Indian J. pure appl. Math., **32**(10) : 1553-1563, October 2001

COMPARISON THEOREMS AND OSCILLATION CRITERIA FOR DIFFERENTIAL EQUATIONS WITH SEVERAL DELAYS*

X. Y. ZENG, B. SHI[@] AND M. J. GAI

Department of Basic Sciences, Naval Aeronautical Engineering Academy, Yantai, Shandong 264001, *P. R. China*

(*Received* 8 *January* 2001; *accepted* 9 *April* 2001)

First, in this paper, we establish the equivalence of the oscillation of the differential equations with several delays of the form:

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(t-\tau_i) = 0 \text{ for } t \geq t_0$$

and the second-order differential equations without delay of the form:

$$y''(t) + \frac{2em}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \left[p_i(t) - \frac{1}{em\,\tau_i} \right] y(t) = 0 \text{ for } t \geq t_0,$$

where $p_i(t) \in C([t_0, \infty), I\!R^+)$ and $\tau_i > 0$ for $i = 1, ..., m$. Next, we obtain some "sharp" conditions for oscillation and nonoscillation of the first equation.

Key Words : Oscillation; Nonoscillation; Differential Equations; Equivalence

INTRODUCTION

Consider the differential equations with several delays of the form:

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0 \text{ for } t \geq t_0, \qquad \text{... (1)}$$

and the second-order differential equations without delay of the form:

$$y''(t) + \frac{2em}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \left[p_i(t) - \frac{1}{em\,\tau_i} \right] y(t) = 0 \text{ for } t \geq t_0, \qquad \text{... (2)}$$

where $p_i(t) \in C([t_0, \infty), I\!R^+)$ and $\tau_i > 0$ for $i = 1, ..., m$.

*Research supported by Distinguished Expert Science Foundation of Naval Aeronautical Engineering Academy;
[@]*Corresponding author; e-mail*: baoshi@public.ytptt.sd.cn)

By a *solution* of eq (1), we mean a function $x(t)$ which is defined for $t \geq t_0 \bar{\tau}$, where $\bar{\tau} := \max\{\tau_1, \ldots, \tau_m\}$ and satisfies eq (1) for $t \geq t_0$. A solution $x(t)$ of eq (1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, the solution $x(t)$ is called *nonoscillatory*.

Recently, there has been increasing interest in the study of oscillation and nonoscillation of differential equations with one delay of the form:

$$x'(t) + p(t)\, x(t-\tau) = 0 \text{ for } t \geq t_0, \qquad \ldots (*)$$

where $p(t) \in C([t_0, \infty), I\!R^+)$ and $\tau > 0$. For example, one is referred to Domshalk[1&2], Elbert and Stavroulakis[3], Kozakiewics[4], Tang and Shen[7] and Tang, Yu and Wang *et al.*[8] But it should be pointed out that there is no result about the equivalence of the oscillation between eqs (1) and (2).

Our aim in this paper is to establish the equivalence of the oscillation of eqs (1) and (2) and to obtain some "sharp" conditions for oscillation and nonoscillation of eq. (1).

LEMMAS

Lemma 1 — Assume that $a_i(t) \geq 0$ ($\not\equiv 0$) for $i = 1, \ldots, m$. Let $v(t)$ be an eventually positive solution of the equation:

$$\left[v(t) - \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t-\tau_i}^{t} v(s)\, ds \right]' + e \sum_{i=1}^{m} a_i(t)\, v(t-\tau_i) = 0 \text{ for large } t. \qquad \ldots(3)$$

Then eventually

$$u'(t) \leq 0 \text{ and } u(t) > 0, \qquad \ldots (4)$$

where

$$u(t) = v(t) - \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t-\tau_i}^{t} v(s)\, ds. \qquad \ldots (5)$$

PROOF : From (3) and (5), there exists a $T_1 > t_0$ such that

$$v(t) > 0 \text{ and } u'(t) = -e \sum_{i=1}^{m} a_i(t)\, v(t-\tau_i) \leq 0 \text{ for } t \geq T_1,$$

which implies that $u(t)$ is nonincreasing. So, if (4) does not hold, then eventually $u(t) < 0$ and there exists a $T_2 \geq T_1$ and an $\alpha > 0$ such that $u(t) \leq -\alpha$ for $t \geq T_2$. Then we have from (5)

$$v(t) \leq -\alpha + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t-\tau_i}^{t} v(s)\, ds. \qquad \ldots (6)$$

Now, we consider following two cases.

Case 1 — $v(t)$ is unbounded, *i.e.* $\limsup_{t\to\infty} v(t)=\infty$. Thus, there exists a sequence $\{t_j\}_{j=1}^{\infty}$ such that $t_j \ge T_2+\bar{\tau}$ and $t_j\to\infty$ as $j\to\infty$ and $\bar{v}(t_j)=\sup_{T_2\le t\le t_j}\{u(t)\}$.

In view of (6), we obtain

$$\bar{v}(t_j)\le -\alpha+\sum_{i=1}^{m}\frac{1}{m\,\tau_i}\int_{t_j-\tau_i}^{t_j} v(s)ds \le -\alpha+\bar{v}(t_j),$$

which is a contradiction.

Case II — $v(t)$ is bounded, *i.e.*, $\limsup_{t\to\infty} v(t)=b<\infty$. Choose a sequence $\{t_j^*\}_{j=1}^{\infty}$ such that $t_j^*\to\infty$ and $v(t_j^*)\to b$ as $j\to\infty$.

Let ξ_j be such that $\bar{v}(\xi_j)$ = max $\{v(s)\mid t_j^*-\bar{\tau}\le s\le t_j^*-1\}$. Then $\xi_j\to\infty$ as $j\to\infty$ and $\limsup_{j\to\infty}\bar{v}(\xi_j)\le b$. So, from (6), we have $v_{t_j^*}\le -\alpha+\bar{v}(\xi_j)$. Taking the upper limits on the both sides, we obtain $b\le -\alpha+\limsup_{j\to\infty}\bar{v}(\xi_j)\le -\alpha+b$, which is also a contradiction. The proof is complete.

MAIN RESULTS

***Theorem* 1** — *Assume that*

$$a_i(t):=p_i(t)-\frac{1}{em\,\tau_i}\ge 0\ (\not\equiv 0)\ \text{for large t.} \qquad \text{... (7)}$$

Then every solution of Eq (1) is oscillatory if and only if every solution of Eq (2) is oscillatory.

PROOF — THE PROOF OF SUFFICIENCY PART

If it is not the case, we let $x(t)$ be an eventually positive solution of eq (1). Then there exists an $T_1>t_0$ such that $x(t)>0$ and $a_i(t)\ge 0\ (\not\equiv 0)$ for $t\ge T_1$.

Let

$$v(t)=x(t)\exp\left\{\sum_{i=1}^{m}\frac{t}{m\,\tau_i}\right\}. \qquad \text{... (8)}$$

Then, $v(t)>0$ for $t\ge T_1$. By (8) and eq. (1), we obtain

$$v'(t)\exp\left\{-\sum_{i=1}^{m}\frac{t}{m\,\tau_i}\right\}+v(t)\exp\left\{-\sum_{i=1}^{m}\frac{t}{m\,\tau_i}\right\}\left(-\sum_{i=1}^{m}\frac{1}{m\,\tau_i}\right)$$

$$+\sum_{i=1}^{m} p_i(t)\, v(t-\tau_i) \exp\left\{-\sum_{i=1}^{m} \frac{t-\tau_i}{m\,\tau_i}\right\} = 0.$$

From (7), we have

$$v'(t) \exp\left\{-\sum_{i=1}^{m} \frac{t}{m\,\tau_i}\right\} + v(t) \exp\left\{-\sum_{i=1}^{m} \frac{t}{m\,\tau_i}\right\}\left(-\sum_{i=1}^{m} \frac{1}{m\,\tau_i}\right)$$

$$+\sum_{i=1}^{m}\left[a_i(t) + \frac{1}{em\,\tau_i}\right] v(t-\tau_i) \exp\left\{-\sum_{i=1}^{m} \frac{t-\tau_i}{m\,\tau_i}\right\} = 0.$$

Then we have

$$v'(t) - v(t) \sum_{i=1}^{m} \frac{1}{m\,\tau_i} + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} v(t-\tau_i) + e \sum_{i=1}^{m} a_i(t)\, v(t-\tau_i) = 0$$

or
$$\left[v(t) - \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t-\tau_i}^{t} v(s)\, ds\right]' + e \sum_{i=1}^{m} a_i(t)\, v(t-\tau_i) = 0 \text{ for } t \ge T_1 + \bar{\tau}. \quad \dots (9)$$

Set
$$u(t) = v(t) - \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t-\tau_i}^{t} v(s)\, ds \text{ for } t \ge T_1 + \bar{\tau}. \quad \dots (10)$$

From Lemma 1, there exists a $T_2 \ge T_1 + \bar{\tau}$ such that (4) holds for $t \ge T_2$. Let $d = \min\{v(t) \mid T_2 - \bar{\tau} \le t \le T_2\}$. Then we have

$$v(t) > d \text{ for } t \ge T_2 + \bar{\tau}. \quad \dots (11)$$

In fact, if (11) does not hold, let $t^* = \inf\{t > T_2 \mid v(t) \le d\}$. Then we have $v(t) > d$ for $T_2 - \bar{\tau} \le t < t^*$ and $v(t^*) = d$. Thus by (10), we get

$$d = v(t^*) = u(t^*) + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t^*-\tau_i}^{t^*} v(s)\, ds > d,$$

which is a contradition. So, (11) holds.

Let $\lim_{t\to\infty} u(t) = l \ge 0$. We consider the following two possible cases.

Case I — $l = 0$. Choose a $T_3 \ge T_2$ such that

$$u(t) < \frac{d}{4\bar{\tau}} \frac{\sum_{i=1}^{m} \tau_i}{m} \quad \text{for } t \geq T_3.$$

Therefore, for any $\bar{T} \geq T_3$, we have

$$\int_{\bar{T}}^{t+r} u(s)ds \leq 2\bar{\tau}\frac{d}{4\bar{\tau}} \sum_{i=1}^{m} \frac{\tau_i}{m} < \frac{\sum_{i=1}^{m} \tau_i}{2m} \upsilon(t) \text{ for } t \in [\bar{T}, \bar{T}+\bar{\tau}].$$

So, we get

$$\upsilon(t) > \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T}^{t+\bar{\tau}} u(s)ds \text{ for } t \in [\bar{T}, \bar{T}+\bar{\tau}].$$

Case II — $l > 0$: Owing to that $u(t)$ is nonincreasing, it follows that $u(t) \geq l$ for $t \geq T_2$. From (10) and (11), we have

$$\upsilon(t) \geq l + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t-\tau_i}^{t} \upsilon(s)\,ds \geq l + d \text{ for } t \geq T_2 .$$

By induction, we have

$$\upsilon(t) \geq nl + d \text{ for } t \geq T_2 + (n-1)\bar{\tau} \text{ and } n = 1, 2, \ldots .$$

Therefore, we get $\lim_{t\to\infty} \upsilon(t) = \infty$. So, there exists a $[\bar{T}' = T_3]$ such that

$$\upsilon(t) > \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{\bar{T}'}^{t+\bar{\tau}} u(s)\,ds \text{ for } t \in [T', T'+\bar{z}].$$

Then from the above two cases, we find that these exists a $T_4 \geq T_3$ such that

$$u(t) > \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{t+\bar{\tau}} u(s)ds \text{ for } t \in [T_4, T_4+\bar{\tau}].$$

In what follows, we shall prove that

$$v(t) > \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{t+\bar{\tau}} u(s)ds \text{ for } t \geq T_4. \quad \ldots (12)$$

If (12) does not hold, we let

$$T^* = \inf \left\{ t \geq T_4 + \bar{\tau} \,\middle|\, v(t) \leq \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{t+\bar{\tau}} u(s)ds \right\}.$$

Then
$$u(t) > \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{t+\bar{\tau}} u(s)ds \text{ for } t \in [T_4, T^*]$$

and
$$v(T^*) \leq \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{T^*+\tau} u(s)ds. \quad \ldots (13)$$

So, from (10) and (13), we have

$$\frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{T^*+\bar{\tau}} u(s)\,ds \geq v(T^*) = u(T^*) + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{T^*-\tau_i}^{T^*} v(s)ds$$

$$> u(T^*) + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{T^*-\tau_i}^{T^*} ds \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{s+\bar{\tau}} u(\xi)\,d\xi$$

$$= u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{T^*-\tau_i}^{T^*} ds \int_{T_4}^{T^*+\bar{z}} u(\xi)\,d\xi$$

$$- \frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{T^*+\bar{\tau}-\tau_i}^{T^*+\bar{\tau}} u(\xi)\,d\xi \int_{T^*-\tau_i}^{\xi-\bar{\tau}} ds$$

$$\geq u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{T^*+\bar{\tau}} u(s)ds$$

$$-\frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\tau_i} \int_{T^*+\bar{\tau}-\tau_i}^{T^*+\bar{\tau}} (\xi - \bar{\tau} - T^* + \tau_i)\, d\xi\, u(T^*)$$

$$= u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{T^*+\bar{\tau}} u(s)ds - \frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\tau_i} \frac{\tau_i^2}{2} u(T^*)$$

$$= \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{T^*+\bar{\tau}} u(s)ds,$$

which is a contradiction. Hence, (12) holds. Thus, we obtain

$$v(t-\tau_i) > \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{t+\bar{\tau}-\tau_i} u(s)ds \geq \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_4}^{t} u(s)ds \text{ for } t \geq T_4 + \bar{\tau}. \qquad \text{... (14)}$$

Write $y(t) = \int_{T_4}^{t} u(s)ds$. Then, $y'(t) = u(t), y''(t) = u'(t)$. From (9), (10) and (14), we have

$$y''(t) + \frac{2em}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} a_i(t) y(t) \leq 0 \text{ for } t \geq T_4 + \bar{\tau}, \qquad \text{... (15)}$$

which implies that (15) has an eventually positive solution. We know from Swason[6] that eq (2) has an eventually positive solution. This is a contradiction.

THE PROOF OF NECESSITY PART : If it is not the case, let $y(t)$ be an eventually positive solution of eq (2). Then there exists an $T_1 > t_0$ such that

$$a_i(t) \geq 0\ (\not\equiv 0),\ y(t) > 0,\ y'(t) > 0 \text{ and } y''(t) \leq 0 \text{ for } t \geq T_1. \qquad \text{... (16)}$$

Set $u(t) = y'(t)$ for $t \geq T_1$. Ther from (16) we get $u(t) > 0, u'(t) \leq 0$ for $t \geq T_1$ and

$$y(t) = \int_{T_1}^{t} u(s)ds + y(T_1) \text{ for } t \geq T_1. \qquad \text{... (17)}$$

Define a function $\upsilon(t)$ as follows :

$$\upsilon(t) = \frac{2m}{\sum_{i=1}^{m} \tau_i} y(T_1) \text{ for } t \in [T_1, T_1 + \bar{\tau}] \qquad \text{... (18)}$$

and
$$\upsilon(t) = u(t) + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{t-\tau_i}^{t} \upsilon(s)\, ds \text{ for } t \geq T_1 + \bar{\tau}. \qquad \text{... (19)}$$

It is easy for one to see from (18) that

$$\upsilon(t) \leq \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{t} u(s)\, ds + y(T_1) \right] \text{ for } t \in [T_1, T_1 + \bar{\tau}].$$

In what follows, we shall show that

$$\upsilon(t) \leq \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{t} u(s)\, ds + y(T_1) \right] \text{ for } t \geq T_1. \qquad \text{... (20)}$$

In fact, if (20) does not hold, let

$$T^* = \inf \left\{ t \geq T_1 \mid \upsilon(t) \leq \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{t} u(s)ds + y(T_1) \right] \right\}.$$

Then, we have

$$\upsilon(t) \leq \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{t} u(s)ds + y(T_1) \right] \text{ for } f \in [T_1, T^*]$$

and
$$\upsilon(T^*) > \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{T^*} u(s)ds + y(T_1) \right].$$

Therefore, we get

$$\frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{T^*} u(s)ds + y(T_1) \right]$$

$$< \upsilon(T^*) = u(T^*) + \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{T^*-\tau_i}^{T^*} \upsilon(s)ds$$

$$\leq u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{T^*-\tau_i}^{T^*} ds \left[\int_{T_1}^{s} u(\xi)\,d\xi + y(T_1) \right]$$

$$= u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \left[\int_{T^*-\tau_i}^{T^*} ds \int_{T_1}^{T^*} u(\xi)\,d\xi \right.$$

$$\left. - \int_{T^*-\tau_i}^{T^*} u(\xi)\,d\xi \int_{T^*-\tau_i}^{\xi} ds \right] + \frac{2m}{\sum_{i=1}^{m} \tau_i} y(T_1)$$

$$\leq u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_1}^{T^*} u(\xi)\,d\xi$$

$$- \frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \int_{T^*-\tau_i}^{T^*} (\xi - T^* + \tau_i)\,d\xi\, u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} y(T_1)$$

$$= u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} \int_{T_1}^{T^*} u(\xi)\,d\xi - \frac{2m}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} \frac{1}{m\,\tau_i} \frac{\tau_i^2}{2} u(T^*) + \frac{2m}{\sum_{i=1}^{m} \tau_i} y(T_1)$$

$$= \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{T^*} u(s)ds + y(T_1) \right],$$

which is a contradiction. So, (20) holds. Therefore, we have

$$v(t-\tau_i) \le \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{t-\tau_i} u(s)ds + y(T_1) \right]$$

$$\le \frac{2m}{\sum_{i=1}^{m} \tau_i} \left[\int_{T_1}^{t} u(s)ds + y(T_1) \right] \text{ for } t \ge T_1 + \bar{\tau}. \quad \dots (21)$$

By (7), (17), (19), (21) and eq (2), we get

$$\left[v(t) - \sum_{i=1}^{m} \frac{1}{m\tau_i} \int_{t-\tau_i}^{t} v(s)\,ds \right]' + \frac{2em}{\sum_{i=1}^{m} \tau_i} \sum_{i=1}^{m} a_i(t)\, v(t-\tau_i) \le 0 \text{ for } t \ge T_1 + \bar{\tau}.$$

Write $x(t) = v(t) \exp\left\{ -\sum_{i=1}^{m} \frac{t}{m\tau_i} \right\}$. Then we have

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(t-\tau_i) \le 0 \text{ for } t \ge T_1 + \bar{\tau}. \quad \dots (22)$$

This shows that (22) has an eventually positive solution. We know from Li[5], it follows that eq (1) has an eventually positive solution, which leads to a contradiction. The proof is complete.

By Theorem 1, we have the following comparison result for oscillation of eq (*).

Corollary 1 — Assume that

$$a(t) := p(t) - \frac{1}{\tau e} \ge 0 \ (\not\equiv 0).$$

Then every solution of Eq (*) is oscillatory if and only if every solution of the following equation :

$$y''(t) + \frac{2e}{\tau}\left[p(t) - \frac{1}{\tau e} \right] y(t) = 0 \text{ for } t \ge t_0$$

is oscillatory.

Now, we give some "sharp" conditions for the oscillation and nonoscillation of eq (1) by using Theorem 1.

Theorem 2 — *Assume that (7) holds. Then the following statements are true :*

(i) If

$$\liminf_{t\to\infty} t \sum_{i=1}^{m} \int_t^{\infty} \left[p_i(s) - \frac{1}{em\,\tau_i} \right] ds > \frac{\sum_{i=1}^{m} \tau_i}{8\,em} ,$$

then every solution of eq (1) oscillates ;

(ii) If

$$t \sum_{i=1}^{m} \int_t^{\infty} \left[p_i(s) - \frac{1}{em\,\tau_i} \right] ds \le \frac{\sum_{i=1}^{m} \tau_i}{8\,em} \quad \textit{for large } t,$$

then eq (1) has a nonoscillatory solution.

One is refered to Swanson[6] for Theorem 2.

REFERENCES

1. Y. Domshlak, In: *Functional Differential Equations, Israel Seminar*, **2** (1994) 59-68.
2. Y. Domshlak, *Sturmian comparison method in investigation of the behavior of solutions for differential-operator equations*, Elm, Baku, USSR (1986) (*in Russian*).
3. A. Elbert and I. P. Stavroulakis, *Proc Amer Math Soc.*, **12** (1995) 1503-10.
4. E. Kozakiewicz, *In: 4th int. Colloq. Diff. Eqns. Usp*, 157-61, 1994.
5. B. Li, *J. math Anal. Appl*, **192** (1995) 312-21.
6. C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, London, 1968.
7. X. H. Tang and J. H. Shen, *J. math. Anal. Appl.* **217** (1998) 32-42.
8. X. H. Tang, J. S. Yu and Z. C. Wang, *Sci. Bull.* **44** (1) (1999) 26-31 (*in Chinese*).