| x_n - x^* \|^{q-1}$$

$$\leq L \, [\beta_n \| T^n x_n - x_n \| + \| \varepsilon_n \|] \| x_n - x^* \|^{q-1}$$

$$\leq L \, [\beta_n (1 + L) \| x_n - x^* \| + \| \varepsilon_n \|] \| x_n - x^* \|^{q-1}$$

$$= L \, (1 + L) \, \beta_n \| x_n - x^* \|^q + L \| \varepsilon_n \| \| x_n - x^* \|^{q-1}$$

$$\leq L \, (1 + L) \, \beta_n \| x_n - x^* \|^q + l \| \varepsilon_n \| \, [1 + \| x_n - x^* \|^q]$$

$$= [L(1 + L) \, \beta_n + L \| \varepsilon_n \|] \| x_n - x^* \|^q + L \| \varepsilon_n \|. \qquad \text{... (37)}$$

Furthermore,

$$\| y_n - x_n \| = \| \beta_n (T^n x_n - x_n) + \varepsilon_n \| \leq \beta_n \| x_n - T^n x_n \| + \| \varepsilon_n \|, \qquad \text{... (38)}$$

$$\| y_n - x_n \|^q \leq 2 \, (\beta_n^q \| x_n - T^n x_n \|^q + c_q \| \varepsilon_n \|^q) \; \text{(using (12))}. \qquad \text{... (39)}$$

Using (38) and (39) in (24) we obtain

$$\| T^n y_n - x_n \|^q \leq \| x_n - T^n x_n \|^q + qL \, [\beta_n \| x_n - T^n x_n \| + \| \varepsilon_n \|] \; \| x_n - T^n x_n \|^{q-1}$$

$$+ \; 2c_q L^q \, [\beta_n^q \| x_n - T^n x_n \|^q + c_q \| \varepsilon_n \|^q]$$

$$= \| x_n - T^n x_n \|^q + qL \, \beta_n \| x_n - T^n x_n \|^q + qL \| \varepsilon_n \| \| x_n - T^n x_n \|^{q-1}$$

$$+ \ 2c_q L^q \beta_n^q \| x_n - T^n x_n \|^q + 2c_q^2 L^q \| \varepsilon_n \|^q$$

$$\leq [1 + qL\beta_n] \| x_n - T^n x_n \|^q + 2c_q^2 L^q \| \varepsilon_n \| (1+L)^{q-1} \| x_n - x^* \|^{q-1}$$

$$+ \ 2c_q L^q \beta_n^q (1+L)^q \| x_n - x^* \|^q + 2c_q^2 L^q \| \varepsilon_n \|^q$$

$$\leq [1 + qL\beta_n] \| x_n - T^n x_n \|^q \, qL \| \varepsilon_n \| (1+L)^{q-1} [1 + \| x_n - x^* \|^{q-1}]$$

$$+ \ 2c_q L^q \beta_n^q (1+L)^q \| x_n - x^* \|^q + 2c_q^2 L^q \| \varepsilon_n \|^q$$

$$= \ [1 + qL\,\beta_n] \| x_n - Tx_n \|^q + qL \| \varepsilon_n \| (1+L)^{q-1}$$

$$+ \ [qL \| \varepsilon_n \| (1+L)^{q-1} + 2c_q L^q \beta_n^q (1+L)^q] \ \| x_n - x^* \|^q + 2c_q^2 L^q \| \varepsilon_n \|^q.$$

As in the proof of Theorem 2 we obtain $(1 + qL\,\beta_n) \leq (1 + L\,\beta_n)^q$, so that

$$\| T^n y_n - x_n \|^q \leq (1 + L\,\beta_n)^q \| x_n - T^n x_n \|^q + qL \| \varepsilon_n \| (1+L)^{q-1}$$

$$+ \ [qL \| \varepsilon_n \| (1+L)^{q-1} + 2c_q L^q \beta_n^q (1+L)^q] \| x_n - x^* \|^q$$

$$+ \ 2c_q^2 L^q \| \varepsilon_n \|^q. \qquad \dots (40)$$

Using inequalities (23), (37) and (40) in (36) we obtain

$$\| x_n - x^* + \alpha_n (T^n y_n - x_n) \|^q \leq \| x_n - x^* \|^q + q\,\alpha_n$$

$$\{[L(1+L)\,\beta_n + L \| \varepsilon_n \|] \| x_n - x^* \|^q + L \| \varepsilon_n \|\}$$

$$- \ q\,\alpha_n \left\{ \frac{1}{2} (1-k)\,(1+L)^{-(q-2)} \| x_n - T^n x_n \|^q - \frac{1}{2} (k_n^2 - 1) \| x_n - x^* \|^q \right\}$$

$$+ \ c_q\,\alpha_n^q \quad \{(1 + L\,\beta_n)^q \| x_n - T^n x_n \|^q + qL \| \varepsilon_n \| (1+L)^{q-1}$$

$$+ \ [qL \| \varepsilon_n \| (1+L)^{q-1} + 2c_q L^q \beta_n^q (1+L)^1] \| x_n - x^* \|^q$$

$$+ \ 2c_q^2 L^q \| \varepsilon_n \|^q\}$$

$$= \left\{ \ 1 + q\,\alpha_n [L\,(1+L)\,\beta_n + L \| \varepsilon_n \|] + \frac{q}{2}\,\alpha_n (k_n^2 - 1) \right.$$

$$+ \ c_q\,\alpha_n^q [qL \| \varepsilon_n \| (1+L)^{q-1} + 2c_q L^q \beta_n^q (1+L)^q]\} \ \| x_n - x^* \|^q$$

$$+ \, q\,L\,\alpha_n \,\|\, \varepsilon_n \,\| - \frac{q}{2}\,\alpha_n\,(1-k)\,(1+L)^{-(q-2)}\,\|\, x_n - T^n x_n \,\|^q$$

$$+ \, c_q\,\alpha_n^q\,(1+L\,\beta_n)^q\,\|\, x_n - T^n x_n \,\|^q + qLc_q\,\alpha_n^q\,\|\, \varepsilon_n \,\|\,(1+L)^{q-1}$$

$$+ \, 2c_q^3\,L^q\,\alpha_n^q\,\|\, \varepsilon_n \,\|^q. \qquad \ldots (41)$$

Using (41) and (35) we obtain

$$\|\, x_{n+1} - x^* \,\|^q$$

$$\leq \{1 + q\,\|\, \delta_n \,\| + (1 + q\,\|\, \delta_n \,\|)\,[q\,\alpha_n\,[L\,(1+L)\,\beta_n + L\,\|\, \varepsilon_n \,\|] + \frac{q}{2}\,\alpha_n\,(k_n^2 - 1)$$

$$+ \, c_q\,\alpha_n^q\,[qL\,\|\, \varepsilon_n \,\|\,(1+L)^{q-1} + 2c_q\,L^q\,\beta_n^q\,(1+L)^q]]\}\ \|\, x_n - x^* \,\|^q$$

$$+ \, (1+q)\,\|\, \delta_n \,\|)\,[qL\,\alpha_n\,\|\, \varepsilon_n \,\| + qLc_q\,\alpha_n^q\,\|\, \varepsilon_n \,\|\,(1+L)^{q-1} + 2c_q^3\,L^q\,\alpha_n^q\,\|\, \varepsilon_n \,\|^q]$$

$$- \, (1 + q\,\|\, \delta_n \,\|)\,\alpha_n\left[\frac{q}{2}\,(1-k)\,(1+L)^{-(q-2)} - c_q\,\alpha_n^{q-1}\,(1+L\,\beta_n)^q\right]\|\, x_n - T^n x_n \,\|^q$$

$$+ \, q\,\|\, \delta_n \,\| + c_q\,\|\, \delta_n \,\|^q$$

$$= \, [1 + \lambda_n]\,\|\, x_n - x^* \,\|^q - \alpha_n\left[\frac{q}{2}\,(1-k)\,(1+L)^{-(q-2)} - c_q\,\alpha_n^{q-1}\,(1+L\,\beta_n)^q\right]$$
$$\|\, x_n - T^n x_n \,\|^q$$

$$- q\,\|\, \delta_n \,\|\,\alpha_n\left[\frac{q}{2}\,(1-k)\,(1+L)^{-(q-2)} - c_q\,\alpha_n^{q-1}\,(1+L\,\beta_n)^q\right]\|\, x_n - T^n x_n \,\|^q + \sigma_n, \qquad \ldots (42)$$

where

$$\lambda_n = q\,\|\, \delta_n \,\| + (1 + q\,\|\, \delta_n \,\|)\left\{q\,\alpha_n\,[L\,(1+L)\,\beta_n + L\,\|\, \varepsilon_n \,\|] + \frac{q}{2}\,\alpha_n\,(k_n^2 - 1)\right.$$

$$\left. + \, c_q\,\alpha_n^q\,[qL\,\|\, \varepsilon_n \,\|\,(1+L)^{q-1} + 2c_q\,L^q\,\beta_n^q\,(1+L)^q]\right\}$$

and

$$\sigma_n = (1 + q\,\|\, \delta_n \,\|)\,[qL\,\alpha_n\,\|\, \varepsilon_n \,\| + qLc_q\,\alpha_n^q\,\|\, \varepsilon_n \,\|\,(1+L)^{q-1}$$

$$+ \, 2c_q^3\,L^q\,\alpha_n^q\,\|\, \varepsilon_n \,\|^q] + q\,\|\, \delta_n \,\| + c_q\,\|\, \delta_n \,\|^q.$$

From condition (i) we obtain

$$\frac{q}{2}\,(1-k)\,(1+L)^{-(q-2)} - c_q\,\alpha_n^{q-1}\,(1+L\,\beta_n)^q \geq 0,$$

so that it follows from (42) that

$$\| x_{n+1} - x^* \|^q \leq [1 + \lambda_n] \| x_n - x^* \|^q - \alpha_n$$

$$\left[\frac{q}{2} (1-k)(1+L)^{-(q-2)} - c_q \alpha_n^{q-1} (1 + L\beta_n)^q \right] \| x_n - T^n x_n \|^q + \sigma_n. \quad \ldots (43)$$

The condition $\sum_{n=0}^{\infty} (k_n^2 - 1) < \infty$ ad condition (ii) imply that $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\sum_{n=0}^{\infty} \sigma_n < \infty$, and it now follows as in the proof of Theorem 2 that $\{x_n\}$ converges strongly to a fixed point of T. □

Remark 5 : It is clear that Lemma 3 and Theorem 3 are true if K is a nonempty closed convex subset of E and T maps K into K, provided the sequence $\{x_n\}$ is assumed to remain in K.

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