Indian J. pure appl. Math., **32**(10) : 1571-1579, October 2001

REFINEMENTS OF HADAMARD'S INEQUALITY FOR *r*-CONVEX FUNCTIONS

GOU-SHENG YANG

Department of Mathematics, Tamkang University, Tamsui, Taiwang 25137, *R.O.C.*

AND

DAH-YAN HWANG*

Department of General Education, Kuang Wu Institute of technology Peito, Taipei, Taiwan, 11271, *R.O.C.*

(*Received* 17 *May* 1999; *after revision* 2 *December* 1999; *accepted* 13 *March* 2001)

In the present note we establish that there is a monotonically increasing function between the integral power mean and the stolarsky mean.

Key Words : Hadamard Inequality; *r*-Convex Function; Integral Power Mean; Logarithmic Mean; Stolarsky Mean

INTRODUCTION

The inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}$$

which holds for all convex functions $f : [a, b] \to R$ are known in the literature as Hadamard inequalities.

Recently, C. E. M. Pearce, J. Pecaric and V. Simic[3] generalize this inequality to r-convex positive function f which on an interval $[a, b]$ if, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \begin{cases} (\lambda f(x)^r + (1-\lambda) f(y)^r)^{1/r}, & \text{if } r \neq 0, \\ f(x)^{\lambda} f(y)^{(1-\lambda)}, & \text{if } r = 0. \end{cases}$$

The definition of r-convexity naturally complements the concept of r-concavity, in which the inequality is reversed (cf. Uhrin[5]) and which plays an important role in statistics.

Corresponding author: *e-mail*: dyhuang@tmail.kwit.edu.tw.

We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

In what follows, we define :

(I) The integral power mean M_p of a positive function f on $[a, b]$ is a functional given by

$$M_p(f) = \begin{cases} \left[\dfrac{1}{b-a} \displaystyle\int_a^b f(t)^p \, dt \right]^{1/p} & , p \neq 0, \\ \exp\left[\dfrac{1}{b-a} \displaystyle\int_a^b \ln f(t) \, dt \right], & p = 0. \end{cases}$$

(II) The extended logarithmic mean L_p of two positive number a, b is given for $a \neq b$,

by
$$L_p(a, b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1))(b-a)} \right]^{1/p} & , p \neq -1, 0, \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a} \right)^{\frac{1}{(b-a)}} & , \quad p = 0 \end{cases}$$

and $Lp(a, a) = a$.

(III) The alternative extended logarithmic mean $F_r(x, y)$ of two positive numbers x, y is given for $x \neq y$ by

$$F_r(x, y) = \begin{cases} \dfrac{r}{r+1} \cdot \dfrac{x^{r+1} - y^{r+1}}{x^r - y^r}, & r \neq 0, -1, \\ \dfrac{x-y}{\ln x - \ln y}, & r = 0, \\ xy\left(\dfrac{\ln x - \ln y}{x-y} \right), & r = -1 \end{cases}$$

and
$$F_r(x, x) = x.$$

(IV) The Stolarsky mean $E(x, y, r, s)$ (see [4]) of two positive numbers x, y is given for $x \neq y$ by

$$E\,(x, y, r, s) = \left[\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \right]^{\frac{1}{(s-r)}}, \; r \neq s \text{ and } rs \neq 0,$$

$$E(x, y, r, 0) = E(x, y, 0, r) = \left[\frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x} \right]^{\frac{1}{r}}, \; r \neq 0,$$

$$E(x, y; r, r) = e^{\frac{-1}{r}} \left(\frac{x^{x^r}}{y^{y^r}} \right)^{\frac{1}{(x^r - y^r)}}, \; r \neq 0,$$

$$E\,(x, y; 0, 0) = \sqrt{xy}$$

and $E\,(x, y, r, s) = x$ if $x = y > 0$.

The following are extensions of Hadamard's inequality:

Theorem A^2 — *If* $f: [a, b] \to R$ *is positive, continuous and convex, then*

$$M_p\,(f) \leq L_p\,(f(a), f(b)),$$

while if f is concave, the inequality is reversed.

Theorem B^1 — *Suppose f is a positive function on [a, b]. If f is r-convex, then*

$$\frac{1}{b-a} \int_a^b f(t)\,dt \leq F_r\,(f(a), f(b)),$$

while if f is r-concave, the inequality is reversed.

Theorem C^3 — *Let f be defined as in theorem B. Then*

$$M_p\,(f) \leq E(f(a), f(b); r, p + r),$$

while if f is r-concave, the inequality is reversed.

Theorem C reduces to Theorem A and Theorem B when $r = 1$ and $p = 1$, respectively.

The main purpose of this note is to establish that there is a monotonically increasing function between

$$M_p\,(f) \text{ and } E\,(f(a), f(b); r, p + r).$$

MAIN RESULT

Theorem — *Suppose f is a positive r-convex function [a, b] and* $G : [0, 1] \to R$ *is defined by*

$$G(t) = \begin{cases} \left\{ \frac{1}{b-a} \int_a^b \left[\frac{x-a}{b-a} f(tb+(1-t)x)^r + \frac{b-x}{b-a} f(ta+(1-t)x^r \right]^{p/r} dx \right\}^{1/p}, & r \neq 0, p \neq 0, \\ \left\{ \frac{1}{b-a} \int_a^b \left[f(tb+(1-t)x)^{\left(\frac{x-a}{b-a}\right)} f(ta+(1-t)x)^{\left(\frac{b-x}{b-a}\right)} \right]^p dx \right\}^{1/p}, & r = 0, p \neq 0, \\ \exp\left\{ \frac{1}{b-a} \int_a^b \ln\left[\frac{x-a}{b-a} f(tb+(1-t)x)^r + \frac{b-x}{b-a} f(ta+(1-t)x)^r \right]^{1/r} dx \right\}, & r \neq 0, p = 0, \\ \exp\left\{ \frac{1}{b-a} \int_a^b \ln\left[f(tb+(1-t)x)^{\left(\frac{x-a}{b-a}\right)} f(ta+(1-t)x)^{\left(\frac{b-x}{b-a}\right)} \right] dx \right\}, & r = 0, p = 0. \end{cases}$$

Then

⟨ *i* ⟩ *G(t) is monotonically increasing on [0, 1] and*

⟨ *ii* ⟩ $G(0) = M_p(f)$ and $g(1) = E(f(a), f(b); r, p+r)$.

PROOF : Let $x \in [a, b]$ and $0 \le s \le t \le 1$. Then

$$sa + (1-s)x = \frac{[bt - as + sx - tx]}{t(b-a)} [ta + (1-t)x] + \frac{[as - at + tx - sx]}{t(b-a)} [tb + (1-t)x] \quad \dots (1)$$

and
$$sb + (1-s)x = \frac{[bt - bs + sx - tx]}{t(b-a)} [ta + (1-t)x] \frac{[bs - at + tx - sx]}{t(b-a)} [tb + (1-t)x]. \quad \dots (2)$$

For $r \neq 0$ and $p \neq 0$, it follows from (1), (2) and the r-convexity of f that

$$G(s) = \left\{ \frac{1}{b-a} \int_a^b \left[\frac{x-a}{b-a} f(sb+(1-s)x)^r + \frac{b-x}{b-a} f(sa+(1-s)x)^r \right]^{p/r} dx \right\}^{1/p}$$

$$\le \left[\frac{1}{b-a} \int_a^b \left[\frac{x-a}{b-1a} \left(\frac{(bt-bs+sx-tx)}{t(b-a)} f(ta+(1-t)x)^r \right. \right. \right.$$

$$\left. + \frac{(bs-at+tx-sx)}{t(b-a)} f(tb+(1-t)x)^r \right)$$

$$+ \frac{b-x}{b-a}\left(\frac{(bt-as+sx-tx)}{t(b-a)} f(ta+1-t)x)^r\right.$$

$$\left.\left.\left. + \frac{(as-at+tx-sx)}{t(b-a)} f(tb+(1-t)x)^r\right)\right]^{p/r} dx\right]^{1/p}$$

$$= \left\{\frac{1}{b-a}\int_a^b \left[\frac{x-a}{b-a} f(tb+(1-t)x)^r + \frac{b-x}{b-a} f(ta+(1-t)x)^r\right] dx\right\}^{1/p}$$

$$= G(t).$$

If $r = 0$ and $p \neq 0$, then f is log-convex, it follows from (1) and (2) that

$$G(s) = \left\{\frac{1}{b-a}\int_a^b \left[f(sb+(1-s)x)^{\left(\frac{x-a}{b-a}\right)} \cdot f(sa+(1-s)x)^{\left(\frac{b-x}{b-a}\right)}\right]^p dx\right\}^{1/p}$$

$$\leq \left[\frac{1}{b-a}\int_a^b \left[f(ta+(1-t)x)^{\frac{(bt-bs+sx-tx)}{t(b-a)}\cdot\left(\frac{x-a}{b-a}\right)}\cdot\right.\right.$$

$$f(tb+(1-t)x)^{\frac{(bs-at+tx-sx)}{t(b-a)}\cdot\left(\frac{x-a}{b-a}\right)}$$

$$\cdot f(ta+(1-t)x)^{\frac{(bt-as+sx-tx)}{t(b-a)}\cdot\left(\frac{b-x}{b-a}\right)}\cdot$$

$$\left.\left. f(tb+(1-t)x)^{\frac{(as-at+tx-sx)}{t(b-a)}\left(\frac{b-x}{b-a}\right)}\right]^p dx\right]^{1/p}$$

$$= \left\{\frac{1}{b-a}\int_a^b \left[f(tb+(1-t)x)^{\left(\frac{x-a}{b-a}\right)} \cdot f(ta+(1-t)x)^{\left(\frac{b-x}{b-a}\right)}\right]^p dx\right\}^{1/p}$$

$$= G(t).$$

If $r \neq 0$ and $p = 0$, using (1) and (2), we have

$$G(s) = \exp\left\{\frac{1}{b-a}\int_a^b \ln\left[\frac{x-a}{b-a} f(sb+(1-s)x)^r + \frac{b-x}{b-a} f(sa+(1-s)x)^r\right]^{1/r} dx\right\}$$

$$\leq \exp\left[\frac{1}{b-a}\int_a^b \ln\left[\frac{x-a}{b-a}\left(\frac{(bt-bs+sx-tx)}{t(b-a)} f(ta+(1-t)x)^r +\right.\right.\right.$$

$$\frac{(bs-at+tx-sx)}{t(b-a)} f(tb+(1-t)x)^r$$

$$+\frac{b-x}{b-a}\left(\frac{(bt-as+sx-tx)}{t(b-a)} f(ta+(1-t)x)^r\right.$$

$$\left.\left.\left.+\frac{(as-at+tx-sx)}{t(t-a)} f(tb+(1-t)x)^r\right)\right] dx\right]$$

$$=\exp\left\{\frac{1}{b-a}\int_a^b \ln\left[\frac{x-a}{b-a} f(tb+(1-t)x)^r+\frac{b-x}{b-a} f(ta+(1-t)x)^r\right]^{1/r} dx\right\}$$

$$= G(t).$$

Finally, if $r = 0$ and $p = 0$, using (1) and (2) again, we have

$$G(s)=\exp\left\{\frac{1}{b-a}\int_a^b \ln\left[f(sb+(1-s)x)^{\left(\frac{x-a}{b-a}\right)}\cdot f(sa+(1-s)x)^{\left(\frac{x-a}{b-a}\right)}\right] dx\right\}$$

$$\leq \exp\left[\frac{1}{b-a}\int_a^b \ln\left[f(ta+1-t)x)^{\frac{(bt-bs+sx-tx}{t(b-a)}\cdot\left(\frac{x-a}{b-a}\right\}}\right.\right.$$

$$\cdot f(tb+(1-t)x)^{\frac{(bs-at+tx-sx)}{t(b-a)}\cdot\left(\frac{x-a}{b-a}\right)}$$

$$\left.\left.\cdot f(ta+(1-t)x)^{\frac{(bt-as+sx-tx)}{t(b-a)}\cdot\left(\frac{b-x}{b-a}\right)}\cdot f(tb+(1-t)x)^{\frac{(as-at+tx-sx)}{t(b-a)}\cdot\left(\frac{b-x}{b-a}\right)}\right] dx\right]$$

$$=\exp\left\{\frac{1}{b-a}\int_a^b \ln\left[f(tb+(1-t)x)^{\left(\frac{x-a}{b-a}\right)}\cdot f(ta+(1-t)x)^{\left(\frac{b-x}{b-a}\right)}\right] dx\right\}$$

$$= G(t).$$

This completes the proof of $\langle i \rangle$.

To prove $\langle ii \rangle$ we observe first that

$$G(0)=M_p(f).$$

To prove $G(1)=E(f(a),f(b);r,p+r)$, suppose first that $f(a) = f(b)$. Then it is obviously $G(1) = f(a), f(b); r, p + r)$, so that we may assume $f(a) \neq f(b)$.

Case 1 — If $r \neq 0$ and $p \neq 0$, then

$$G(1)=\left\{\frac{1}{b-a}\int_a^b\left[\frac{x-a}{b-a}f(b)^r+\frac{b-x}{b-a}f(a)^r\right]^{p/r}dx\right\}^{1/p}$$

$$=\left\{\int_{f(a)^r}^{f(b)^r}\frac{t^{p/r}}{f(b)^r-f(a)^r}dt\right\}^{\frac{1}{p}}$$

$$=\left\{\frac{r}{p+r}\cdot\frac{f(b)^{p+r}-f(a)^{p+r}}{f(b)^r-f(a)^r}\right\}^{1/p}$$

$$=E(f(a),f(b);r,p+r)$$

if $r + p = 0$, and

$$G(1)=\left\{\frac{1}{b-a}\int_a^b\left[\frac{x-a}{b-a}f(b)^r+\frac{b-x}{b-a}f(a)^r\right]^{-1}dx\right\}^{1/p}$$

$$=\left\{\int_{f(a)^r}^{f(b)^r}\frac{1/t}{f(b)^r-f(a)^r}dt\right\}^{1/p}$$

$$=\left\{\frac{\ln f(b)^r-\ln f(a)^r}{f(b)^r-f(a)^r}\right\}^{-1/r}$$

$$=\left\{\frac{1}{r}\cdot\frac{f(b)^r-f(a)^r}{\ln f(b)-\ln f(a)}\right\}^{1/r}$$

$$=E(f(a),f(b);r,0)$$

if $r + p = 0$.

Case 2 — If $r = 0$ and $p \neq 0$, then

$$G(1)=\left\{\frac{1}{b-a}\int_a^b\left[f(b)^{\left(\frac{x-a}{b-a}\right)}\cdot f(a)^{\left(\frac{b-x}{b-a}\right)}\right]^p dx\right\}^{1/p}$$

$$=\left\{\frac{-1}{p}f(b)^p\int_a^b\left(\frac{f(a)}{f(b)}\right)^{p\cdot\left(\frac{b-x}{b-a}\right)}dp\left(\frac{b-x}{b-a}\right)\right\}^{1/p}$$

$$= \left\{ \frac{1}{p} \cdot \frac{f(a)^p - f(b)^p}{\ln f(a) - \ln f(b)} \right\}^{1/p}$$

$$= E(f(a), f(b); 0, p).$$

Case 3 — If $r \neq 0$ and $p = 0$, then

$$G(1) = \exp\left\{ \frac{1}{b-a} \cdot \frac{1}{r} \int_a^b \ln\left(\frac{x-a}{b-a} f(b)^r + \frac{b-x}{b-a} f(a)^r \right) dx \right\}$$

$$= \exp\left[\frac{1}{r} \cdot \frac{1}{f(b)^r - f(a)^r} \left[\left(\frac{x-a}{b-a} f(b)^r + \frac{b-x}{b-a} f(a)^r \right) \cdot \ln\left(\frac{x-a}{b-a} f(b)^r + \frac{b-x}{b-a} f(a)^r \right) - \left(\frac{x-a}{b-a} f(b)^r + \frac{b-x}{b-a} f(a)^r \right) \right]_a^b \right]$$

$$= e^{-1/r} \left(\frac{f(b)^{f(b)^r}}{f(a)^{f(a)^r}} \right)^{\frac{1}{f(b)^r - f(a)^r}}$$

$$= E(f(a), f(b); r, r).$$

Case 4 — If $r = 0$ and $p = 0$, then

$$G(1) = \exp\left\{ \frac{1}{b-a} \int_a^b \left(\frac{x-a}{b-a} \ln f(b) + \frac{b-x}{b-a} \ln f(a) \right) dx \right\}$$

$$= \exp\left\{ \frac{1}{(b-a)^2} \left[(x^2/2 - ax) \ln f(b) + (bx - x^2/2) \ln f(a) \right]_a^b \right\}$$

$$= \sqrt{f(a), f(b)}$$

$$= E(f(a), f(b), 0, 0).$$

This completes the proof of $\langle ii \rangle$.

Remark 1 : The Theorem is refinements of Theorem C and then Theorem A and Theorem B.

Remark 2 : If f is a positive r-concave function, then $G(t)$ is monotonically decreasing on [0, 1].

ACKNOWLEDGEMENT

The second author acknowledges support of the Kans Wa Institute of Technology under Project No. KW-87-CC02.

REFERENCES

1. P. M. Gill, C. E. M. Pearce and J. Peccaric, *J. math. Anal. Appl.* **215** (1997), 461-70.
2. C. E. M. Pearce and J. Peccaric, *Bull. Austral. Math. Soc.* **53** (1996) 229-33.
3. C. E. M. Pearce, J. Peccaric and V. Simic, *J. math. Anal. Appl.* **220** (1998) 99-109.
4. K. B. Stolarsky, *Math. Mag.* **48** (1975) 87-92.
5. B. Uhrin, *Ann. Probab.* **12** (1984) 640-45.