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INTERPOLATION PROBLEMS OF LOEWNER TYPE WITH INFINITELY MANY NODES

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Two versions of Loewner type boundary interpolation problems with infinitely many nodes are considered, where also (estimates of) radial residues of the unknown function are prescribed at some points of the real line. Necessary and sufficient conditions for solvability of these problems are established, in terms of properties (nonnegativity, continuity etc.) of a kernel (called Pick kernel) built from interpolation data.

Key Words : Loewner Type Interpolation Problem; Pick Kernel; Radial Boundary Value; Radial Derivative

INTRODUCTION

This paper should be regarded as a continuation of [1]. Here we study the same interpolation problems as in [1], but with infinitely many nodes. Namely, we consider the existence aspect of the problem of recapturing a Pick function from its radial boundary values and (estimates of) radial derivatives at points of a subset L of the real axis R and from (some estimates of) its radial residues at points of another subset M of R, where the set $L \cup M$ is infinite.

Using results of [1] and [6], we establish criteria for solvability of these problems. So we show, in Theorem 1, that nonnegativity of the Pick kernel $\mathcal{K}$ is necessary and sufficient for solvability of the estimates version of the problem, except the case $L = \phi$ where an additional conditions is needed (see (3)). The most natural way to produce interpolants with infinitely many nodes from those with finitely many nodes is the use of normal families. However, if a Pick function ψ is the limit, in the sense of uniform convergence on compacta, of a sequence (ψ_n) of Pick functions and if each ψ_n has radial derivative $\psi_n{}'(x)$ (resp. radial residue $\psi_n^{(-1)}(x)$) at a point $x \in L$ (resp. $x \in M$), equal $\varphi^1(x)$ (resp. $\varphi(x)$), then $\psi'(x)$ (resp. $\psi^{(-1)}(x)$) need not equal $\varphi^1(x)$ (resp. $\varphi(x)$). We found conditions on $\mathcal{K}$ being necessary and ensuring $\psi'(x) = \varphi^1(x), x \in L$, and $\psi^{(-1)}(x) = \varphi(x), x \in M$ (Theorem 2), though under some additional assumptions. Our Theorem 2 can be regarded as an extension of the Loewner interpolation theorem[2].

To avoid repetitions, we refer to [1] for notes on genesis and development of the Loewner type interpolation problems. The problems with infinitely many nodes have been studied in [2], [3] and [4].

Infinite Interpolation

In order to introduce boundary value interpolation problems of Loewner and Pick-Nevanlinna type with infinitely many nodes, we shall adapt the formulations of the problems I and I' given in [1].

Let L and M be disjoint subsets of $\boldsymbol{R}$, not both empty, and $L \cup M =: E$. Let φ be a function from E into $\boldsymbol{R}$, such that $\varphi(x) \neq 0$ for $x \in M$, and let φ^1 be a function from L into $\boldsymbol{R}$.

Denote the following problem by $I_\infty = I_\infty(\varphi)$.

Find a function $\psi : C^+ \to C$ satisfying

(i) ψ is analytic in C^+;

(ii) $\operatorname{Im}\psi(z) \geq 0, \ z \in C^+$;

(iii) for any $x \in L$ there exist the radial boundary value

$$\lim_{y \downarrow 0} \psi(x+iy) =: \psi(x)$$

and the radial derivative

$$\lim_{y \downarrow 0} \psi'(x+iy) =: \psi'(x)$$

and $\psi(x) = \varphi(x), \ \psi'(x) = \varphi^1(x)$; and

(iv) for any $x \in M$ the radial residue

$$\lim_{y \downarrow 0} iy\, \psi(x+iy) =: \psi^{(-1)}(x)$$

equals $\varphi(x)$.

Let $I'_\infty = I'_\infty(\varphi)$ denote the problem which is obtained from I_∞ by replacing the equality $\psi'(x) = \varphi^1(x)$ in (iii) by the inequality

$$\psi'(x) \leq \varphi^1(x), \ \ x \in L \qquad \text{... (1)}$$

and the equality $\psi^{(-1)}(x) = \varphi(x)$ in (iv) by the inequality

$$\psi^{(-1)}(x) \leq \varphi(x), \ \ x \in M. \qquad \text{... (2)}$$

The problem I_∞ is a Loewner type boundary interpolation problem, whereas I'_∞ is a boundary interpolation problem of Pick-Nevanlinna type.

Given, L, M, φ and φ^1 as in problems I_∞ and I'_∞, we define a kernel $\mathcal{K} = \mathcal{K}(\varphi; \cdot, \cdot)$:

$$\mathcal{K}(u, x) := \frac{\varphi(x) - \varphi(u)}{x - u}, \ \ u, x \in L, \ \ u \neq x;$$

$$\mathcal{K}(u, u) := \varphi^1(u), \ \ u \in L;$$

$\mathcal{K}(u, x) = \mathcal{K}(x, u) := \dfrac{\varphi(u)}{u-x}, \ x \in L, \ u \in M;$

$\mathcal{K}(u, x) := 0, \ u, x \in M, \ u \neq x;$

$\mathcal{K}(u, u) := -\varphi(u), \ u \in M.$

We will call $\mathcal{K}$ the *Pick kernel* to the problem I_∞, as well as to the problem I'_∞.

The cardinal number of the largest linearly independent subset of the set of functions $\mathcal{K}(u, \cdot), u \in E$, will be called *rank* of $\mathcal{K}$ and denoted by *rank*$\mathcal{K}$.

If L_1 and M_1 are some subsets of L and M respectively, not both empty, and $L_1 \cup M_1 =: E_1$, $\varphi/E_1 =: \varphi_1$, $\varphi^1/L_1 := \varphi_1^1$, then the kernel $\mathcal{K}(\varphi_1; \cdot, \cdot)$ will be called a Pick restriction of $\mathcal{K}(\varphi_1; \cdot, \cdot)$.

In the case when E_1 is finite, the points of E_1 can be arranged as a sequence $x_1, \ldots, x_l, x_{l+1}, \ldots, x_n$ such that $x_\lambda \in L_1, 1 \le \lambda \le l$ (= *card* L_1) and $x_{l+\mu} \in M_1, 1 \le \mu \le n-l$ (= *card* M_1) (n = *card* E_1). The kernel $\mathcal{K}(\varphi_1; \cdot, \cdot)$, together with this arrangement of E_1, determines a matrix $\mathcal{P}_1$. It is transparent that $\mathcal{P}_1$ is the Pick matrix (in the sense of [1]) to the problems $I(\varphi_1)$ and $I'(\varphi_1)$ (with respect to the above arrangement of E_1). As well, it is clear that $\mathcal{P}_1 \ge 0$ iff $\mathcal{K}(\varphi_1; \cdot, \cdot) \ge 0$.

The kernel $\mathcal{K}(\varphi_1; \cdot, \cdot)$ will be said to have the property (*) iff *rank* $\mathcal{K}(\varphi_1; \cdot, \cdot)$ = *rank* $\mathcal{K}(\varphi_1; \cdot, \cdot)$ for any Pick restriction $\mathcal{K}(\varphi_1; \cdot, \cdot)$ of $(\varphi_1; \cdot, \cdot)$ such that *rank* $(\varphi_1; \cdot, \cdot) <$ *card* $E_1 < \infty$ (if such a restriction exists).

It is obvious that $\mathcal{K}(\varphi_1; \cdot, \cdot)$ has the property (*) if and only if for any finite subset E_1 of E the corresponding Pick matrix $\mathcal{P}_1$ has the property (*).

First, we consider the problem I'_∞.

Theorem 1 — *Necessary and sufficient for the problem I'_∞ to have a solution is that the kernel $\mathcal{K}$ be nonnegative and that*

$$\sum_{u \in M} \frac{-\varphi(u)}{u^2+1} < +\infty. \qquad \ldots (3)$$

PROOF : *Necessity* — Nonnegativity of $\mathcal{K}$ follows from the necessity part of Theorem 2 of [1]. Let ψ be a solution of the problem I'_∞ and show that (3) must hold. As ψ satisfies (i) and (ii), there exists a unique functional Hilbert space H whose reproducing kernel is

$$K(w, z) := \frac{\psi(z) - \overline{\psi(w)}}{z - \overline{w}}, \ \ z, w \in C^+,$$

[5] (Th. 5). According to [6] (Lemma 1 and remark 3), the function

$$\frac{\partial^{-1}}{\partial \overline{w}^{-1}} K(u, z) := \frac{\psi^{(-1)}(u)}{u-z}, \ z \in C^+,$$

belongs to H for any $u \in M$, all such functions form an orthogonal set in H and $\| \partial^{-1}/\partial \overline{w}^{-1} K(u, \cdot) \|_H^2 = -\psi^{(-1)}(u)$, $u \in M$. This implies that

$$\sum_{u \in M} \left| \left\langle K(i, \cdot), \frac{\partial^{-1}}{\partial \overline{w}^{-1}} k(u, \cdot) \right\rangle_H \right|^2 \cdot \left\| \frac{\partial^{-1}}{\partial \overline{w}^{-1}} K(u, \cdot) \right\|^{-2} \leq \| K(i, \cdot) \|_H^2 \quad \text{i.e.,}$$

$$\sum_{u \in M} \frac{-\psi^{(-1)}(u)}{|u - i|^2} \leq \operatorname{Im} \psi(i).$$

Since ψ satisfies (2), it follows that (3) really holds.

Sufficiency — The case when E is at most countable : If E is finite, then a solution exists by [1] (Th. 2). If E is countable and $L = \phi$, then we can set

$$\psi(z) := \sum_{u \in M} \frac{1 + zu}{u - z} \frac{-\varphi(u)}{u^2 + 1}, \; z \in C^+.$$

Since $(1 + zu)/(u - z) = (1 + z^2)/(u - z) + z$ is bounded by $|1 + z^2|/\operatorname{Im} z + |z|$, and (3) holds, it follows that the above series converges uniformly on compacta in C^+. Thus, ψ satisfies (*i*). Since any function (in z) $(1 + zu)/(u - z)$ satisfies (*ii*), and $\varphi(u) < 0, u \in < M$, (by nonnegativity of $\mathcal{K}$), it follows that ψ satisfies (*ii*). To show that ψ satisfies also (2), let $x \in M$ and set

$$\theta_x(z) := \frac{1 + zx}{x - z} \frac{-\varphi(x)}{x^2 + 1}, \quad z \in C^+,$$

and $\psi_x := \psi - \theta_x$. The functions θ_x and ψ_x, evidently, satisfy (*i*) and (*ii*). Since $\psi_x^{(-1)}(x) \leq 0$, [6] (Lemma 1), it follows that $\psi^{(-1)}(x) \leq \theta_x^{(-1)}(x) = \varphi(x)$. This shows that ψ satisfies (2) and is a solution of the problem I'_∞.

Now let $L \neq \phi$. Since E is countable, there exists a sequence (E_n) of finite subsets of E, such that $E_n \subset E_{n+1}, n \in N$, and $\bigcup_{n=1}^{\infty} E_n = E$. For any $n \in N$, set $\varphi/E_n =: \varphi_n$ and $\varphi^1/L_n =: \varphi_n^1$, where L_n denotes $E_n \cap L$. According to [1] (Th. 2), every problem $I'_\infty(\varphi_n)$ has a solution ψ_n. Since the functions $(\psi_n - i)/(\psi_n + i), n \in N$, form a normal family (as they are bounded by 1), we can choose a subsequence of the sequence (ψ_n) converging uniformly on compact subsets of C^+, where the limit can be ∞ in the whole C^+ or nowhere in C^+. Assume, as we may, that the sequence (ψ_n) itself is such a subsequence. We shall show that the limit ψ of the sequence is a solution of the problem $I'_\infty(\varphi)$.

For any $n \in N$, there exists a unique functional Hilbert space H_n whose reproducing kernel is [5] (Th. 5).

$$K_n(w, z) := \frac{\psi_n(z) - \overline{\psi_n(w)}}{z - \overline{w}}, \; w, \; z \in C^+,$$

If $x \in L$, then $x \in E_n$ for each sufficiently large n, and the function

$$K_n(x, z) := \frac{\psi_n(z) - \psi_n(x)}{z - x} = \frac{\psi_n(z) - \varphi(x)}{z - x}, \; z \in C^+,$$

belongs to H_n and

$$\langle f, K_n(x, \cdot) \rangle_{H_n} = \lim_{y \downarrow 0} f(x + iy) = : f(x), \; f \in H_n,$$

[6] (Th. 1 and Remark 1). If $y > 0$ and $x \in E_n$, then we have

$$K_n(x + iy, x + iy) \leq K_n(x, x) \qquad \text{... (4)}$$

(see [6] (Remark 2)) and therefore

$$| K_n(x, x + iy) |^2 \leq K_n(x, x) \cdot K_n(x + iy, x + iy) \leq K_n(x, x)^2 = \psi_n'(x)^2 \leq \varphi^1(x)^2,$$

which implies $| \psi_n(x + iy) - \varphi(x) | \leq y \, \varphi^1(x)$. ... (5)

This shows that $\psi_n(x + iy)$ can not tend to ∞ as $n \to +\infty$, and, consequently, that the limit ψ of the sequence (ψ_n) is a function from C^+ into C.

It is clear that ψ satisfies (*i*) and (*ii*). Let $n \to +\infty$ in (5). Then we obtain

$$| \psi(x + iy) - \varphi(x) \leq \varphi^1(x), \; y > 0, \qquad \text{... (6)}$$

which shows that

$$\lim_{y \downarrow 0} \psi(x + iy) = \varphi(x),$$

i.e., that ψ satisfies $\psi(x) = \varphi(x), x \in L$.

Since $K_n(x, x) = \psi_n'(x) \leq \varphi^1(x)$, it follows from (4) that $K_n(x + iy, x + iy) \leq \varphi^1(x)$. Let $n \to +\infty$:

$$K(x + iy, x + iy) \leq \varphi^1(x). \qquad \text{... (7)}$$

By Theorem 1 of [6], there exists the radial derivative $\psi'(x)$ and

$$\lim_{y \downarrow 0} K(x + iy, x + iy) = \psi'(x),$$

so that (1) is satisfied.

Now, let $x \in M$ and $\psi_n^* := -1/\psi_n, n \in N$. Then $x \in E_n$ for each sufficiently large n and $\psi_n^{(-1)}(x) \leq \varphi(x)$. Since ψ_n^* satisfies (*i*) and (*ii*) and since $\psi_n^{(-1)}(x) = -\psi_n^{*\prime}(x)^{-1}$, we have $\psi_n^{*\prime}(x) \leq -1\ \varphi(x)$. Putting ψ_n^* instead of ψ_n in (5), we obtain $|\psi_n^*(x+iy)| \leq -y/\varphi(x)$ (as $\psi_n^*(x) = 0$), which yields $|iy\,\psi_n(x+iy)| \geq -\varphi(x)$. Let $n \to +\infty$: $|iy\,\psi(x+iy)| \geq -\varphi(x), y > 0$. This implies $|\psi^{(-1)}(x)| \geq -\varphi(x)$, i.e. $\psi^{(-1)}(x) \leq \varphi(x)$, $\psi^{-1}(x)$ is a negative real number.

Thus, ψ is a solution of the problem $I'_\infty(\varphi)$.

The case when E is uncountable. We shall reduce this case to the preceding. It follows from (3) and $-\varphi(u) > 0, u \in M$, that M must be at most countable.

We shall extract from E a countable subset G, such that $M \subset G$, that $G \cap L$ is everywhere dense in L and that

$$\liminf_{G \cap L \ni u \to x} \varphi^1(u) \leq \varphi^1(x), x \in L \setminus G. \qquad \text{... (8)}$$

For any $n \in N$ and $k \in \mathbf{Z}$, denote the set

$$L \cap \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

by L_{nk}. Let S be the set of all ordered pars $(n, k) \in N \times Z$ for which $L_{nk} \neq \phi$. If $(n, k) \in S$, then let

$$s_{nk} := \inf\left\{\varphi^1(x) : x \in L_{nk}\right\}$$

and let x_{nk} be a point in L_{nk} such that $\varphi^1(x_{nk}) < s_{nk} + 1/n$. Finally, let $G := \{x_{nk} : (n, k) \in S\} \cup M$.

It is clear that G is countable and $M \subset G$. Since for each $x \in L$ and $n \in N$ there is a (unique) $k(n) = k \in \mathbf{Z}$ such that $x \in [k/2^n, (k+1)/2^n)$ and $G \cap L_{nk} \neq \phi$, it follows that $G \cap L$ is everywhere dense in L. If we fix $x \in L \setminus G$, then the sequence $(s_{n,k(n)})$ is nondecreasing and $s_{n,k(n)} \leq \varphi^1(x), n \in N$, so that there exists the limit $\lim_{n \to +\infty} s_{n,k(n)} =: s(x)$ and $s(x) \leq \varphi^1(x)$. Since $s_{n,k(n)} \leq \varphi^1(x_{n,k(n)}) < s_{n,k(n)} + 1/n, n \in N$, it follows that $\lim_{n \to +\infty} \varphi^1(x_{n,k(n)}) = s(x)$, which shows that (8) is also true.

Set $\varphi/G =: \varphi_G$, $\varphi^1/G \cap L =: \varphi_G^1$, and consider the corresponding problem $I'_\infty(\varphi_G)$. Since G is countable, this problem has a solution ψ, by the preceding case. We shall show that ψ is also a solution of the problem $I'_\infty(\varphi)$.

As $M \subset G$, we have only to show that ψ satisfies $\psi(x) = \varphi(x)$ and (1) at each point $x \in L \setminus G$. Let x be a point of $L \setminus G$ and (u_n) a sequence of points of $G \cap L$ such that $\lim_{n \to +\infty} u_n = x$ and

$$\lim_{n \to +\infty} \varphi^1 (u_n) = \liminf_{G \cap L \ni u \to x} \varphi^1 (u) =: l(x).$$

Since $$| \mathcal{K}(\varphi; x, u_n) |^2 \leq \mathcal{K}(\varphi; x, x)\, \mathcal{K}(\varphi; u_n, u_n) = \varphi^1 (x)\, \varphi^1 (u_n),$$

i.e. $$| \varphi (x) - \varphi (u_n) |^2 \leq | x - u_n |^2 \varphi^1 (x)\, \varphi^1 (u_n), n \in N, \quad \text{... (9)}$$

it follows that $\lim \varphi (u_n) = \varphi (x)$. Put u_n instead of x in (6) and (7), and let $n \to +\infty$, to obtain : $| \psi (x + iy) - \varphi (x) | \leq y l (x), y > 0$ and $K (x + iy, x + iy) \leq l (x), y > 0$.

Now, let $y \downarrow 0 : \psi (x) = \varphi (x), K (x, x) \leq l(x)$. As $K(x, x) = \psi' (x)$ and $l(x) \leq \varphi^1 (x)$ (see (8)), it follows that $\psi' (x) \leq \varphi^1 (x)$.

Thus, ψ is a solution of the problem $I'_\infty (\varphi)$.

The proof is finished.

Clearly, if E is finite, then the condition (3) comes for free and the above theorem reduces to Theorem 2 in [1].

The proof of Theorem 1 shows that in the case $L \neq \phi$ the inequality (3) is a consequence of nonnegativity of $\mathcal{K}$. Indeed, the inequality (3) has been used only in the case $L = \phi$ of the sufficiency part of the proof. Notice that the same fact can be established also directly.

The next theorem concerns the problem I_∞. We establish a solvability condition for the problem, under some additional assumptions.

***Theorem* 2** — *Let $E \subset L'$ in the problem I_∞. Suppose also that*

$$\limsup_{L \ni u \to x} \varphi^1 (u) < +\infty, \; x \in L \quad \text{... (10)}$$

and that $$\limsup_{L \ni u \to x} \frac{\varphi^1 (u)}{\varphi (u)^2} < +\infty, \; x \in M, \quad \text{... (11)}$$

Necessary and sufficient for the problem I_∞ to have a solution is that the kernel $\mathcal{K}$ be nonnegative, have the property (*), that $\mathcal{K}/L^2$ be continuous as a function of one variable at any point of L^2 and that

$$\lim_{L \ni u \to x} (u - x)\, \varphi (u) = \varphi (x), \; x \in M. \quad \text{... (12)}$$

If L contains an interval (a, b) then the solution of the problem I_∞ is unique and can be extended analytically across (a, b).

PROOF : *Necessity* : Theorem 1 of [1] easily implies that $\mathcal{K} \geq 0$ and (*) are necessary for solvability of the problem. In order to show that the continuity condition and (12) also are necessary, let ψ be a solution of the problem I_∞ and H the corresponding functional Hilbert space whose reproducing kernel is $K (w, z) = [\psi (z) - \overline{\psi (w)}]/(z - \overline{w}),\; w, z \in C^+$, [5] (Th.5). As in the proof of Theorem 1, any function $K (x, \cdot), x \in L$, belongs to H and

$$\langle f, K (x, \cdot) \rangle_H = f (x), f \in H,$$

[6] (Th. 1 and Remark 1). As well, for any $x \in M$ the function $\partial^{-1}/\partial \overline{w}^{-1} K(x, z) := \psi^{(-1)}(x)/(x-z)$, $z \in C^{+}$, belongs to H and

$$\left\langle f, \frac{\partial^{-1}}{\partial \overline{w}^{-1}} K(x, \cdot) \right\rangle_H = f^{-1}(x), f \in H,$$

[6] (Lemma 1 and Remark 3).

Let $x \in L$. The assumption (10) implies that $\varphi^{1}(u)$ is bounded for $u \in L$ in a neighbourhood V of x. Since $K(u, u) = \psi'(u) = \varphi^{1}(u)$, this means that the family of H functions $K(u, \cdot), u \in L \cap V$, is bounded in norm. Putting $u \in L \cap V$ instead of u_n in (9) and using once more the boundedness of $\varphi^{1}(u), u \in L \cap V$, we conclude that $\lim_{L \cap V \ni u \to x} \varphi(u) = \varphi(x)$, i.e., that $\lim_{\cap V \ni u \to x} \psi(u) = \psi(x)$. This implies that for any $z \in C^{+}$ we have $\lim_{L \cap V \ni u \to x} K(u, z) = K(x, z)$, which means that

$$\lim_{L \cap V \ni u \to x} \langle K(u, \cdot), K(z, \cdot) \rangle_H = \langle K(x, \cdot), K(x, \cdot) \rangle_H.$$

Since the functions $K(z, \cdot), z \in C^{+}$, are complete in H, it follows that $K(u, \cdot)$ converges weakly in H to $K(x, \cdot)$ as $L \cap V \ni u \to x$. In particular, $K(u, v) = \mathcal{K}(u, v)$ tends to $K(x, v) = K(x, v)$ as $L \cup V \ni u \to x$, for any $(x, v) \in L^2$. This shows that $\mathcal{K}(u, v)$ is continuous in u at each point $(x, v) \in L^2$.

Now, let $x \in M$. The assumption (11) implies that $\varphi(u) = \psi(u) \neq 0$ and that $\psi'(u)/\psi(u)^2$ is bounded for $u \in L$ in a neighbourhood V of x (as $\psi'(a) = \varphi^{1}(u)$). Since $\| K(u, \cdot) \|_H^2 = \psi'(u)$, this means that the following set of H functions: $\psi(u)^{-1} K(u, \cdot), u \in L \cap V$ is bounded in norm. Since

$$\left| \left\langle \frac{\partial^{-1}}{\partial \overline{w}^{-1}} K(x, \cdot) K(u, \cdot) \right\rangle_H \right|^2 \leq \left\| \frac{\partial^{-1}}{\partial \overline{w}^{-1}} K(x, \cdot) \right\|^2 \| K(u, \cdot) \|^2,$$

i.e.,

$$\left| \frac{\psi^{(-1)}(x)}{x-u} \right|^2 \leq \frac{\psi'(u)}{\psi(u)^2} [-\psi^{(-1)}(x)] \psi(u)^2,$$

it follows that $\lim_{L \cap V \ni u \to x} |\psi(u)| = +\infty$, and therefore $\lim_{L \cap V \ni u \to x} \psi(u)^{-1} K(u, z) = 1/(x-z) = 1/(x-z)$, $z \in C^{+}$, which means that

$$\lim_{L\cap V\ni u\to x}\left\langle \frac{1}{\psi(u)}K(u,\cdot),K(z,\cdot)\right\rangle_H=\left\langle \frac{1}{\psi^{(-1)}(x)}\frac{\partial^{-1}}{\partial\overline{w}^{-1}}K(x,\cdot),K(z,\cdot)\right\rangle_H \quad \ldots (13)$$

$z\in C^+$. As the functions $K(z,\cdot), z\in C^+$, are complete in H, it follows that $\psi(u)^{-1}k(u,\cdot)$ converges weakly in H to $\psi^{(-1)}(x)^{-1}\partial^{-1}/\partial\overline{w}^{-1}k(x,\cdot)$ as $L\cap V\ni u\to x$. Thus, we may put $\partial^{-1}/\partial\overline{w}^{-1}$ $K(x,\cdot)$ in (13) instead of $K(z,\cdot)$, to obtain

$$\lim_{L\ni u\to x}\frac{1}{\psi(u)}\frac{\psi^{(-1)}(x)}{x-u}=-1, \quad \ldots(14)$$

which is equivalent to (12).

Sufficiency : If *rank* $\mathcal{K}=: r<+\infty$, then for any finite subset G of E with *card* $G>r$ the corresponding problem $I_\infty(\varphi_G)$ $(\varphi_G:=\varphi/G, \varphi_G^1:=\varphi^1/G)$ has a unique solution, by Theorem 1 of [1]. It is easily seen that then the problem $I_\infty(\varphi)$ also has a (unique) solution.

Let *rank* $\mathcal{K}=\infty$. Since $\mathcal{K}\geq 0$, there exists a solution ψ of the problem $I'_\infty(\varphi)$. We shall show that ψ is a solution of the problem I_∞.

We have only to check that $\psi'(x)=\varphi^1(x), x\in L$, and $\psi^{(-1)}(x)=\varphi(x), x\in M$. Let $x\in L$. Since $\mathcal{K}(u,x)$ is continuous in u at the point (x,x) and $\mathcal{K}(u,x)=K(u,x)$ for $u\in L\backslash\{x\}$, it follows that $\mathcal{K}(x,x)=\lim_{L\ni u\to x}K(u,x)$. On the other hand, we have $\lim_{L\ni u\to x}K(u,x)=K(x,x)$, by the necessity part of this proof. Thus, it must be $K(x,x)=\mathcal{K}(x,x)$, i.e. $\psi'(x)=\varphi^1(x)$.

Now, let $x\in M$. Since $\psi(u)=\varphi(u), u\in L$, and (12) holds, it follows that $\lim_{L\ni u\to x}(u-x)$ $\psi(u)=\varphi(x)$. On the other hand, we have $\lim_{L\ni u\to x}(u-x)\psi(u)=\psi^{(-1)}(x)$, by the necessity part of this proof (see 14). Thus, $\psi^{-1}(x)=\varphi(x)$.

If L contains an interval (a,b) and ψ is a solution of the problem I_∞, then we have $|K(u,z)|^2\leq K(u,u)K(x,x)$, i.e. $|\psi(z)-\psi(u)|^2\leq|z-u|^2\varphi^1(u)\varphi^1(x)$ for each $u\in L$ and $z=x+iy$ such that $x\in(a,b), y\geq 0$. Together with (10), this shows that $\lim_{D\ni z\to u}\psi(z)=\psi(u)$ for any $u\in(a,b)$, where $D:=\{z:\mathbf{Re}\,z\in(a,b),\operatorname{Im}z\geq 0\}$.

Extend ψ by the symmetry principle: $\psi(z):=\overline{\psi(\overline{z})}, z\in C^-$. The so extended ψ is analytic on C^+ and C^- and continuous on (a,b). Thus, it is analytic also at points of (a,b), i.e. ψ is extended analytically across (a,b).

Any two (extended) solutions of the problem I_∞ coincide on (a,b) and therefore they must coincide everywhere.

The proof is finished.

The classical Loewner theorem[2] is the special case of the last theorem when $M=\phi$, E is an interval on $\mathbf{R}$ and φ^1 is supposed to be the derivative of φ and continuous. M. Rosenblum and J. Rovnyak[3] generalized the Loewner theorem to Borel subsets Δ of R such that neither Δ nor $\mathbf{R}\backslash\Delta$ are zero sets with respect to the Lebesgue measure.

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