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SOME q-BERNOULLI NUMBERS OF HIGHER ORDER ASSOCIATED WITH THE p-ADIC q-INTEGRALS

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The purpose of this paper is to give a new definition of the extension of q-Bernoulli numbers by using a p-adic q-integral in the p-adic number field.

Key Words: q-Bernoulli, Numbers, p-acid, q-integrals; Vumer Conquence

INTRODUCTION

Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of rational integers, the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$.

Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{v_p(p)} = p^{-1}$. If $q \in \mathbb{C}_p$, we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. In this paper, we use the notation :

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q \to 1} [x] = x$.

For any positive integer N, it was known (see [3]) that $\mu_q(x + p^N \mathbb{Z}_p) = \frac{q^x}{[p^N]}$ can be extended to a distribution on $\mathbb{Z}_p$. This distribution yields an integral for each non-negative integer m (see [3]):

$$\beta_m = \int_{\mathbb{Z}_p} [a]^m \, d\mu_q(a) = \frac{1}{(1-q)^m} \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{i+1}{[i+1]},$$

where β_m is the mth Carlitz q-Bernoulli number, which reduces to B_k when $q = 1$.

To define a generalized q-Bernoulli number with order n, which reduces the generalized ordinary Bernoulli number of higher order, we first use the multiple p-acid q-integral.

In this paper, the aim is to define the extension number of q-Bernoulli number with order n and to give a new explicit formula by this number.

EXTENSION OF q-BERNOULLI NUMBER

For $h_i\,(i=1,2,\ldots,k)\in\mathbb{Z}_+$, we define a sequence of p-adic rational numbers as generalized Carlitz's q-Bernoulli numbers, polynomials with order k by

$$\beta_n^{(\overbrace{h_1,\ldots,h_k}^{k\text{ times}}:k)}=\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{ times}}[x_1+\ldots+x_k]^n\,q^{\sum_{i=1}^{k}x_i(h_i-1)}\,d\mu_q(x_1)\ldots d\mu_q(x_k), \quad \ldots (1)$$

and

$$\beta_n^{(\overbrace{h_1,\ldots,h_k}^{k\text{ times}}:k)}(x)=\beta_n^{(\overbrace{h_1,\ldots,h_k}^{k\text{ times}}:k)}(x,q)$$

$$=\underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{ times}}[x_1+\ldots+x_k]^n\,q^{\sum_{i=1}^{k}x_i(h_i-1)}\,d\mu_q(x_1)\ldots d\mu_q(x_k). \quad \ldots (2)$$

It is easy to see in [3] that

$$\beta_n^{(\overbrace{h_1,\ldots,h_k}^{k\text{ times}}:k)}=\frac{1}{(1-q)^n}\sum_{i=1}^{n}\binom{n}{i}(-1)^i\frac{(i+h_1)(i+h_2)\ldots(i+h_k)}{[i+h_1][i+h_2]\ldots[i+h_k]},$$

and

$$\beta_n^{(\overbrace{h_1,\ldots,h_k}^{k\text{ times}}:k)}(x,q)=\frac{1}{(1-q)^n}\sum_{i=1}^{n}\binom{n}{i}(-1)^i\frac{(i+h_1)(i+h_2)\ldots(i+h_k)}{[i+h_1][i+h_2]\ldots[i+h_k]}\,q^{ix}$$

$$=\sum_{j=0}^{n}\binom{n}{J}[x]^{n-j}q^{xj}\beta_n^{(\overbrace{h_1,\ldots,h_k}^{k\text{ times}}:k)},$$

for $n>0$.

Note that

$$\beta_n^{(\overbrace{1,1,\ldots,1}^{k\text{ times}}:k)}(x,q)=\beta_n^{(k)}(x,q)\ \text{and}\ \lim_{q\to 1}\beta_n^{(k)}(x,q)=B_n^{(k)}(x), \quad \ldots (5)$$

where $B_n^{(k)}(x)$ is called the mth Bernoulli polynomial of order k.

We may now mention the following formulas which are easy to prove :

$$q^{h_1}\,\beta_n^{(h_1\,:\,1)}(x+1,q)-\beta_n^{(h_1\,:\,1)}(x,q)=n\,[x]^{n-1}+h_1\,(q-1)\,[x]^n, \qquad \text{... (6)}$$

$$\beta_n^{(\overbrace{h_1+1,\,h_2+1,\,\ldots,\,h_k+1}^{k\text{ times}}:\,k)}(x,\,q)$$

$$=(q-1)\,\beta_{n+1}^{(\overbrace{h_1,\,h_2,\,\ldots,\,h_k}^{k\text{ times}}:\,k)}(x,\,q)+\beta_n^{(\overbrace{h_1,\,h_2,\,\ldots,\,h_k}^{k\text{ times}}:\,k)}(x,\,q), \qquad \text{... (7)}$$

and

$$[l]^{n-k}\sum_{s_1,\,s_2,\,\ldots,\,s_k=0}^{l-1} q^{s_1h_1+s_2h_2+\ldots+s_kh_k}\,\beta_n^{(\overbrace{h_1,\,h_2,\,\ldots,\,h_k}^{k\text{ times}}:\,k)}\left(x+\frac{s_1+s_2+\ldots+s_k}{l},\,q^l\right)$$

$$=\beta_n^{(\overbrace{h_1,\,h_2,\,\ldots,\,h_k}^{k\text{ times}}:\,k)}(lx,\,q), \qquad \text{... (8)}$$

where just as in formula (8) we get

$$[l]^{n-k}\sum_{r=0}^{\infty} t_{r,k}\,q^k\,\beta_n^{(k)}\left(x+\frac{s}{k},\,q^k\right)=\beta_n^{(k)}(xk,\,q),$$

where

$$\left(\frac{1-x^k}{1-x}\right)^z=\sum_{s=0}^{\infty} T_{k,s}\,x^s.$$

Let d be a fixed integer and let p be a fixed prime number. We set

$$X=\lim_{\substack{\leftarrow\\ N}}\,(\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^*=\bigcup_{\substack{0<a<dp\\ (a,p)=1}} a+dp\,\mathbb{Z}_p,$$

$$a+dp^N\mathbb{Z}_p=\{x\in X\mid x\equiv a\pmod{dp^N}\},$$

where $a\in\mathbb{Z}$ lies in $0\le a<dp^N$.

Let χ be a primitive Dirichlet character with conductor $d\in\mathbb{Z}_+$. Then we define the generalized q-Bernoulli number of higher order with χ as follows : for $m\ge 0$,

$$\beta_{n,\chi}^{(\overbrace{h_1, h_2, \dots, h_k}^{k\text{ times}} : k)}$$

$$= \underbrace{\int_X \int_X \dots \int_X}_{k\text{ times}} \prod_{i=1}^{k} \chi(x_i)\, [x_1 + \dots + x_k]^n \; q^{\sum_{i=1}^{k} x_i (h_i - 1)} \, d\mu_q(x_1) \dots d\mu_q(x_k). \quad \dots (9)$$

From (9), we easily get

$$\beta_{n,\chi}^{(\overbrace{h_1, h_2, \dots, h_k}^{k\text{ times}} : k)} \quad \dots (10)$$

$$= [d]^{n-k} \sum_{i_1, i_2, \dots, i_k = 0}^{d-1} q^{\sum_{j=1}^{k} h_j i_j} \beta_n^{(\overbrace{h_1, h_2, \dots, h_k}^{k\text{ times}} : k)} \left(\frac{i_1 + i_2 + \dots + i_k}{d}, q^d \right) \prod_{j=1}^{k} \chi(i_j). \dots (10)$$

Note that

$$\lim_{q \to 1} \beta_n^{(\overbrace{h_1, h_2, \dots, h_k}^{k\text{ times}} : k)} = B_n^{(\overbrace{h_1, h_2, \dots, h_k}^{k\text{ times}})},$$

where $B_n^{(\overbrace{h_1, h_2, \dots, h_k}^{k\text{ times}})}$ was defined in [2].

By the simple calculation, we see :

$$[x_1 + x_2 + \dots + x_m + d(t_1 + t_2 + \dots + t_m) : q^d]^n$$

$$= \left(\left[t_1 + \frac{x_1}{d} : q^d \right] + q^{dt_1 + x_1} \left[t_2 + \frac{x_2}{d} : q^q \right] + \dots \right.$$

$$\left. + \; q^{d(t_1 + \dots + t_{m-1}) + \sum_{i=1}^{m-1} x_i} \left[t_m + \frac{x_m}{d} : q^d \right] \right)^n$$

$$= \sum_{\substack{i_1, \dots, i_m \geq 0 \\ n = i_1 + \dots + i_m}} \sum_{k_1 = 0}^{n - i_1} \sum_{k_2 = 0}^{n - i_1 - i_2} \dots \sum_{k_{m-1} = 0}^{n - i_1 - \dots - i_{m-1}} \binom{n}{i_1, \dots, i_m} \binom{n - i_1}{k_1} \binom{n - i_1 - i_2}{k_2}$$

$$\binom{n-i_1-\ldots-i_{m-1}}{k_{m-1}}(q^d-1)^{\sum_{i=1}^{m-1} k_i}\left(\prod_{j=1}^{m-1}[t_j+x_j:q^d]^{k_j+i_j}\right)[t_m+x_m:q^d]^{i_m}, \quad \ldots (11)$$

where $\binom{n}{i_1,\ldots,i_m}$ is the multinomial coefficient.

By the definition of $\beta_n^{(\overbrace{1,1,\ldots,1}^{m \text{ times}}:m)}(x,q)=\beta_n^{(m)}(x,q)$, we have the following :

$$\beta_n^{(m)}\left(\frac{x_1+x_2+\ldots+x_m}{d},q^d\right)$$

$$=\sum_{i_1,\ldots,i_m\geq 0}\sum_{k_1=0}^{n-i_1}\cdots\sum_{k_{m-1}=0}^{n-i_1-\ldots-i_{m-1}}\binom{n}{i_1,\ldots,i_m}\binom{n-i_1}{k_1}\cdots\binom{n-i_1-\ldots-i_{m-1}}{k_{m-1}}$$

$$\cdot\left(\prod_{j=1}^{m-1}\beta_{k_j+i_j}\left(\frac{x_j}{d},q^d\right)\right)\beta_{i_m}\left(\frac{x_m}{d},q^d\right)(q^d-1)^{\sum_{i=1}^{m-1}k_i} \quad \ldots (12)$$

From (10), (12), we easily get

$$\beta_{m\,\chi}^{(m)}=\beta_{n,\chi}^{(\overbrace{1,1,\ldots,1}^{k \text{ times}}:m)}$$

$$=[d]^{n-m}\sum_{x_1,\ldots,x_m=0}^{d-1}q^{\sum_{j=1}^{m}x_j}\beta_n^{(m)}\left(\frac{x_1+x_2+\ldots+x_m}{d},q^d\right)\prod_{j=1}^{m}\chi(x_j)$$

$$=\sum_{\substack{i_1,\ldots,i_m\geq 0\\ n=i_1+\ldots+i_m}}\sum_{k_1=0}^{n-i_1}\cdots\sum_{k_{m-1}=0}^{n-i_1-\ldots-i_{m-1}}\binom{n}{i_1,\ldots,i_m}\binom{n-i_1}{k_1}\cdots\binom{n-i_1-\ldots-i_{m-1}}{k_{m-1}}$$

$$\cdot\left(\prod_{j=1}^{m-1}\beta_{k_j+i_j,\chi}\right)(q^d-1)^{\sum_{i=1}^{m-1}k_i}\beta_{i_m,\chi}. \quad \ldots (13)$$

Let $\mu_k=\mu_{k;\,q}$ be given by

$$\mu_k(a+dp^N\mathbb{Z}_p)=[dp^N:q]^{k-1}q^a\,bet_k\left(\frac{a}{dp^N},q^{dp^N}\right).$$

Then μ_k extends to a $\mathbb{Q}(q)$-valued distribution ont he compact open sets $U\subset X$ (see [5]).

Now, we define

$$\mu_n^{(m)}(x_1+x_2+\ldots+x_n+dp^N\mathbb{Z}_p)$$

$$=\sum_{\substack{i_1,\ldots,i_m\geq 0\\ n=i_1+\ldots+i_m}}\ \sum_{k_1=0}^{n-i_1}\cdots\sum_{k_{m-1}=0}^{n-i_1-\ldots-i_{m-1}}\binom{n}{i_1,\ldots,i_m}\binom{n-i_1}{k_1}\cdots\binom{n-i_1-\ldots-i_{m-1}}{k_{m-1}}$$

$$\cdot(q^d-1)^{\sum_{i=1}^{m-1}k_i}\left(\prod_{j=1}^{m-1}\mu_{k_j+i_j}(x_j+dp^N\mathbb{Z}_p)\right)\mu_{i_m}(x_m+dp^N\mathbb{Z}_p).$$

Then we have

$$\underbrace{\int_X\int_X\cdots\int_X}_{m\text{ times}}\prod_{i=1}^{m}\chi(x_i)\,d\mu_n^{(m)}(x_1+x_2+\ldots+x_m)=\beta_{n,\chi}^{(m)}, \qquad \ldots (14)$$

$$\underbrace{\int_{pX}\int_{pX}\cdots\int_{pX}}_{m\text{ times}}\prod_{i=1}^{m}\chi(x_i)\,d\mu_n^{(m)}(x_1+x_2+\ldots+x_m)=[p]^{n-m}\chi(p^m)\,\beta_{n,\chi}^{(m)}(q^p). \qquad \ldots (15)$$

By using (14), (15), the Kummer type congruence for $\beta_{n,\chi}^{(m)}$ can be proved in [5]. In this paper, we remain for the reader to prove the Kummer type congruence for $\beta_{n,\chi}^{(m)}$.

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