*Indian J. pure appl. Math.*, 32(10) : 1591-1599, October 2001


# DIOPHANTINE INEQUALITIES AND IRRATIONALITY MEASURES FOR CERTAIN TRANSCENDENTAL NUMBERS

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(*Received* 20 *April* 1999; *after revision* 26 *November* 1999; *accepted* 23 *April* 2001)

This paper describes diophantine inequalities and irrationality measures for certain transcedental numbers.

**Key Words : Duophantine Inequalities; Irrationality Measures; Trouscendental Numbers; Number Theory; Combinations**

## INTRODUCTION

A real number $\alpha$ is said to be a *Liouville number* if there exists an infinite sequence of rational numbers $p_1/q_1, p_2/q_2, \ldots$ such that

$$\left| \alpha - \frac{p^N}{q^n} \right| \leq q_N^{-N}$$

for all $N \geq 1$. The first and most notable property of these numbers was discovered by Liouville who proved in 1844 that all such numbers are transcendental.

In 1932, Mahler[12] discovered a classification of complex numbers in terms of certain diophantine approximation properties. We now recall part of Mahler's classification (see also [3] or [10]). Let $\alpha$ be a complex number. For $N, H \in \mathbb{Z}^+$, suppose that $\beta$ is an algebraic number of degree $N$ with $h(\beta) \leq H$ (here the height of $\beta$ is the maximum absolute value of the coefficients of the minimal polynomial for $\beta$ in $\mathbb{Z}[X]$) that minimizes the quantity $|\alpha - \beta|$. Let $\omega(N, H)$ be defined so that

$$|\alpha - \beta| = H^{-N\omega(N,H)}.$$

If there exists an $N$ such that $\omega(N, H)$ is unbounded then $\alpha$ is called a *U-number.* If $N$ is the smallest such integer for which $\omega(N, H)$ is unbounded, then $\alpha$ is called a $U_N$*-number*. Thus, $U_N$-numbers have excellent approximation by algebraic numbers of degree $N$. We remark that $U$-numbers are transcendental and furthermore that the set of $U_1$-numbers is precisely the set of Liouville numbers.

†This work was completed while the author was an Ulam Visiting Professor of Mathematics at the University of Colorado at Boulder. The author wishes to thank the Department of Mathematics for their warm hospitality and support.

With respect to Liouville numbers, Erdos[7] proved the following attractive result.

***Theorem*** (Erdos) — *Every real number is the sum of two Liouville numbers.*

Various refinements and generalizations have been given in the work of Alniacik[1&2] Pollington[13] and the author[4].

In a different direction, using the following elementary counting argument we show that given any transcendental number $\alpha \in \mathbb{R}$, there exists a decomposition of $\alpha$ into the sum of two *algebraically independent* Liouville numbers. Implicit in Erdos proof is the fact that there exist uncountably many distinct pairs of Liouville numbers $(x, y)$ such that $\alpha = x + y$. We denote the set of all such pairs $(x, y)$ by $\mathcal{L}_\alpha \subseteq \mathbb{R}^2$. If we assume that the transcendental number $\alpha$ is not expressible as a sum of two algebraically independent Liouville numbers, then for each pair $(x, y) \in \mathcal{L}_\alpha$, $x$ and $y$ are algebraically dependent. As there are only countably many polynomials in $\mathbb{Z}[X, Y]$, it follows that there must exist a polynomial $P(X, Y) \in \mathbb{Z}[X, Y]$ so that $P(x, y) = 0$ for infinitely many $(x, y) \in \mathcal{L}_\alpha$. By construction, each pair $(x, y) \in \mathcal{L}_\alpha$ is a solution to $X + Y - \alpha = 0$. Thus the curve $P(X, Y) = 0$ contains the line $X + Y - \alpha = 0$ and hence $\alpha$ must be algebraic which is a contradiction. Therefore, every transcendental number may be decomposed as a sum of two algebraically independent Liouville numbers.

Conversely, since algebraically dependent numbers belong to the same Mahler class (see [3]), the only transcendental numbers that could possibly be expressed as the sum of two *algebraically dependent* Liouville numbers are $U$-numbers. It is obvious that every $U_1$-number (Liouville number) has such a decomposition.

In [6] it was shown that there exists an uncountable collection of $U_2$-numbers that are expressible as the sum of two algebraically dependent Liouville numbers. This was accomplished by proving that for any set $\mathcal{S}$ of nonnegative integers, there exist uncountably many $U_2$-numbers $\alpha$ such that $(\alpha + k)^2$ is a Liouville number for all $k \in \mathcal{S}$. A precise statement of the quantitative version of this result is stated here as Theorem 5. Thus selecting $\mathcal{S} = \{0, 1\}$, the previous result implies that there exist $U_2$-numbers $\alpha$ so that both $(\alpha + 1)^2/2$ and $-(\alpha^2 + 1)/2$ are Liouville numbers. Plainly these Liouville numbers are algebraically dependent and their sum is $\alpha$.

Suppose that $\alpha$ is a $U_2$-number such that $(\alpha + k)^2$ is a Liouville number for each $k \in \mathcal{S}$. Here we wish to study the behaviour of $(\alpha + m)^2$ for natural numbers $m \notin \mathcal{S}$. In this direction, let $f(x)$ be an unbounded, increasing positive function. We say that $\alpha$ is a *Liouville number of type f* if there exists an infinite sequence of distinct rationals $p_1/q_1, p_2/q_2, \ldots$ so that

$$\left| \alpha - \frac{p_n}{q_n} \right| = O\left(q_n^{-f(q_n)}\right),$$

where the implied constant only depends upon $\alpha$. As an aside we remark that with only minor modifications to the proof of Erdos[7] one has

***Theorem*** 1 — *Let $\alpha$ be a real number and $f(x)$ be an unbounded, increasing positive function. Then $\alpha$ is the sum of two Liouville numbers of type $f$.*

Before stating our main theorem, we give the following qualitative formulation.

***Theorem*** 2 — *Let $\varepsilon > 0$ and $\mathcal{S}$ be any nonempty set of nonnegative integers. Then there exists uncountably many $U_2$-numbers $\alpha$ such that for any nonnegative integer $k$, $(\alpha + k)^2$ is Liouville of type $q(\log q)^{1+\varepsilon}$ if and only if $k \in \mathcal{S}$.*

In fact, all implied constants may be explicitly given and thus we are able to deduce effective measures of irrationality for the $U$-numbers $(\alpha+k)^2$ where $k$ is any nonnegative integer. This leads to our main result which also allows us to control, indirectly, the measure of irrationality of $\alpha$ via its continued fraction expansion. For $\alpha \in \mathbb{R}$, we denote its (simple) continued fraction expansion by $\alpha = [a_0, a_1, \ldots]$.

***Theorem* 3** — *Let $\mathcal{S}$ be any nonempty set of nonnegative integers and $a_0 \in \mathbb{Z}^+$. Let $\mathcal{A}'$ be any nonempty set of positive odd integers and let $\mathcal{A} = \mathcal{A}' \cup \{2(a_0 + \mathcal{S})\}$. Then given $\varepsilon$, $0 < \varepsilon \le 1$, there exists uncountably many $U_2$-numbers $\alpha = [a_0, a_1, \ldots]$ such that $a_n \in \mathcal{A}$ for all $n \ge 1$ and so that for any nonnegative integer $k$, $(\alpha+k)^2$ is Liouville of type $q(\log q)^{1+\varepsilon}$ if and only if $k \in \mathcal{S}$. Specifically, for each nonnegative integer $n$, there exists an explicitly computable constant $c_n = c_n(a_0, \mathcal{A}, \varepsilon) > 0$, such that for each $k \in \mathcal{S}$, there exist an infinite collection of distinct rationals $p/q$ such that*

$$\left| (\alpha+k)^2 - \frac{p}{q} \right| < c_k\, q^{-q(\log q)^{1+\varepsilon}},$$

*and for each nonnegative integer $m \notin \mathcal{S}$ and rational $p/q$ satisfying*

$$q > \max\{504,\ \exp(1 + (\alpha+m)^2)^{1/(2\varepsilon)}\},$$

*it follows that* $c_m\, q^{-q(\log q)^{1+\varepsilon}} < \left| (\alpha+m)^2 - \frac{p}{q} \right|.$

*As will become evident in the proof of Theorem 3, if we let $\mu$ denote the smallest element of $\mathcal{A}$, then the constant $c_n(a_0, \mathcal{A}, \varepsilon)$ may be taken to be*

$$c_n = ((a_0 + n + 1)(\mu+1)^{(\mu+1)^{52}})^{-2}. \qquad \ldots (1.1)$$

*If the set $\mathcal{A}$ is finite, then the $\alpha$'s occurring in the theorem would have bounded partial quotients and hence would be badly approximable numbers. In this case we may improve the Liouville type in Theorem 3. In particular we have the following.*

***Theorem* 4** — *Given the hypotheses of Theorem 3, if we further assume that the set $\mathcal{A}$ has finite cardinality, then there exist uncountably many badly approximable $U_2$-numbers $\alpha = [a_0, a_1, \ldots]$ such that $a_n \in \mathcal{A}$ for all $n \ge 1$ and such that for any nonnegative integer $k$, $(\alpha+k)^2$ is Liouville of type $C\,q\,(\log q)^{\varepsilon}$, where $C = C(\alpha) = 36 \log(M+1)$, and $M$ denotes the maximum element of $\mathcal{A}$. That is, for each $k \in \mathcal{S}$, the inequality*

$$\left| (\alpha+k)^2 - \frac{p}{q} \right| < c_k\, q^{-Cq(\log q)^{\varepsilon}}$$

*has infinitely many rational solutions $p/q$, and for each nonnegative integer $m \notin \mathcal{S}$ and rational $p/q$ satisfying $q > \exp(1 + (\alpha+m)^2)^{1/(2\varepsilon)}$, it follows that*

$$c_m q^{-Cq(\log q)^{\varepsilon}} < \left| (\alpha+m)^2 - \frac{p}{q} \right|.$$

ACKNOWLEDGEMENT

The author wishes to thank Dr. T. Struppeck for several helpful discussions regarding this work.

## PREVIOUS RESULTS AND CONTINUED FRACTIONS

We begin by recalling the main result of [6] (Theorem 1).

***Theorem* 5** — *Suppose that $\mathcal{S}$ is a set of nonnegative integers and $a_0 \in \mathbb{Z}^+$. Let $\mathcal{A} \subseteq \mathbb{Z}^+$ be a set of cardinality card $(\mathcal{A}) \geq 2$ with $2(\mathcal{S}+a_0) \subseteq \mathcal{A}$. Let $\mathcal{P}: \mathcal{A} \to [0, 1]$ be a probability measure on $\mathcal{A}$ and $g: \mathbb{Z}^+ \to \mathbb{Z}^+$ be a monotonically increasing function tending to infinity. Then there exists an uncountable collection of $U_2$-numbers such that for each $\alpha = [a_0, a_1, \ldots]$ in the collection :-*

(*i*) *$a_n \in \mathcal{A}$ for all $n \geq 1$.*

(*ii*) *For each $a \in \mathcal{A}$, the asymptotic density of $a$ in the sequence $\{a_0, a_1, \ldots\}$ is $\mathcal{P}(a)$.*

(*iii*) *For sufficiently large $n$, $a_n \leq g(n)$.*

(*iv*) *$(\alpha+k)^2$ is a Liouville number for each $k \in \mathcal{S}$.*

We remark that the function field analogue of this result is given in [5]. The proof of Theorem 5 involves generating certain diophantine approximations by quadratic irrational having trace zero. In essence, we construct $\alpha$ by generating an infinite sequence of "words" (finite strings of elements of $\mathcal{A}$) each of which is a palindrome. These words are then used to build the trace zero quadratic irrational approximates.

We will utilize the following elementary lemma about continued fractions.

***Lemma* 6** — Let $\alpha = [a_0, a_1, \ldots]$ and $\beta = [b_0, b_1, \ldots]$ be two irrational numbers and let $q_N$ be the denominator of the $N$th convergent of $\alpha$. If

$$|\alpha - \beta| < \frac{1}{q_N q_{N+1}}$$

then $\quad a_n = b_n$ for $n = 0, 1, 2, \ldots, N-2$.

PROOF : For convenience we will assume that $\alpha < \beta$. We first consider the case when $\alpha < p_N/q_N < \beta$. Thus we have

$$|\alpha - \beta| = \left| \alpha - \frac{p_N}{q_N} \right| + \left| \beta - \frac{p_N}{q_N} \right| < \frac{1}{q_N q_{N+1}}. \quad \ldots (2.1)$$

By a well-known result, $|\alpha - p_N/q_N| > 1/(2q_N q_{N+1})$, and hence (2.1) yields

$$\left| \beta - \frac{p_N}{q_N} \right| < \frac{1}{2q_N q_{N+1}} < \frac{1}{2q_N^2}.$$

Thus by Legendre's theorem, $p_N/q_N$ is a convergent of $\beta$ and therefore we have $a_n = b_n$ for all $n = 1, 2, ..., N - 1$.

Next we examine the case when $pN/qN < \alpha < \beta$. In this case we consider $p_N - 1/q_{N-1}$ and note that if $\alpha < pN - 1/qN - 1 < \beta$, then in view of the inequalities

$$|\alpha - \beta| < \frac{1}{q_N q_{N+1}} < \frac{1}{q_N - 1\, q_N},$$

the previous argument shows that $a_n = b_n$ for all $n = 0, 1, 2, ..., N - 2$. If, on the other hand, $\alpha < \beta < p_{N-1}/q_{N-1}$, then as $\alpha = [a_0, a_1, ..., a_{N-1}, ...]$ and $p_N - 1/q_{N-1} = [a_0, a_1, ..., a_{N-1}]$, we have that $a_n = b_n$ for all $n = 0, 1, 2, ..., N - 2$.

Finally, if $\alpha < \beta < p_N/q_N$, then by the previous remarks we conclude that $a_n = b_n$ for all $n = 0, 1, 2, ..., N - 1$, which completes the proof.

It is well-known that if $p/q > 1$ is a rational such that $\sqrt{p/q}$ is irrational, then the continued fraction expansion for $\sqrt{p/q}$ is periodic and has the form

$$\sqrt{p/q} = [a, \overline{W; 2a}],$$

where the word $W$ is a palindrome (see [9]). Upper bounds for the period length of $\sqrt{N}$, where $N \in \mathbb{Z}^+$ is not a perfect square, have been studied extensively and are important in the study of real quadratic number fields (see, for example, [8], [11], [14] and [15]). In particular, Stanton, Sudler and Williams[14] showed that the period length of $\sqrt{N}$ is bounded from above by $3.76\sqrt{N}\log N$. Here we will require an upper bound for the period length of $\sqrt{p/q}$. We denote the period length of $\sqrt{p/q}$ as $\boldsymbol{p}(\sqrt{p/q})$. The next lemma follows almost directly from the arguments of Stanton, Sudler and Williams (Section 5, [14]).

*Lemma* 7 — If $p/q > 1$ is a rational such that $\sqrt{p/q}$ is irrational, then

$$\boldsymbol{p}(\sqrt{p/q}) \le 3.76\sqrt{p/q}\,\log(pq).$$

## THE PROOF OF THEOREM 3

We begin with an observation regarding the proof of Theorem 5 (Theorem 1 in [6]). The construction allows us to make $(\alpha + k)^2$ Liouville of any type we desire for all $k \in S$. That is, we may construct $\alpha$ so that given $k \in S$, there exists an infinite sequence of rational numbers $r_1/s_1, r_2, s_2, ...$ such that

$$\left| (\alpha + k)^2 - \frac{r_n}{s_n} \right| < \frac{1}{(s_n)^{\Omega_n}},$$

where $\{\Omega_n\}$ is a positive monotonically increasing sequence tending to infinity. In particular, an important point which is implicit in the construction of $\alpha$ (see inequality (3.3) of [6]) is that the parameter $\Omega_n$ may be selected *after* the rational $r_n/s_n$ has been determined. Thus $\Omega_n$ may grow as a function of the denominator $s_n$.

Next we define the function $g : \mathbb{Z}^+ \cup \{0\} \to [1, \infty)$ by

$$g(n) = \begin{cases} a_0 & \text{if } n = 0 \\ \mu & \text{if } 1 \le n \le (\mu+1)^{52} \\ n^{1/52} - 1 & \text{if } n \supset (\mu+1)^{52} \end{cases}$$

(recall that $\mu$ is the smallest element of $\mathcal{A}$)). For convenience we define $f_\varepsilon(q) = q\,(\log q)^{1+\varepsilon}$.

We now construct $\alpha = [a_0, a_1, \ldots]$ as in the proof of Theorem 5 where $a_n \in \mathcal{A}$ for all $n \ge 1$. We remark that as $\mu \le g(n)$ for all $n$, we are able to arrange that the inequality $a_n \le g(n)$ holds for *all* $n \ge 0$ rather than just for all sufficiently large $n$. Finally we note that by our comments at the beginning of the proof, it follows that for each $k \in \mathcal{S}$, there exists an infinite sequence of distinct rationals $\{r_n/s_n\}$ so that

$$\left| (\alpha + k)^2 - \frac{r_n}{s_n} \right| < c_k\, s_n^{-f_\varepsilon(s_n)}.$$

It is plain that $(\alpha+k)^2$ is Liouville of type $f_\varepsilon(q)$ for all $k \in \mathcal{S}$. it now remains to show that for a nonnegative integer $m, m \notin \mathcal{S}, (\alpha+m)^2$ is not Liouville of type $f_\varepsilon(q)$ and that the second inequality of the theorem holds.

We select and fix a nonnegative integer $m, m \notin \mathcal{S}$. We let $q_N$ denote the denominator of the $N$th convergent of $\alpha + m$. It is easy to see that the continued fraction expansion for $\alpha + m$ is given by $\{a_0 + m, a_1, a_2, a_3, \ldots]$. By an elementary inequality from the theory of continued fractions and our bounds on the partial quotients of $\alpha$ we have

$$q_{N+1}\, q_{N+2} \le \prod_{n=0}^{N+1} (a_n + 1) \prod_{n=0}^{N+2} (a_n + 1)$$

$$< (a_0 + m + 1)^2 \prod_{n=1}^{N+2} (g(n) + 1)^2 \qquad \ldots (3.1)$$

$$\le (a_0 + m + 1)^2\, (\mu + 1)^{2(\mu+1)^{52}} \prod_{n=1}^{N+2} n^{1/26}$$

$$= (a_0 + m + 1)^2\, (\mu + 1)^{2(\mu+1)^{52}}\, ((N+2)\,!)^{1/26}.$$

Thus

$$q_{N+1}\, q_{N+2} < c'\, ((N+2)\,!)^{1/26},$$

where $c' = (a_0 + m + 1)^2\, (\mu + 1)^{2(\mu+1)^{52}}$. An application of Sterling's formula gives

$$q_{N+1}\, q_{N+2} < c'\, (2\pi/e^2)^{1/52}\, (N+2)^{(N+2)/26} < c'\, (N+2)^{(N+2)/26}. \qquad \ldots (3.2)$$

Let $p/q$ be an arbitrary rational, $p/q > 1$, satisfying $\sqrt{p/q} \notin \mathbb{Q}$,

$$q > \max\ \{504,\ \exp\ (1+(\alpha+m)^2)^{1/(2\varepsilon)}\},$$

and

$$\left| (\alpha+m)^2 - \frac{p}{q} \right| \le 1.$$

As we have $q > \exp\ (1+(\alpha+m)^2)^{1/(2\varepsilon)}$, it follows that

$$p = | p + (\alpha+m)^2 q - (\alpha+m)^2 q |$$

$$\le q \left| (\alpha+m)^2 - \frac{p}{q} \right| + (\alpha+m)^2 q$$

$$\le q\,(1+(\alpha+m)^2) \le q\,(\log q)^{2\varepsilon}.$$

The previous inequality along with Lemma 7 reveal

$$p\,(\sqrt{p/q}) \le 11.28 f_\varepsilon(q).$$

If we now set $N = [11.28\ f_\varepsilon(q)] + 1$, then inequality (3.2) together with the fact that $q > 504$ yield

$$q_{N+1}\, q_{N+2} < c'\,(13\,q\,(\log q)^{1+\varepsilon})^{(13\,q\,(\log q)^{1+\varepsilon})/26}$$

$$\le c'\, q^{q\,(\log q)^{1+\varepsilon}}. \qquad \ldots (3.3)$$

Given the value of $C'$, we have

$$q_{N+1}\, q_{N+2} < c_m^{-1}\, q^{q\,(\log q)^{1+\varepsilon}}, \qquad \ldots (3.4)$$

where the constant $c_m$ is as defined in (1.1). We now claim that the last inequality of the theorem holds. We prove this by contradiction and suppose that the inequality does not hold for $p/q$. Thus

$$\left| (\alpha+m)^2 - \frac{p}{q} \right| \le c_m\, q^{-q\,(og\, q)^{1+\varepsilon}},$$

which by (3.4) reveals that

$$\left| (\alpha+m)^2 - \frac{p}{q} \right| < \frac{1}{q_{N+1}\, q_{N+2}}.$$

Plainly $| \alpha+m+\sqrt{p/q} | \ge 1$, so we conclude

$$\left| \alpha+m-\sqrt{\frac{p}{q}} \right| < \frac{1}{q_{N+1}\, q_{N+2}}.$$

We recall that $q_N$ is the $N$th connersent for $\alpha+m$ and hence by Lemma 6 we have that the first $N-1$ partial quotients of $\alpha+m$ must agree with the first $N-1$ partial quotients of $\sqrt{p/q}$. By our choice of $N$ we see that $N-1 \geq p\,(\sqrt{p/q})$, so $\alpha+m$ must contain at least one complete period of the periodic continued fraction for $\sqrt{p/q}$. Given our construction of $\alpha$, we see that

$$\sqrt{p/q} = [a_0+m, \overline{W; 2\,(a_0+m)}],$$

for some word $W$, hence $2(a_0+m)$ is a partial quotient of $\alpha+m$. Furthermore, by our construction we have

$$2\,(a_0+m) \in \mathcal{A} = \mathcal{A}' \cup \{2(a_0+\mathcal{S})\}.$$

Since $\mathcal{A}'$ contains only odd integers, $2\,(a_0+m) \in 2(a_0+\mathcal{S})$ and thus it follows that $m \in \mathcal{S}$, which is a contradiction. Therefore we must have

$$c_m\, q^{-q\,(\log q)^{1+\varepsilon}} < \left| (\alpha+m)^2 - \frac{p}{q} \right|,$$

which is the last inequality of the theorem.

It remains only to show that for $m \notin \mathcal{S}$, $(\alpha+m)^2$ is *not* Liouville of type $f_\varepsilon(q) = q\,(log\ q)^{1+\varepsilon}$. If $(\alpha+m)^2$ were Liouville of this type then there would exist a constant $\tilde{c} = \tilde{c}\,(\alpha, m) > 0$ and an infinite collection of distinct rationals $\{\tilde{p}_n/\tilde{q}_n\}$ so that

$$\left| (\alpha+m)^2 - \frac{\tilde{p}_n}{\tilde{q}_n} \right| \leq \tilde{c}\, \tilde{q}_n^{-f_\varepsilon(\tilde{q}_n)}.$$

Since $\alpha+m$ is a $U_2$-number, there must exist an infinite subsequence of $\{\tilde{p}_n/\tilde{q}_n\}$ consisting of rationals which are not perfect square rationals. Thus without loss of generality we may assume that none of the rationals $\tilde{p}_n/\tilde{q}_n$ are perfect square rationals.

We select an integer $T$ so large that for all $n \geq T$,

$$\tilde{q}_n > \max\left\{ 504, \exp\,(1+(\alpha+m)^2)^{1/(2\varepsilon)} \right\},$$

and

$$\tilde{c}\, \tilde{q}_n^{-f_\varepsilon(\tilde{q}_n)} < (q_{N+1}\, q_{N+2})^{-1},$$

where $N = [11.28 f_\varepsilon(\tilde{q})] + 1$. (this is possible by the weak inequality in (3.3)). Thus, as before, we have

$$\left| \alpha+m - \sqrt{\frac{\tilde{p}_n}{\tilde{q}_n}} \right| < \frac{1}{q_{N+1}\, q_{N+2}},$$

and by our choice of $N$ (which is at least $p\,(\sqrt{\tilde{p}_n/\tilde{q}_n})$), we conclude that $2\,(a_0+m) \in 2\,(a_0+\mathcal{S})$ and hence $m \in \mathcal{S}$ which is a contradiction. Therefore for $m \notin \mathcal{S}$, $(\alpha+m)^2$ is not Liouville of the type $q\,(\log q)1+\varepsilon$ which completes the proof.

## BADLY APPROXIMABLE $U_2$-NUMBERS

Here we provide a sketch of the proof of Theorem 4. If the set $\mathcal{A}$ has finite cardinality then we may replace inequalities (3.1) and (3.2) by

$$q_{N+1}\, q_{N+2} \le (M+1)^{3N},$$

where $M = \max\ \{a : a \in \mathcal{A}\}$. Given the choice of $N$ in the proof of Theorem 3, we have

$$(M+1)^{3N} \le (M+1)^{36\, q\, (\log q)^{1+\varepsilon}} = q^{C\, q\, (\log q)^{\varepsilon}},$$

where $C = 36 \log (M + 1)$. The proof of Theorem 4 now follows from the proof of Theorem 3 and the above observations.

## CONCLUDING REMARKS

Suppose now that we are given a probability measure $\mathcal{P}$ on $\mathcal{A}$. By adopting similar methods as in [6], we may insure that for each $a \in \mathcal{A}$, the asymptotic density of $a$ in the sequence $\{a_0, a_1, \ldots\}$ (recall that $\alpha = [a_0, a_1, \ldots])$ is equal to $\mathcal{P}(a)$. As an illustration, suppose that $\mathcal{S} = \{0\}$ and $\mathcal{A} = \{1, 2\}$. Then Theorem 4 along with the previous remark imply that there exists a badly approximable $U_2$-number $\alpha = [a_0, a_1, \ldots]$ such that all the partial quotients are either 1 or 2 and the asymptotic density of 1 in $\{a_0, a_1, \ldots\}$ is 1 while the asymptotic density of 2 is 0. Furthermore, if $k$ is a nonnegative integer then $(\alpha + k)^2$is Liouville of type 36 (log 3) $q\,(\log q)^{\varepsilon}$ if and only if $k = 0$. It is interesting to note that the partial quotients of $\alpha$ statistically resemble those of $\frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$ and thus $\alpha$ is a transcendental number which, in some vague sense, is as "badly approximable as possible."

Finally, inspired by our main results, we close by posing the following open question: *Given a set of nonnegative integers $\mathcal{S}$, do there exist $U_2$-numbers $\alpha$ so that for a nonnegative integer $k$, $(\alpha + k)^2$ is Liouville if and only if $k \in \mathcal{S}$ ?*

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